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A note on positive definite norm dependent functions

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Abstract: Let K be an origin symmetric star body in $\mathbb{R}^n$. We prove, under very mild conditions on the function $f : [0, \infty) \to \mathbb{R}$, that if the function $f(\|x\|_K)$ is positive definite on $\mathbb{R}^n$, then the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds isometrically in L_0. This generalizes the solution to Schoenberg's problem and leads to progress in characterization of n-dimensional versions, i.e. random vectors $X = (X_1, \dots, X_n)$ in $\mathbb{R}^n$ such that the random variables $\sum a_i X_i$ are identically distributed for all $a \in \mathbb{R}^n$, up to a constant depending on $\|a\|_K$ only.

1. Introduction

In 1938, Schoenberg [26] posed the problem of finding the exponents $0 < p < 2$ for which the function $\exp(-\|x\|_q^p)$ is positive definite on $\mathbb{R}^n$, where

$$\|x\|_q = (|x_1|^q + \cdots + |x_n|^q)^{1/q}$$

is the norm the space ℓ_q^n with $2 < q \le \infty$. Recall that a complex valued function f defined on $\mathbb{R}^n$ is called *positive definite* on $\mathbb{R}^n$ if, for every finite sequence $\{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ and every choice of complex numbers $\{c_i\}_{i=1}^m$, we have

$$\sum_{i=1}^m \sum_{j=1}^m c_i \bar{c}_j f(x_i - x_j) \ge 0.$$

For $q = \infty$, the problem was solved in 1989 by Misiewicz [21], and for $2 < q < \infty$, the answer was given in [11] in 1991 (note that, for $1 \le p \le 2$, Schoenberg's question was answered earlier by Dor [5], and the case $n = 2,\ 0 < p \le 1$ was established in [7, 9, 16]). The answers turned out to be the same in both cases: the function $\exp(-\|x\|_q^p)$ is not positive definite if the dimension of the space is greater than 2, and for $n = 2$ the function is positive definite if and only if $0 < p \le 1$. Different and independent proofs of Schoenberg's problems were given by Lisitsky [17] and Zastavnyi [28, 29] shortly after the paper [11] appeared.

For an origin symmetric star body K in $\mathbb{R}^n$, let $E_K = (\mathbb{R}^n, \|\cdot\|_K)$ be the space whose unit ball is K, where $\|x\|_K = \min\{a \ge 0 :\ x \in aK\}$ is the Minkowski functional of K. Note that the class of star bodies includes convex bodies, and E_K is a normed space if and only if K is convex (see [12], p. 13). Denote by $\Phi(K) = \Phi(E_K)$ the class of continuous functions $f : [0, \infty) \to \mathbb{R}$ for which $f(\|\cdot\|_K)$ is a positive definite function on $\mathbb{R}^n$.

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The classes $\Phi(K)$ admit an interesting probabilistic interpretation. Following Eaton [6], we say that a random vector X in $\mathbb{R}^n$ is an n-dimensional version if all linear combinations of its coordinates have the same distribution, up to a constant, namely for any vector $a \in \mathbb{R}^n$ the random variables

$$\sum_{i=1}^{n} a_i X_i \quad \text{and} \quad \|a\|_K X_1$$

are identically distributed. The result of Eaton is that a random vector is an n-dimensional version if and only if its characteristic functional has the form $f(\|x\|_K)$. Hence, by Bochner's theorem, the problem of finding all n-dimensional versions is equivalent to characterizing the classes $\Phi(K)$. Note that, by the classical result of P. Lévy [15], if K is the unit ball of a finite dimensional subspace of L_q, $0 < q \le 2$, then the function $\exp(-|t|^q) \in \Phi(K)$, and the corresponding n-dimensional versions are the classical q-stable vectors.

The classes $\Phi(K)$ have been studied by a number of authors. Schoenberg [27] proved that $f \in \Phi(B_2^n)$ if and only if

$$f(t) = \int_0^\infty \Omega_n(tr)\, d\lambda(r)$$

where B_2^n is the unit Euclidean ball in $\mathbb{R}^n$, $\Omega_n(|\cdot|_2)$ is the Fourier transform of the uniform probability measure on the sphere S^{n-1}, and λ is a finite measure on $[0,\infty)$. In the same paper, Schoenberg proved an infinite dimensional version of this result: $f \in \Phi(\ell_2)$ if and only if

