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An almost sure limit theorem for Wick powers of Gaussian differences quotients

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Abstract: Let $G=\{G(x),x\in R_+\}$, $G(0)=0$, be a mean zero Gaussian process with $E(G(x)-G(y))^2=\sigma^2(x-y)$. Let $\rho(x)=\frac{1}{2}\frac{d^2}{dx^2}\sigma^2(x)$, $x\neq 0$. When ρ^k is integrable at zero and satisfies some additional regularity conditions,

$$\lim_{h\downarrow 0}\int :\left(\frac{G(x+h)-G(x)}{h}\right)^k : g(x)\,dx =: (G')^k : (g) \qquad a.s.$$

for all $g\in\mathcal{B}_0(R^+)$, the set of bounded Lebesgue measurable functions on R_+ with compact support. Here G' is a generalized derivative of G and $:(\,\cdot\,)^k:$ is the k–th order Wick power.

1. Introduction

Let $G=\{G(x),x\in R_+\}$, $G(0)=0$, be a mean zero Gaussian process with stationary increments, and set

$$E(G(x)-G(y))^2=\sigma^2(x-y)=\sigma^2(|x-y|). \tag{1.1}$$

(The function σ^2 is referred to as the increment's variance of G.) We assume that

$$\sigma^2(h) \text{ is a convex function that is regularly varying at zero;} \tag{1.2}$$

$$\lim_{h\to 0}\frac{h^2}{\sigma^2(h)}=0 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\sigma^2(h)}{h}=0; \tag{1.3}$$

$$\frac{\sigma^2(s+h)+\sigma^2(s-h)-2\sigma^2(s)}{h^2}\le C\frac{\sigma^2(s)}{s^2} \quad \text{for } h\le\frac{s}{8}; \tag{1.4}$$

$$\sigma^2(s) \text{ has a second derivative for each } s\neq 0. \tag{1.5}$$

Note that by (1.2)

$$\rho(s):=\frac{1}{2}\frac{d^2}{ds^2}\sigma^2(s)\ge 0. \tag{1.6}$$

It follows from the second condition in (1.3) that G has a continuous version; (see [6, Lemma 6.4.6]). We work with this version. However, when $\lim_{x\to 0}\rho(x)=\infty$,

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G is not differentiable; it is not even mean square differentiable. It is a natural question to ask whether the weak limit

$$\lim_{h\to 0}\int\left(\frac{G(x+h)-G(x)}{h}\right)g(x)\,dx \tag{1.7}$$

exists in some sense. Here $g\in\mathcal{B}_0(R_+)$, the set of bounded Lebesgue measurable functions on R_+ with compact support.

We show in [7, Theorem 2.1] that when G satisfies the second condition in (1.3) there exists a mean zero Gaussian field $\{G'(g), g\in\mathcal{B}_0(R_+)\}$ with covariance

$$E\left(G'(g)G'(\widetilde{g})\right)=\int\int\rho(t-s)\,g(s)\,\widetilde{g}(t)\,ds\,dt \tag{1.8}$$

such that

$$\lim_{h\to 0}\int\left(\frac{G(x+h)-G(x)}{h}\right)g(x)\,dx=G'(g)\qquad\text{in } L^2. \tag{1.9}$$

Because of this we think of G' as a generalized derivative of G.

More generally, one may consider

$$\lim_{h\to 0}\int\left(\frac{G(x+h)-G(x)}{h}\right)^k g(x)\,dx \tag{1.10}$$

for any integer $k\geq 1$. However, when k is even, the expectation of the square of the integral in (1.10) contains terms in $\sigma^2(h)/h^2$ which goes to infinity as h goes to zero by (1.3). To obtain a finite limit in (1.10) we replace $(\frac{G(x+h)-G(x)}{h})^k$ by a k–th order polynomial

$$\sum_{j=0}^{k}a_j(h)\left(\frac{G(x+h)-G(x)}{h}\right)^j \tag{1.11}$$

where, $a_j(h)$ is a non–random function of h, which, necessarily, has the property that, at least for some $0\leq j\leq k$, $\lim_{h\to 0}|a_j(h)|=\infty$. We call this process renormalization. The renormalization we use is known as the k–th Wick power.

The k–th Wick power of a mean zero Gaussian random variable X is

$$:X^k:=\sum_{j=0}^{[k/2]}(-1)^j\binom{k}{2j}E(X^{2j})\;X^{k-2j}. \tag{1.12}$$

When $X=N(0,1)$,

$$:X^k:=\sqrt{k!}\,H_k(X), \tag{1.13}$$

where H_k is the k–th Hermite polynomial. One advantage of Wick powers over Hermite polynomials is that they are homogeneous, i.e., for $a\in R^1$,

$$:(aX)^k:=a^k:X^k:. \tag{1.14}$$

Therefore, when X has variance σ_X^2,

$$:X^k:=\sqrt{k!}\,\sigma_X^k H_k\left(\frac{X}{\sigma_X}\right). \tag{1.15}$$

When ρ^k is locally integrable and bounded away from the origin we construct a k–th order Wick power Gaussian chaos from the mean zero Gaussian field $G' = \{G'(f), f \in \mathcal{B}_0(R_+)\}$ in the following way: For each $\delta \in (0, \delta_0]$, for some $\delta_0 > 0$, let $f_\delta(s)$ be a continuous positive symmetric function on $(s,\delta) \in R_+ \times (0,1]$, with support in the ball of radius δ, with $\int f_\delta(y)\,dy = 1$. That is, f_δ is a continuous approximate identity. In [7, (3.25) and (3.26)] we show that for the Gaussian processes G considered here, for all $g \in \mathcal{B}_0(R_+)$,

$$:(G')^k:(g) := \lim_{\delta\to 0} \int :(G'(f_{x,\delta}))^k: g(x)\,dx \quad \text{in } L^2 \tag{1.16}$$

and

$$E\left(:(G')^k:(g)\right)^2 = k! \int\int \rho^k(x-y) g(x) g(y)\,dx\,dy. \tag{1.17}$$

In [7, Theorem 3.1] we show that

$$\lim_{h\to 0} \int :\left(\frac{G(x+h)-G(x)}{h}\right)^k: g(x)\,dx = :(G')^k:(g) \qquad \text{in } L^2. \tag{1.18}$$

(For example, if $\sigma^2(h) = h^r$, in order for ρ^k to be locally integrable and to have $\lim_{x\to 0}\rho(x) = \infty$ as required by the first condition in (1.3), it is necessary that $\frac{2k-1}{k} < r < 2$.)

