SOME REMARKS ON ALGEBRAIC INVERSE PROBLEMS

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During my five-week visit to CMA I primarily worked in two problem areas:

a) algorithms for the computation of functionals defined on the solution of a discrete ill-posed problem

b) hyperbolic approximations for a Cauchy problem for the heat equation

Below I give a brief account of that work.

Functionals. Consider the linear system of equations

$$Ax \simeq b, \tag{1}$$

where A is an $m \times n$ matrix, $m \geq n$, and let the *singular value decomposition (SVD)* of A be

$$\begin{aligned} A &= U\Sigma V^T, \quad U^TU = V^TV = I, \\ \Sigma &= \mathrm{diag}(\sigma_1, \ldots, \sigma_n), \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0. \end{aligned} \tag{2}$$

If the condition number of A is very large, typically of the same order of magnitude as the reciprocal of the unit round off of the computer where we want to solve (1), then we call (1) a *discrete ill-posed problem.*

In recent years several authors have pointed out that in many cases one is not primarily interested in the solution of the ill-posed problem but rather a functional defined on the solution, see [1], [2], [8]. For example, in geophysical exploration it may be more important to know the total mass or the centre of gravity of a mineral deposit than the exact boundaries.

Taking into account also that the data are most often contaminated with random errors, we are led to the problem of computing a confidence interval for a functional

$$w^T x, \tag{3a}$$

subject to the constraint

$$x \in S_0 = \{x \mid \|Ax - b\| \leq \epsilon\}, \tag{3b}$$

where ϵ is a constant related to properties of the random error, see [9]. The norm is the Euclidean vector norm. In the sequel we only discuss the problem of computing the minimum of the functional (3a).

Even though (3) may be much better conditioned than the problem of solving (1), it may still not be well-conditioned enough, see [9]. Then it is necessary to formulate mathematically some a priori information about the problem and use it in the numerical solution. In [9] a nonnegativity constraint is imposed. We assume that there is a trial solution x_0, from which the solution is not allowed to deviate more than a certain amount. Thus we consider

$$\min_{x \in S} w^T x, \quad S = \{x \mid \|Ax - b\| \le \epsilon, \ \|x - x_0\| \le \delta\}. \tag{4}$$

In [5] we study numerical algorithms for computing the solution of (4). It turns out not to be practical to solve (4) as it stands: the set S is the intersection of an ellipsoid and a sphere in R^n and it is not easy to find points on the boundary of S. What we are looking for is a scalar ϕ such that the hyperplane $w^T x = \phi$ intersects S in exactly one point.

From the point of view of computational efficiency it is better to reformulate the problem so that we fix the position of the hyperplane and consider its intersection with the sphere. This is a sphere of dimension $n-1$, which is easier to treat mathematically than S. Then we determine an ellipsoid that intersects the $(n-1)$-dimensional sphere in exactly one point. Thus we solve

$$L(\phi) = \min_{x \in T} \|Ax - b\|^2, \quad T = \{x \mid \|x - x_0\| \le \delta, \ w^T x = \phi\} \qquad (5).$$

To get the solution of (4) we then solve the equation $L(\phi) = \epsilon^2$.

Apart from the geometric arguments above, (5) can be seen to be easier to deal with numerically than (4), since the linear constraint can be eliminated to give an equivalent problem

$$\bar{L}(\phi) = \min_{y \in \bar{T}} \|\bar{A}y - \bar{b}\|, \quad \bar{T} = \{y \mid \|y - \bar{x}_0\| \le \bar{\delta}\}.$$

This can be solved easily using the methods in [4]. We have also studied algorithms for the problem when there are several different vectors w, as is the case in the applications in [9].

The inspiration to do this work comes from several discussions with R.S. Anderssen.