$$f(t) = \int_0^\infty \exp(-t^2 r^2)\, d\lambda(r).$$

Bretagnolle, Dacunha-Castelle and Krivine [2] proved a similar result for the classes $\Phi(\ell_q)$ for all $q \in (0,2)$ (one just has to replace 2 by q in the formula), and showed that for $q > 2$ the classes $\Phi(\ell_q)$ (corresponding to infinite dimensional ℓ_q-spaces) are trivial, i.e. contain constant functions only. Cambanis, Keener and Simons [3] obtained a similar representation for the classes $\Phi(B_1^n)$. Richards [24] partially characterized the classes $\Phi(B_q^n)$ for $0 < q < 2$. Aharoni, Maurey and Mityagin [1] proved that if E is an infinite dimensional Banach space with a symmetric basis $\{e_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \frac{\|e_1 + \cdots + e_n\|}{n^{1/2}} = 0,$$

then the class $\Phi(E)$ is trivial. Misiewicz [21] proved that for $n \ge 3$ the classes $\Phi(\ell_\infty^n)$ are trivial, and Lisitsky [17] and Zastavnyi [28, 29] proved the same the classes $\Phi(\ell_q^n)$, $q > 2$, $n \ge 3$. One can find more detailed information and references in [23, 22, 4, 8, 12].

In all the results mentioned above the classes $\Phi(K)$ appear to be non-trivial only if K is the unit ball of a subspace of L_q with $0 < q \le 2$. It was conjectured by Misiewicz [20] that the latter condition on K is necessary for $\Phi(K)$ to be non-trivial. In support of this conjecture, Misiewicz [20] and Kuritsyn [14] proved that if $f \in \Phi(K)$ is a non-constant function and its inverse Fourier transform ν (which is a finite measure on $\mathbb{R}$, by Bochner's theorem) has a finite moment or the order $q \in (0,2]$, then K is the unit ball of a subspace of L_q. Lisitsky [18] showed that if $f \in \Phi(K)$ is a non-constant function and $\int_{\mathbb{R}} |\log|t||\, d\nu(t) < \infty$ then, $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 (the definition of embedding in L_0 was given later in [10]; see below), and formulated a weaker conjecture that if $\Phi(K)$ is non-trivial then K is the unit ball of a subspace of L_0.

The purpose of this note is to provide simple conditions on the function f itself (rather than on its inverse Fourier transform) under which $f(\|\cdot\|_K)$ can be positive definite only if K is the unit ball of a subspace of L_q, $0 \le q \le 2$. We prove that if f is a continuous non-constant function satisfying $|f(0)-f(t)| \le C|t|^q$ in a neighborhood of the origin, where $C>0$, $q \in (0,2)$, and $f(\|\cdot\|_K)$ is positive definite, then K is the unit ball of a subspace of L_q. We also prove that if $\lim_{t\to\infty} t^\epsilon |f(t)| < \infty$ for some $\epsilon \in (0,1)$, and $f(\|\cdot\|_K)$ is positive definite, then K is the unit ball of a subspace of L_0. This shows that, in order to defy the conjectures of Misiewicz and Lisitsky, the function f must exhibit rather odd behaviour at both the origin and infinity. Finally, we combine these facts with known results about embedding in L_q to further generalize the solution of Schoenberg's problem.

2. Proofs and examples

As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$ (Schwartz test functions), and by $\mathcal{S}'(\mathbb{R}^n)$ the space of distributions over $\mathcal{S}(\mathbb{R}^n)$. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ is a locally integrable function with power growth at infinity, then

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)\, dx.$$

We say that a distribution is positive (negative) outside of the origin in $\mathbb{R}^n$ if it assumes non-negative (non-positive) values on non-negative Schwartz's test functions with compact support outside of the origin.

The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function ϕ.

We need a Fourier analytic criterion of embedability in L_q that applies to every $q>0$ which is not an even integer; see [K], Th. 6.10.

Proposition 1. *Let K be an origin-symmetric star body in $\mathbb{R}^n$, and $q>0$ is not an even integer. Then the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds isometrically in L_q if and only if $\Gamma(-q/2)(\|\cdot\|_K^q)^\wedge$ is a positive distribution on $\mathbb{R}^n \setminus \{0\}$.*

We now prove our first result.

Theorem 1. *Let K be an origin symmetric star body in $\mathbb{R}^n$, and f a non-constant continuous function on $[0,\infty)$. Suppose that there exist $C>0$, $0<q<2$, $u>0$ such that*

$$|f(0)-f(t)| \le Ct^q \tag{1}$$

for every $t \in (0,u)$. If $f(\|\cdot\|_K)$ is a positive definite function, then the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds isometrically in L_q.

Proof. A positive definite function $f(\|x\|_K)$ has absolute maximum at zero (see [19] or [25], p. 21) and is bounded on $\mathbb{R}^n$, hence $f(0) \ge f(t)$ for every $t>0$ and f is bounded on $[0,\infty)$.

Let $0<\alpha<q$. The condition (1) and the remark above imply that the integral

$$c = \int_0^\infty t^{-1-\alpha}(f(0)-f(t))\, dt$$

converges and is positive (f is not a constant).