In this paper we obtain the rather remarkable result that, under some additional mild regularity conditions on ρ, the limit in (1.18) is almost sure.

Theorem 1.1. *Let $G = \{G(x), x \in R_+\}$, $G(0) = 0$ be a mean zero Gaussian process with stationary increments satisfying (1.1)–(1.6). Fix an integer $k \ge 1$ and assume that there exists a $0 < \delta < 1/2$ and an $M > 0$ such that*

$$\rho(x) \le \frac{C_M}{|x|^{(1-\delta)/k}} := C_M\,\varphi(|x|), \qquad 0 < |x| \le M \tag{1.19}$$

and

$$|\rho(x+h) - \rho(x)| \le C_M \frac{|h|}{|x|}\,\rho(x), \qquad 4|h| \le |x| \le M. \tag{1.20}$$

Then for all $g \in \mathcal{B}_0(R_+)$,

$$\lim_{h\downarrow 0} \int :\left(\frac{G(x+h)-G(x)}{h}\right)^k: g(x)\,dx = :(G')^k:(g) \qquad a.s. \tag{1.21}$$

(Note that by using Wick powers it is clear that (1.21) deals with a generalized derivative of G. This point would be obscured if we expressed (1.21) in terms of Hermite polynomials.)

For a fixed $g \in \mathcal{B}_0(R_+)$ both the left-hand side and right-hand side of (1.21) are k–th order Gaussian chaoses. Let $\{:X_h^k:(g), h \in (0,1]\}$, denote the left-hand side of (1.21) and $:X_0^k:(g)$ denote the right-hand side of (1.21). Theorem 1.1 is the statement that for all $g \in \mathcal{B}_0(R_+)$, the k-th order Gaussian chaos process

$$\mathcal{X} := \{X_h, h \in [0,1]\} := \{:X_h^k:(g), h \in [0,1]\}, \tag{1.22}$$

has a continuous version. Of course there is no problem in choosing the version. The process on the left in (1.21) is continuous in $h \in (0,1]$. We can take $:(G')^k:(g)$ to be its limit on the set of probability one for which the limit exists, and to be zero otherwise.

Theorem 1.1 is proved in Section 4 using a majorizing measure result for the continuity of Gaussian chaoses. Technically, this is an interesting application of this theory, because the proof consists of obtaining continuity at a single point. To prove (1.21) we need a majorizing measure condition for exponential Orlicz spaces based on the function $\exp x^q - 1$ for $q \leq 1$. Whereas it is known that such results exist we could not find a reference, so we provide proofs in Section 3. We prove Theorem 1.1 in Section 4

2. L^2 results

We list here some L^2 estimates we need in this paper that are obtained in [7]. To better motivate these results we state the main result in [7] and explain how it led to our consideration of Theorem 1.1 in this paper.

Theorem 2.1 (Theorem 1.1, [7]). *Let f be a function with $Ef^2(\eta) < \infty$, where $\eta = N(0,1)$. Then under the hypotheses of Theorem 1.1*

$$\int_a^b f\left(\frac{G(x+h)-G(x)}{\sigma(h)}\right) dx = \sum_{j=0}^{k} (h/\sigma(h))^j \, \frac{E(H_j(\eta)f(\eta))}{\sqrt{j!}} \, :(G')^j:(I_{[a,b]}) + o\left(\frac{h}{\sigma(h)}\right)^k \tag{2.1}$$

in L^2. Here H_j is the j-th Hermite polynomial and $:(G')^j:(I_{[a,b]})$ is a j-th order Wick power Gaussian chaos as described in (1.16).

We wondered whether (2.1) could be almost sure. In Theorem 1.1 we show that when $f(\,\cdot\,) = H_k(\,\cdot\,)$ it is. Note that in this case the right-hand side of (2.1) is

$$\frac{(h/\sigma(h))^k}{\sqrt{k!}} \, :(G')^k:(I_{[a,b]}) + o\left(\frac{h}{\sigma(h)}\right)^k. \tag{2.2}$$

and by (1.13) and (1.14) the left-hand side of (2.1) is

$$\frac{(h/\sigma(h))^k}{\sqrt{k!}} \int_a^b :\left(\frac{G(x+h)-G(x)}{h}\right)^k: \, dx. \tag{2.3}$$

Thus Theorem 2.1 gives the limit in L^2 of (1.21) when $g = I_{[a,b]}$.

The next lemma which is part of [7, Lemma 4.2] provides part of the L^2 metric estimates that are needed in proof of continuity of $\mathcal{X}$.