Cauchy problems for the heat equation. The problem of determining the temperature at an inaccessible side of a thick wall from measurements inside the wall can be formulated as a Cauchy problem for the heat equation in a quarter plane

$$\begin{aligned} u_{xx} &= u_t, && 0 \le x, \ 0 \le t, \\ u(x,0) &= 0, && 0 \le x, \\ u(1,t) &= g(t), && 0 \le t. \end{aligned} \tag{6}$$

We want to find the solution at $x = 0$. This problem is ill-posed in the sense that the solution does not depend continuously on the data. It can be stabilized if we impose a bound on the solution at $x = 0$ and allow for some imprecision in the data:

$$\begin{aligned} &u_{xx} = u_t, \qquad 0 \le x,\ 0 \le t, \\ &u(x,0) = 0, \qquad 0 \le x, \\ &\|u(0,\cdot)\| \le M, \\ &\|u(1,\cdot) - g(\cdot)\| \le \epsilon, \end{aligned} \tag{7}$$

for some constants M and ϵ, which are assumed to be known a priori. $\|\cdot\|$ is the L^2-norm on $(0,\infty)$. Using the results of [7] it is easy to show that any two solutions of (7), u_1 and u_2, may differ in norm by

$$\|u_1(x,\cdot) - u_2(x,\cdot)\| \le 2\epsilon^x M^{1-x}, \tag{8}$$

at most.

We are interested in the computation of an approximate solution of (7). Carasso [3] obtained an error estimate of the type (8) for a regularization method. In applications related to oil reservoir modelling modified equations of the form

$$u_{xx} = u_t + \mu u_{tt}, \tag{9}$$

are used. The extra term makes the equation hyperbolic and the Cauchy problem for this equation is well-posed. Such modifications have several advantages. The equation can be discretized and solved using standard methods for hyperbolic equations. These methods can also be used for problems with non-constant coefficients.

On the other hand it turns out that the error estimates for the modified equation are worse than (8). In [6] it is shown that essentially we only get logarithmic continuity:

$$\|u(x,\cdot) - v(x,\cdot)\| \le \frac{C}{(\log(M/\epsilon))^2},$$

for small ϵ (u is the "exact" solution and v is the solution of the Cauchy problem for the modified equation (9)).

This work was finished and written up during my visit to CMA. There I also started a joint project with another visitor, T.I. Seidman. We hope to get improved error estimates for a modified equation, where the modification is made using pseudodifferential operators.

REFERENCES.

1. R.S. Anderssen, On the use of Linear Functionals for Abel-type Integral Equations in Applications, in *The Application and Numerical Solution of Integral Equations*, R.S. Anderssen, F.R. de Hoog, and M.A. Lukas (eds.), Sijthoff and Nordhoff, Alphen and den Rijn, 1980.

2. M. Bertero, C. De Mol and G.A. Viano, The Stability of Inverse Problems, *Inverse Scattering Problems in Optics*, H.P. Bates (ed.), Springer Verlag, Berlin 1980.

3. A.S. Carasso, Determining surface temperatures from interior observations, SIAM J. Appl. Math. 42(1982), 558-574.

4. L. Eldén, Algorithms for the Regularization of Ill-Conditioned Least squares Problems, BIT 17(1977), 134-145.

5. L. Eldén, Algorithms for the Computation of Functionals Defined on the Solution of a Discrete Ill-posed Problem, in preparation.

6. L. Eldén, Hyperbolic approximations for a Cauchy problem for the heat equation, Centre for Mathematical Analysis, ANU, Canberra, Research Report CMA-R44-86, 1986.

7. H.A. Levine, Continuous data dependence, regularization, and a three lines theorem for the heat equation with data in a space like direction, Ann. Mat. Pura Appl. (IV) Vol CXXXIV (1983), 267-286.

8. P. Linz, Approximate Evaluation of Functionals Defined on Solutions of Ill-posed Problems, Univ. Calif. Davis, Computer Science Division, Report CSE-85-3, 1985.

9. D.P. O'Leary, B.W. Rust, Confidence Intervals for Inequality-Constrained Least Squares Problems, with Applications to Ill-Posed Problems, SIAM J. Sci. Stat. Comput. 7(1986), 473–489

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