Making a change of variables $u = t\|x\|_K$, we see that for every $x \in \mathbb{R}^n \setminus \{0\}$

$$c\|x\|_K^\alpha = \int_0^\infty t^{-1-\alpha} \left(f(0) - f(t\|x\|_K)\right)\, dt. \tag{2}$$

Let ϕ be an even non-negative test function supported outside of the origin. Then

$$\int_{\mathbb{R}^n} \hat{\phi}(x)\, dx = (2\pi)^n \phi(0) = 0. \tag{3}$$

Applying the definition of the Fourier transform of a distribution and equalities (2) and (3), we get

$$\begin{aligned}\langle (\|\cdot\|_K^\alpha)^\wedge, \phi\rangle = \langle \|x\|_K^\alpha, \hat{\phi}\rangle &= \int_{\mathbb{R}^n} \|x\|_K^\alpha \hat{\phi}(x)\, dx \\ &= -\frac{1}{c}\int_0^\infty t^{-1-\alpha}\left(\int_{\mathbb{R}^n} f(t\|x\|_K)\hat{\phi}(x)\, dx\right) dt \\ &= -\frac{1}{c}\int_0^\infty t^{-1-\alpha}\langle (f(t\|\cdot\|_K))^\wedge, \phi\rangle dt \le 0,\end{aligned}$$

because $f(t\|\cdot\|_K)$ is a positive definite function on $\mathbb{R}^n$ for every fixed $t \in \mathbb{R}$, and, by Bochner's theorem, $(f(t\|\cdot\|_K))^\wedge$ is a finite measure on $\mathbb{R}^n$.

For every $0 < \alpha < q$ and $x \in \mathbb{R}^n$, we have

$$\|x\|_K^\alpha\ |\hat{\phi}(x)| \le \max(1, \|x\|_K^q)\ |\hat{\phi}(x)|,$$

where the function of $x \in \mathbb{R}^n$ in the right-hand side is integrable, so by the dominated convergence theorem,

$$\begin{aligned}\langle (\|\cdot\|_K^q)^\wedge, \phi\rangle &= \int_{\mathbb{R}^n} \|x\|_K^q \hat{\phi}(x)\, dx \\ &= \lim_{\alpha\to q}\int_{\mathbb{R}^n} \|x\|_K^\alpha \hat{\phi}(x)\, dx = \lim_{\alpha\to q}\langle (\|\cdot\|_K^\alpha)^\wedge, \phi\rangle \le 0.\end{aligned}$$

Now the result follows from Proposition 1 with $0 < q < 2$. $\square$

The concept of embedding in L_0 was introduced in [10].

Definition 1. *We say that a space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if there exist a finite Borel measure μ on the sphere S^{n-1} and a constant $C \in \mathbb{R}$ so that, for every $x \in \mathbb{R}^n$,*

$$\ln \|x\|_K = \int_{S^{n-1}} \ln |(x,\xi)| d\mu(\xi) + C. \tag{4}$$

Embedding in L_0 also admits a Fourier analytic characterization, as established in [10], Th. 3.1.

Proposition 2. *Let K be an origin symmetric star body in $\mathbb{R}^n$. The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if the Fourier transform of $\ln\|x\|_K$ is a negative distribution outside of the origin in $\mathbb{R}^n$.*

We use the latter statement to prove our next result.

Theorem 2. *Let K be an origin symmetric star body in $\mathbb{R}^n$, and f a continuous function on $[0,\infty)$ such that*

$$\lim_{t\to\infty} t^\epsilon |f(t)| = 0 \tag{5}$$

for some $\epsilon \in (0,1)$. If $f(\|\cdot\|_K)$ is a positive definite function, then the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0.

Proof. By the condition (5) and since f is a bounded function, for every $0<\delta<\epsilon$, the integral

$$c = \int_0^\infty t^{-1+\delta} f(t)\ dt$$

converges absolutely. We need to show that $c>0$. In fact, making a change of variables $z = tu$ and expressing the resulting integral in terms of the Γ-function, we get

$$\int_0^\infty u^{-\delta} \exp(-t^2u^2/2)\ du = t^{-1+\delta}\Gamma((1-\delta)/2).$$

The function $f(|\cdot|)$ is positive definite on $\mathbb{R}$ as the restriction to $\mathbb{R}$ of a positive definite function. By Bochner's theorem, $f(|\cdot|) = \hat{\nu}$ for some finite measure ν on $\mathbb{R}$. We have

$$\begin{aligned} c &= \frac{1}{\Gamma((1-\delta)/2)} \int_0^\infty u^{-\delta} \langle f(|t|), \exp(-t^2u^2/2)\rangle du \\ &= \frac{1}{\Gamma((1-\delta)/2)} \int_0^\infty u^{-\delta} \langle \nu, (\exp(-t^2u^2/2))^\wedge\rangle du > 0, \end{aligned}$$

since ν is a non-negative measure and the Fourier transform of a Gaussian density is also a Gaussian density, up to a positive constant.