Lemma 2.1. *Let $G = \{G(x), x \in R_+\}$, $G(0) = 0$, be a mean zero Gaussian process with stationary increments and set $\sigma^2(|x-y|) = E(G(x)-G(y))^2$. Set $\rho(s) = \frac{1}{2}\frac{d^2}{ds^2}\sigma^2(s)$. Fix an integer $j_0 \geq 1$ and assume that there exists a $0 < \delta < 1$ and an $M > 0$ such that (1.20) holds, and (1.19) holds with k replaced by j_0. Then for $1 \leq j \leq j_0$ and any $g \in \mathcal{B}_0(R_+)$,*

$$\| :X_h^j:(g) - :X_0^j:(g) \|_2 \leq C(|h|\varphi^j(h))^{1/2}. \tag{2.4}$$

3. Continuity conditions for stochastic processes in exponential Orlicz spaces

Let $\|\cdot\|_{\psi_q}$ denote the norm in the Orlicz space $L^{\psi_q}(dP)$, where

$$\psi_q(x) = \begin{cases} \exp(x^q) - 1 & 1 \le q < \infty \\ \exp\exp(x) - e & q = \infty. \end{cases} \quad x \in R^+ \tag{3.1}$$

For $0 < q < 1$, we define

$$\psi_q(x) = \begin{cases} K_q\, x & 0 \le x < \left(\frac{1}{q}\right)^{1/q} \\ \exp(x^q) - 1 & x \ge \left(\frac{1}{q}\right)^{1/q} \end{cases} \tag{3.2}$$

where

$$K_q = \frac{\exp(x_0^q) - 1}{x_0} \qquad \text{and} \qquad x_0 := x_0(q) = (1/q)^{1/q}, \tag{3.3}$$

so that $\psi_q(x)$ is continuous.

Lemma 3.1. *For $0 < q < \infty$, $\psi_q(x)$ is convex and increasing and there exists a constant $C_q < \infty$, for which*

$$\psi_q(x) \le C_q\,(\exp(x^q) - 1) \tag{3.4}$$

and

$$\exp(x^q) \le C_q\,(\psi_q(x) + 1)\,. \tag{3.5}$$

In addition $C_q = 1$ for $1 \le q < \infty$.

Proof. This is trivial when $1 \le q < \infty$. We consider the other cases. To show $\psi_q(x)$ is convex we show that its derivative is increasing. It is easy to check that the derivative of $\psi_q(x)$ from the left at x_0 is less than the derivative from the right at x_0. It is also easy to check that the second derivative of $\psi_q(x)$ is positive for $x \ge ((1-q)/q)^{1/q}$. Therefore, the derivative of $\psi_q(x)$ is increasing on $[x_0, \infty)$, so $\psi_q(x)$ is convex.

Since $\exp(x^q) - 1 \ge x^q$ for all $x \ge 0$, we see that (3.4) holds for $0 \le x \le 1$ with $C_q = K_q$. Similarly, choosing m so that $mq \ge 1$, $\exp(x^q) - 1 \ge x^{mq}/m!$, so that (3.4) holds for $1 \le x \le x_0$ with $C_q = m!K_q$. It is then clear that (3.4) with $C_q = \max(m!K_q, 1)$ holds for all x . By further increasing C_q it is easy to see that (3.5) also holds. □

We note the following obvious relationships:

Lemma 3.2.

$$\psi_q^{-1}(x) = \begin{cases} \log\log(e + x) & q = \infty \\ (\log(1+x))^{1/q} & 1 \le q < \infty, \end{cases} \tag{3.6}$$

and for $0 < q < 1$

$$\psi_q^{-1}(x) = \begin{cases} x/K_q & 0 \le x \le K_q x_0 \\ (\log(1+x))^{1/q} & K_q x_0 < x \le \infty. \end{cases} \tag{3.7}$$

For each $0 < q \le \infty$ let $L^{\psi_q}(\Omega, P)$ denote the set of random variables $\xi : \Omega \to C$ such that $E\psi_q(|\xi|/c) < \infty$ for some $c > 0$. $L^{\psi_q}(\Omega, P)$ is a Banach space with norm given by

$$\|\xi\|_{\psi_q} = \inf \{c > 0 : E\psi_q (|\xi|/c) \le 1\}. \tag{3.8}$$

Let (T, d) be a pseudometric space. We use $B_d(t, u)$, or simply $B(t, u)$, to denote a closed ball of radius u in (T, d).

Theorem 3.1. *Let $X = \{X(t) : t \in T\}$ be a measurable separable stochastic process on a separable metric or pseudometric space (T, d) with finite diameter D. Suppose that $X(t) \in L^{\psi_q}(\Omega, P)$ and $\|X(t)-X(s)\|_{\psi_q} \le d(t, s)$ for all $s, t \in T$. Let $0 < q < \infty$ and suppose also that there exists a probability measure μ on T such that*

$$\sup_{t\in T} \int_0^D \left(\log \frac{1}{\mu(B(t,u))}\right)^{1/q} du < \infty. \tag{3.9}$$

Then there exists a version $X' = \{X'(t), t \in T\}$ of X such that

$$E \sup_{t\in T} X'(t) \le C \sup_{t\in T} \int_0^D \left(\log \frac{1}{\mu(B(t,u))}\right)^{1/q} du \tag{3.10}$$

for some $C < \infty$. Furthermore, if

$$\lim_{\epsilon\to 0} \sup_{t\in T} \int_0^\epsilon \left(\log \frac{1}{\mu(B(t,u))}\right)^{1/q} du = 0, \tag{3.11}$$

then X' is uniformly continuous on T almost surely and there exists a positive random variable $Z \in L^{\psi_q}(\Omega, P)$ such that

$$\sup_{\substack{s,t\in T\\ d(s,t)\le\delta}} |X'(s,\omega) - X'(t,\omega)| \le Z(\omega) \sup_{s\in T} \int_0^\delta \left(\log \frac{1}{\mu(B(s,u))}\right)^{1/q} du \tag{3.12}$$

almost surely. When $q = \infty$ the results continue to hold when the above integrands are replaced by

$$\log^+ \log \left(\frac{1}{\mu(B(t,u))}\right). \tag{3.13}$$

(The statement $Z \in L^{\psi_q}(\Omega, P)$ means that $\|Z\|_{\psi_q} \le K_q$, a constant depending only on q.)

We get the following useful corollary of Theorem 3.1

Corollary 3.1. *Under the hypotheses of Theorem 3.1 there exists a constant C_q for which*

$$\Big\| \sup_{\substack{s,t\in T\\ d(s,t)\le\delta}} |X'(s) - X'(t)| \Big\|_{\psi_q} \le C_q \sup_{s\in T} \int_0^\delta \left(\log \frac{1}{\mu(B(s,u))}\right)^{1/q} du \tag{3.14}$$

and for any $t_0 \in T$,

$$\Big\|\sup_{s\in T} |X'(s)| \Big\|_{\psi_q} \le \Big\|X'(t_0)\Big\|_{\psi_q} + C_q \sup_{s\in T} \int_0^D \left(\log \frac{1}{\mu(B(s,u))}\right)^{1/q} du. \tag{3.15}$$

Proof. The statement in (3.14) follows immediately from (3.12). The statement in (3.15) follows from (3.12) by writing

$$\begin{aligned}\sup_{s\in T}|X'(s)| &\le \sup_{s\in T}|X'(s)-X'(t_0)|+|X'(t_0)| \\ &\le \sup_{s,t\in T}|X'(s)-X'(t)|+|X'(t_0)|,\end{aligned} \tag{3.16}$$

and using the triangle inequality with respect to $\|\cdot\|_{\psi_q}$. □

The hypotheses of Theorem 3.1 are satisfied by Gaussian processes when $q=2$. In this case it contains ideas which originated in an important early paper by Garcia, Rodemich and Rumsey Jr., [3] and were developed further by Preston, [9, 10] and Fernique, [1]. The fact that it can be extended to processes in exponential Orlicz spaces for $1\le q\le\infty$ is, no doubt, understood by many researchers in the field of probability on Banach spaces. For lack of a suitable reference a proof was given in [8].

In this paper we need an extension to $\psi_q(x)$ for $0<q\le\infty$. Here too we're sure many researchers are aware that this can be done, but, once again, we have no reference. When $0<q<1$, $\exp(x^q)-1$ is not convex, so a bit more care is necessary. The key point is the following lemma:

Lemma 3.3. *For $0<q\le\infty$, let $X=\{X(t):t\in T\}$ be a measurable separable stochastic process on a precompact metric space (T,d) such that $\|X(t)\|_{\psi_q}\le 1$ for all $t\in T$. Then there exists a random variable Z with $\|Z\|_{\psi_q}\le C'_q$, such that for every probability measure m on T and function $h:T\mapsto\mathbf{R}_+$ with $\int_T h(v)\,m(dv)<\infty$*

$$\begin{aligned}&\int_T|X(t)|h(t)\,m(dt) \\ &\qquad\le Z\int_T h(t)\Phi_q\left(\left(\int_T h(v)\,m(dv)\right)^{-1}h(t)\right)m(dt)\end{aligned} \tag{3.17}$$

where

$$\Phi_q=\begin{cases}\log\log(e+x) & q=\infty\\ (\log(1+x))^{1/q} & 1\le q<\infty\\ (2\log(1+(x/G_q)))^{1/q} & 0\le q<1,\end{cases} \tag{3.18}$$

where $G_q>0$.

Proof. Define

$$\widetilde{Z}(\omega)=\inf\left\{\alpha>0:\int_T\psi_q(\alpha^{-1}|X(t)|)\,m(dt)\le 1\right\}. \tag{3.19}$$

We first show that

$$\|\widetilde{Z}\|_{\psi_q}\le C'_q<\infty \qquad 0<q\le\infty. \tag{3.20}$$

Let $0<q<\infty$; then for $u\ge 1$,

$$P\left(\widetilde{Z}>u\right)\le P\left(\int_T\psi_q(u^{-1}|X(t)|)\,m(dt)>1\right) \tag{3.21}$$

and by (3.4)

$$
\begin{aligned}
P&\left(\int_T \psi_q(u^{-1}|X(t)|)\, m(dt) > 1\right)\\
&\le P\left(C_q \int_T \exp(u^{-q}|X(t)|^q)\, m(dt) > 1 + C_q\right)\\
&= P\left(\left(C_q \int_T \exp(u^{-q}|X(t)|^q)\, m(dt)\right)^{u^q} > (1+C_q)^{u^q}\right)\\
&\le P\left(C_q^{u^q} \int_T \exp(|X(t)|^q)\, m(dt) > (1+C_q)^{u^q}\right)\\
&\le \left(\frac{1+C_q}{C_q}\right)^{-u^q} E\int_T \exp(|X(t)|^q)\, m(dt)\\
&\le \left(\frac{1+C_q}{C_q}\right)^{-u^q} C_q\left(E\int_T \psi_q(|X(t)|)\, m(dt) + 1\right)\\
&\le 2C_q\left(\frac{1+C_q}{C_q}\right)^{-u^q}.
\end{aligned}
$$

The fourth line follows from Jensen's inequality, the sixth from (3.5), and the last because $\|X(t)\|_{\psi_q} \le 1$. Thus we get (3.20) when $0 \le q < \infty$.

Now let $q = \infty$. Note that for each $u \ge 1$, the function $\phi_u(x) = \exp((\log x)^u)$ is convex for $x \ge e$. Using Jensen's inequality again we get that for $u \ge 1$

$$
\begin{aligned}
(3.22)\quad P\{\widetilde{Z} > u\} &\le P\left\{\int_T \exp(\exp(u^{-1}|X(t)|))\, m(dt) > e+1\right\}\\
&= P\left\{\phi_u\Big(\int_T \exp(\exp(u^{-1}|X(t)|))\, m(dt)\Big) > \phi_u(e+1)\right\}\\
&\le P\left\{\int_T \exp(\exp(|X(t)|))\, m(dt) > \exp(\exp(cu))\right\}\\
&\le \exp(-\exp(cu))\int_T E\exp(\exp(|X(t)|))\, m(dt)\\
&\le (1+e)\exp(-\exp(cu))
\end{aligned}
$$

where $c = \log(\log(e+1)) > 0$. Thus we get (3.20) when $q = \infty$.