Now for any $x \in R^n\setminus\{0\}$, we have

$$c\|x\|_K^{-\delta} = \int_0^\infty t^{-1+\delta} f(t\|x\|_K)\ dt.$$

For every even non-negative test function ϕ,

$$\begin{aligned} \langle \|x\|_K^{-\delta}, \hat{\phi}\rangle &= \frac{1}{c}\int_0^\infty t^{-1+\delta}\langle f(t\|\cdot\|_K), \hat{\phi}\rangle dt \\ &= \frac{1}{c}\int_0^\infty t^{-1+\delta}\langle (f(t\|\cdot\|_K))^\wedge, \phi\rangle dt \geq 0, \end{aligned}$$

since the function $f(t\|\cdot\|_K)$ is positive definite for any fixed $t\in\mathbb{R}$.

Suppose that, in addition, ϕ is supported outside of the origin, then by (3)

$$\left\langle \frac{\|x\|_K^{-\delta} - 1}{\delta}, \hat{\phi}\right\rangle = \frac{1}{\delta}\langle \|x\|_K^{-\delta}, \hat{\phi}\rangle \geq 0.$$

Sending δ to zero, we get that

$$-\langle \log\|x\|_K, \hat{\phi}\rangle = -\langle (\log\|\cdot\|_K)^\wedge, \phi\rangle \geq 0,$$

and by Proposition 2, the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. □

Let us show several applications. For normed spaces X and Y and $q \in \mathbb{R},\ q \geq 1$, the q-sum $(X \oplus Y)_q$ of X and Y is defined as the space of pairs $\{(x, y) :\ x \in X, y \in Y\}$ with the norm

$$\|(x, y)\| = (\|x\|_X^q + \|y\|_Y^q)^{1/q}.$$

It was proved in [13] (see also [12], Th. 6.11, Th. 4.21) that if $q > 2$ and X is any two-dimensional normed space, then the three dimensional space $(X \oplus \mathbb{R})_q$ does not embed in $L_p,\ 0 < p \leq 2$. Combining this fact with Theorem 1, we get

Corollary 1. *If a function f satisfies the conditions of Theorem 1 and $(\mathbb{R}^n, \|\cdot\|)$ is a space containing a three-dimensional subspace $(X \oplus \mathbb{R})_q$, where $q > 2$ and X is any two-dimensional normed space, then the function $f(\|\cdot\|)$ is not positive definite.*

Recall that an *Orlicz function M* is a non-decreasing convex function on $[0, \infty)$ such that $M(0) = 0$ and $M(t) > 0$ for every $t > 0$. The norm $\|\cdot\|_M$ of the n-dimensional Orlicz space ℓ_M^n is defined implicitly by the equality $\sum_{k=1}^n M(|x_k|/\|x\|_M) = 1,\ x \in \mathbb{R}^n \setminus \{0\}$. It was proved in [13] that the spaces $\ell_M^n,\ n \geq 3$ do not embed in $L_p,\ 0 < p \leq 2$ if the Orlicz function $M \in C^2([0, \infty))$ satisfies the condition $M'(0) = M''(0) = 0$.

Corollary 2. *If a function f satisfies the conditions of Theorem 1 and $(\mathbb{R}^n, \|\cdot\|)$ is a space containing ℓ_M^3, where M is an Orlicz function such that $M \in C^2([0, \infty))$ and $M'(0) = M''(0) = 0$, then the function $f(\|\cdot\|)$ is not positive definite.*

The concept of embedding of a normed space in L_0 was studied in [10]. In particular, every finite dimensional subspace of $L_p,\ 0 < p \leq 2$ embeds in L_0. Every three-dimensional normed space embeds in L_0. On the other hand, every space that embeds in L_0 also embeds in every $L_p,\ p < 0$.

It follows from the latter fact, combined with Theorems 4.21 and 4.22 from [12], that

Corollary 3. *If a function f satisfies the conditions of Theorem 2 and $(\mathbb{R}^n, \|\cdot\|)$ is a space containing a four-dimensional space $(X \oplus \mathbb{R})_q$, where $q > 2$ and X is any three-dimensional normed space, or $(\mathbb{R}^n, \|\cdot\|)$ contains a space ℓ_M^4, where M is an Orlicz function such that $M \in C^2([0, \infty))$ and $M'(0) = M''(0) = 0$, then the function $f(\|\cdot\|)$ is not positive definite.*

Corollaries 1-3 generalize the solution of Schoenberg's problem.

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