We now prove (3.17). For $1 \le q \le \infty$ we have

$$
(3.23)\qquad xy \le \psi_q(x) + y\psi_q^{-1}(y) \quad x, y \ge 0.
$$

To obtain (3.23) we first note that $\psi_q'(x) \ge \psi_q(x)$. To see this set $h(x) := \psi_q'(x) - \psi_q(x)$. We get the desired inequality because $h(0)$=0 and $h'(x) > 0$ for all $x \in R_+$. To prove this last point it suffices to show that

$$
(3.24)\qquad g(x) := \frac{q-1}{x} + qx^{q-1} - 1 \ge 0.
$$

To verify (3.24) note that the minimum of $g(x)$ takes place at $x_1 = (1/q)^{1/q}$ and $g(x_1) > 0$.

The inequality in (3.23) follows from Young's inequality since $\psi_q(x)$ is convex and $\psi_q'(x) \ge \psi_q(x)$. (Recall that the final term in (3.23) can be taken to be

$\int_0^y (\psi_q')^{-1}(y)\,dy$. Since $\psi_q'(x) \geq \psi_q(x)$, $(\psi_q')^{-1}(y) \leq (\psi_q)^{-1}(y)$, and since $(\psi_q)^{-1}(y)$ is increasing we get (3.23).)

When $0 < q < 1$ it follows from Lemma 3.4, which is given at the end of this section, that

$$xy \leq \psi_q(x) + y\,(2\log(1 + y/G_q))^{1/q} \quad x, y \geq 0, \tag{3.25}$$

for some constant $G_q > 0$. Therefore it follows from (3.23), (3.25) and Lemma 3.2 that

$$xy \leq \psi_q(x) + y\Phi_q(y) \quad x, y \geq 0. \tag{3.26}$$

Let $h : T \mapsto R_+$ be as in the lemma. Putting $x = \widetilde{Z}^{-1}|X(t)|$ and $y = (\int_T h(v) \times m(dv))^{-1}h(t)$ in (3.26) we get

$$\begin{aligned} |X(t)|h(t) \leq \widetilde{Z} \int_T h(v)\, m(dv)\psi_q(\widetilde{Z}^{-1}|X(t)|) \\ + \widetilde{Z}h(t)\Phi_q\Big(\Big(\int_T h(v)\, m(dv)\Big)^{-1} h(t)\Big). \end{aligned} \tag{3.27}$$

Integration with respect to m, and using the definition (3.19), gives

$$\begin{aligned} &\int_T |X(t)|h(t)\, m(dt) \\ &\quad \leq \widetilde{Z} \int_T h(t)\, m(dt) + \widetilde{Z} \int_T h(t)\Phi_q\left(\Big(\int_T h(v)\, m(dv)\Big)^{-1} h(t)\right) m(dt). \end{aligned} \tag{3.28}$$

It is easy to check that $x\,\Phi_q(x/\beta)$, or equivalently, $x\,\Phi_q(x)$, is a convex function for all $0 \leq q < \infty$. Consequently, it follows from Jensen's inequality that

$$\begin{aligned} \int_T h(t)\Phi_q\left(\Big(\int_T h(v)\, m(dv)\Big)^{-1} h(t)\right) m(dt) \\ \geq \int_T h(t)\, m(dt)\Phi_q(1). \end{aligned} \tag{3.29}$$

Using this in (3.28) yields the inequality

$$\begin{aligned} &\int_T |X(t)|h(t)\, m(dt) \\ &\quad \leq D_q\widetilde{Z} \int_T h(t)\Phi_q\left(\Big(\int_T h(v)\, m(dv)\Big)^{-1} h(t)\right) m(dt). \end{aligned}$$

where $D_q = 1 + (1/\Phi_q(1))$. Changing $D_q\widetilde{Z}$ to Z gives (3.17). $\square$

Proof of Theorem 3.1. Using Lemma 3.3 it is easy to complete the proof of Theorem 3.1 by following the proof of [2, Theorem 5.2.6] or [6, Theorem 6.3.3]. We make some comments regarding the proof in [6, Theorem 6.3.3]. In place of (6.73) we have that for some $\alpha < \infty$

$$\begin{aligned} E|X(t) - M_k(t)| &\leq \frac{1}{\mu_k(t)} \int_{B(t,D2^{-k})} E|X(t) - X(u)|\,\mu(du) \\ &\leq \frac{1}{\mu_k(t)} \int_{B(t,D2^{-k})} \alpha\|X(t) - X(u)\|_{\psi_q}\,\mu(du) \\ &\leq \frac{1}{\mu_k(t)} \int_{B(t,D2^{-k})} \alpha d(u,t)\,\mu(du) \leq \alpha D2^{-k}, \end{aligned} \tag{3.30}$$

which is all we need to proceed with the proof. This follows because by Jensen's Inequality, for any convex function Ψ,

$$E\Psi\left(\frac{\alpha|X(t)-X(u)|}{E|X(t)-X(u)|}\right)\geq\Psi\left(\alpha\right). \tag{3.31}$$

Let $\Psi(x)=\psi_q(x)$. Therefore, when $\psi_q(\alpha)\geq 1$

$$E|X(t)-X(u)|\leq\alpha\|X(t)-X(u)\|_{\psi_q}. \tag{3.32}$$

It is easy to see that we can take $\alpha=1$ when $1\leq q\leq\infty$. When $q<1$ the reader can check that it suffices to take $\alpha=x_0$.

When $1\leq q\leq\infty$ the rest of the adaptation of the proof of Theorem 6.3.3 in [6] is completely apparent. When $0<q<1$ one gets as far as the expression on the bottom of page 261 but with the measures multiplied by G_q, (and a different constant following Z). We need only be concerned if $G_q<1$, In this case we proceed as in [6, (6.85)] and note that

$$\log\left(1+\frac{x}{G_q}\right)\leq\frac{\log\left(1+(2/G_q)\right)}{\log 2}\log x \qquad x\geq 2. \tag{3.33}$$

Using this the proof can be completed. $\square$

Lemma 3.4. *For $0<q<1$, there exists a constant $G_q>0$ such that*

$$xy\leq\psi_q(x)+y\left(2\log(1+y/G_q)\right)^{1/q} \quad x,y\geq 0. \tag{3.34}$$

Proof. It is easy to see that for all $p>0$ there exists a constant $D_p>0$ for which

$$\frac{e^s}{s^p}\geq D_p\left(e^{s/2}-1\right) \qquad \forall s\in R_+. \tag{3.35}$$

Taking $s=x^q$ this shows that there exists a constant $G_q>0$ such that

$$\frac{\exp(x^q)}{x^{1-q}}\geq\frac{G_q}{q}\left(\exp(x^q/2)-1\right) \qquad \forall x\in R_+. \tag{3.36}$$

By (3.2)

$$\psi_q'(x)=\frac{q\exp(x^q)}{x^{1-q}} \qquad x>x_0. \tag{3.37}$$

Consequently

$$\psi_q'(x)\geq G_q\left(\exp(x^q/2)-1\right) \qquad x>x_0. \tag{3.38}$$

Let $\Lambda_q(y)$ be the right continuous inverse of $\psi_q'(x)$. By (3.2) we have $\Lambda_q(y)=0$ for $y<K_q$ and $\Lambda_q(y)=x_0$ for $K_q\leq y\leq D_+\psi_q(x_0)=q\exp(x_0^q)/x_0^{1-q}$, the right hand derivative of $\psi_q(x)$ at x_0. In addition, by (3.38) we see that

$$\Lambda_q(y)\leq\left(2\log(1+y/G_q)\right)^{1/q} \qquad y>\frac{q\exp(x_0^q)}{x_0^{1-q}}. \tag{3.39}$$

Therefore, decreasing G_q if necessary, we have that

$$\Lambda_q(y)\leq\left(2\log(1+y/G_q)\right)^{1/q} \qquad \forall\, y\in R_+, \tag{3.40}$$

from which we get (3.34) by Young's Inequality and the obvious fact that $\int_0^y\Lambda_q(s)\,ds\leq y\Lambda_q(y)$ since $\Lambda_q(s)$ is non-decreasing. $\square$

4. Proof of Theorem 1.1

Consider the Gaussian chaos $\mathcal{X} = \{: X_h^k : (g), h \in (0,1]\}$ defined in (1.22). It is clear that this process is continuous on $(0,1]$. Therefore, to show that it is continuous on $[0,1]$ it suffices to show that it is continuous on $[0,h_0]$ for some $0 < h_0 << 1$. For $h,h' \in [0,h_0]$ set

$$d(h,h') := \| : X_h^k : (g) - : X_{h'}^k : (g)\|_2. \tag{4.1}$$

It follows from (1.18) that

$$\lim_{h,h' \to 0} d(h,h') = 0. \tag{4.2}$$

Therefore, by [4, Theorem 3.2.10]

$$\lim_{h,h' \to 0} \| : X_h^k : (g) - : X_{h'}^k : (g)\|_{\psi_{2/k}} = 0. \tag{4.3}$$

Furthermore, the same theorem states that the L^2 and $L_{\psi_{2/k}}$ are equivalent. Consequently

$$\lim_{h \to 0} : X_h^k : (g) =: X_0^k : (g) \qquad \text{in } L_{\psi_{2/k}} \tag{4.4}$$

and

$$\| : X_h^k : (g) - : X_{h'}^k : (g)\|_{\psi_{2/k}} \le C d(h,h') \tag{4.5}$$

for all h, $h' \in [0,1]$.

We use Theorem 3.1 to show that $\mathcal{X}$ is continuous on $([0,h_0],d)$. To do this we need estimates for d. We get one estimate from Lemma 2.1. Th next lemma gives another estimate for d.

Lemma 4.1. *Under the hypotheses of Theorem 1.1, for any $h,h' > 0$*

$$d(h,h') = \| : X_h^k : (g) - : X_{h'}^k : (g)\|_2 \le C \left(\frac{|h-h'|}{hh'} \right)^{1/2}. \tag{4.6}$$

Proof. Note that by (1.12)

$$X_h(g) \stackrel{def}{=} : X_h^1(g) : = \int \left(\frac{G(x+h) - G(x)}{h} \right) g(x)\, dx. \tag{4.7}$$

In addition it is not hard to see that it follows from the definition of ρ in (1.6), that for $x' \le x$, and $y' \le y$

$$E\left(G(x) - G(x')\right)\left(G(y) - G(y')\right) = \int_{x'}^{x} \int_{y'}^{y} \rho(t-s)\, ds\, dt. \tag{4.8}$$

(Details are given in [7, Lemma 2.2].) Therefore

$$\begin{aligned} (4.9)\quad & E\left(X_h(g) X_{h'}(\widetilde{g})\right) \\ &= \frac{1}{h}\frac{1}{h'} \int\int E\left(\left(G(x+h) - G(x)\right)\left(G(y+h') - G(y)\right)\right) g(x)\, dx\, \widetilde{g}(y)\, dy \\ &= \int\int \left\{ \frac{1}{h} \int_x^{x+h} \frac{1}{h'} \int_y^{y+h'} \rho(t-s)\, ds\, dt \right\} g(x)\, \widetilde{g}(y)\, dx\, dy. \end{aligned}$$

Let (X,Y) be a two dimensional Gaussian random variable. By [5, Theorem 3.9]

$$E(:X^k::Y^j:) = k!(E(XY))^k \delta_{k,j}. \tag{4.10}$$

Using this and (4.9) we see that

$$\begin{aligned} &E(:X_h^k:(g):X_{h'}^k:(\widetilde{g})) \\ &\quad = k! \int\int \left(\frac{1}{h}\int_x^{x+h} \frac{1}{h'}\int_y^{y+h'} \rho(s-t)\,dt\,ds \right)^k g(x)\widetilde{g}(y)\,dx\,dy. \end{aligned} \tag{4.11}$$

Set

$$\begin{aligned} B_z(h,h') &= \frac{1}{h}\int_0^h \frac{1}{h'}\int_0^{h'} \rho(z+s-t)\,dt\,ds \\ &= \frac{\sigma^2(z+h)+\sigma^2(z-h')-\sigma^2(z+h-h')-\sigma^2(z)}{2hh'}. \end{aligned} \tag{4.12}$$

By (4.11) and a change of variables we have

$$\begin{aligned} &\| :X_h^k:(g) - :X_{h'}^k:(g)\|_2^2 \\ &\quad = \int\int \left\{ (B_z(h,h))^k - (B_z(h,h'))^k - (B_z(h',h))^k + (B_z(h',h'))^k \right\} \\ &\qquad\qquad g(x)g(y)\,dx\,dy. \end{aligned} \tag{4.13}$$

In addition

$$\begin{aligned} &B_z(h,h) - B_z(h,h') \\ &\quad = \left(\frac{1}{h^2} - \frac{1}{hh'}\right) h^2 B_z(h,h) + \frac{1}{hh'}\left(h^2 B_z(h,h) - hh' B_z(h,h')\right) \\ &\quad = \frac{1}{2hh'}\left(\sigma^2(z+h-h') + \sigma^2(z-h) - \sigma^2(z-h') - \sigma^2(z)\right) \\ &\qquad\quad + \left(\frac{1}{2h^2} - \frac{1}{2hh'}\right)\left(\sigma^2(z+h) + \sigma^2(z-h) - 2\sigma^2(z)\right). \end{aligned} \tag{4.14}$$

We write this as

$$\begin{aligned} &B_z(h,h) - B_z(h,h') \\ &\quad = \frac{1}{2hh'}\left(\sigma^2(z+h-h') - \sigma^2(z) + \sigma^2(z-h) - \sigma^2(z-h')\right) \\ &\qquad\quad + \frac{(h-h')}{2h'h^2}\left(2\sigma^2(z) - \sigma^2(z+h) - \sigma^2(z-h)\right). \end{aligned} \tag{4.15}$$

Since σ^2 and $(\sigma^2)'$ are bounded we need only use the mean value theorem, on four differences, to see that for $h,h'>0$

$$|B_z(h,h) - B_z(h,h')| \le C\frac{|h-h'|}{h'h}. \tag{4.16}$$

Note that

$$B_z^k(h,h) - B_z^k(h,h') = \sum_{j=0}^{k-1} B_z^j(h,h)(B_z(h,h) - B_z(h,h'))B_z^{k-j-1}(h,h'). \tag{4.17}$$

Therefore

$$|B_z^k(h,h) - B_z^k(h,h')| \le C\frac{|h-h'|}{h'h}\sum_{j=0}^{k-1} B_z^j(h,h)B_z^{k-j-1}(h,h'). \tag{4.18}$$

Let $f_h(x) = \frac{1}{h}1_{[0,h]}(x)$ so that the first line in the definition (4.12) of $B_z(h,h')$ can be written as

$$B_z(h,h') = \int\int \rho(z+s-t)f_h(s)f_{h'}(t)\,ds\,dt. \tag{4.19}$$

Using Fubini's Theorem we see that

$$\begin{aligned}
&\int\int B_z^j(h,h)B_z^{k-j-1}(h,h')\,g(x)g(y)\,dx\,dy \qquad (4.20)\\
&= \iint \Big(\int \cdots \int\Big) \prod_{i=1}^{k-1} \rho(x+v_i-y-w_i) \prod_{i=1}^{j} f_h(v_i)f_h(w_i)\,dv_i\,dw_i \\
&\qquad \prod_{i=j+1}^{k-1} f_h(v_i)f_{h'}(w_i)\,dv_i\,dw_i\; g(x)g(y)\,dx\,dy \\
&= \int \cdots \int \Big(\iint \prod_{i=1}^{k-1} \rho(x-y+v_i-w_i)g(x)g(y)\,dx\,dy \Big) \\
&\qquad \prod_{i=1}^{j} f_h(v_i)f_h(w_i)\,dv_i\,dw_i \prod_{i=j+1}^{k-1} f_h(v_i)f_{h'}(w_i)\,dv_i\,dw_i \\
&\le C
\end{aligned}$$

where C is a finite constant that is independent of h and h'. In the last step we use the generalized Holder's inequality and the fact that $\rho \in L^k_{loc}$ and $g \in \mathcal{B}_0(R_+)$, to get

$$\iint \prod_{i=1}^{k-1} \rho(x-y+v_i-w_i)g(x)g(y)\,dx\,dy \le C. \tag{4.21}$$

Using (4.20) together with (4.18) we obtain

$$\int\int \Big| (B_z(h,h))^k - (B_z(h,h'))^k \Big| g(x)g(y)\,dx\,dy \le C'\frac{|h-h'|}{h'h}. \tag{4.22}$$

Clearly the integral of the other two terms in (4.13) has the same bound. Thus we get (4.6). □

Proof of Theorem 1.1. It follows from (2.4) that for any $h > 0$

$$d(h,0) = \| :X_h^k:(g) - :X_0^k:(g)\|_2 \le Ch^{\delta/2} \tag{4.23}$$

(The constant C actually depends on k, but we take k fixed.) We use this bound as well as the one in (4.6).

By Theorem 3.1 to prove that $\mathcal{X}$ is continuous it suffices to show that

$$\sup_{h\in[0,h_0]} \int_0^{Kh_0^{\delta/2}} \left(\log\frac{1}{\lambda(B_d(h,u))}\right)^{k/2} du < \infty, \tag{4.24}$$

and

$$\lim_{\epsilon\to 0}\sup_{h\in[0,h_0]}\int_0^{\epsilon}\left(\log\frac{1}{\lambda(B_d(h,u))}\right)^{k/2}du=0, \tag{4.25}$$

where λ is Lebesgue measure. (Theorem 3.1 requires a probability measure. Rather than bothering to renormalize we need only observe that its conclusions also hold for positive measures with mass less than one.)

We now verify (4.24) and (4.25). We pick h_0 so that $Kh_0^{\delta/2}$ is very small. Let $h\in(0,h_0]$. Note that by (4.6) we have $h'\in B_d(h,u)$ when $|(h-h')/(hh')|^{1/2}\le u/C$, or, equivalently, when $|h-h'|\le hh'(u/C)^2$. (We take $C\ge 1$.) Since $h'\le 1$ and $u\le Kh_0^{\delta/2}$, we see that on $\{h':|h-h'|\le hh'u^2\}$ we have $h'>h/2$. Therefore

$$B_d(h,u)\supseteq\{h':|h-h'|\le h^2u^2/(2C^2)\}, \tag{4.26}$$

so that the Lebesgue measure of $B_d(h,u)$ is at least $h^2u^2/(2C^2)$. Consequently for any $h\in(0,h_0]$ and $u\le Kh_0^{\delta/2}$

$$\log\frac{1}{\lambda(B_d(h,u))}\le 2\left(\log\frac{1}{h}+\log\frac{1}{u}+\log C\right). \tag{4.27}$$

Therefore for any $h\in(0,h_0]$ and $w\le Kh_0^{\delta/2}$

$$\begin{aligned}\int_0^{w}&\left(\log\frac{1}{\lambda(B_d(h,u))}\right)^{k/2}du\\ &\le C'w\left(\log\frac{1}{h}+\log\frac{1}{w}+\log C\right)^{k/2}\le C''w\left(\log\frac{1}{h}+\log\frac{1}{w}\right)^{k/2}.\end{aligned} \tag{4.28}$$

Let $h\in(0,h_0]$ and $v\le Kh_0^{\delta/2}$ and suppose that $h^{\delta/4}\ge v$. Then by (4.28) and the monotonicity of $\log 1/h$

$$\int_0^{v}\left(\log\frac{1}{\lambda(B_d(h,u))}\right)^{k/2}du\le Cv\left(\log\frac{1}{v}\right)^{k/2}. \tag{4.29}$$

(The constants are not necessarily the same at each stage.) Now suppose that $h^{\delta/4}<v$. In this case using (4.28) with $w=h^{\delta/4}$ we have

$$\begin{aligned}\int_0^{h^{\delta/4}}&\left(\log\frac{1}{\lambda(B_d(h,u))}\right)^{k/2}du\\ &\le C_\delta h^{\delta/4}\left(\log 1/h\right)^{k/2}\\ &\le C'_\delta h^{\delta/4}\left(\log 1/h^{\delta/4}\right)^{k/2}\le K_\delta v\left(\log\frac{1}{v}\right)^{k/2}.\end{aligned} \tag{4.30}$$

(Here we use the monotonicity of $x(\log 1/x)^{k/2}$.)

Now consider

$$\int_{h^{\delta/4}}^{v}\left(\log\frac{1}{\lambda(B_d(h,u))}\right)^{k/2}du. \tag{4.31}$$

Since $d(h,x)<d(h,0)+d(x,0)$, we see by (4.23) that $\{x\in B_d(h,u)\}$ when $Ch^{\delta/2}+Cx^{\delta/2}\le u$, or, equivalently, when $x\le C'(u-Ch^{\delta/2})^{2/\delta}$. Since $u\ge h^{\delta/4}$, we see that

for small h, (which we can always achieve by taking $Kh_0^{\delta/2}$ sufficiently small) we have $x \leq C'(u - Ch^{\delta/2})^{2/\delta}$ whenever $x \leq C''u^{2/\delta}$, for some $C'' > 0$. Consequently $\lambda(B_d(h,u)) \geq Ku^{2/\delta}$ and

$$\int_{h^{\delta/4}}^{v} \left(\log \frac{1}{\lambda(B_d(h,u))}\right)^{k/2} du \leq Cv\left(\log 1/v\right)^{k/2}. \tag{4.32}$$

Using (4.23) it is elementary to see that

$$\int_{0}^{v} \left(\log \frac{1}{\lambda(B_d(0,u))}\right)^{k/2} du \leq Cv\left(\log 1/v\right)^{k/2}. \tag{4.33}$$

Combining (4.29), (4.30), (4.32) and (4.33) we get that for any $v \leq Kh_0^{\delta/2}$

$$\sup_{h\in[0,h_0]} \int_{0}^{v} \left(\log \frac{1}{\lambda(B_d(h,u))}\right)^{k/2} du \leq Cv\left(\log 1/v\right)^{k/2}. \tag{4.34}$$

The statements in (4.24) and (4.25) follow immediately. □

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