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VISIBLE ACTIONS ON GENERALIZED FLAG VARIETIES AND A GENERALIZATION OF THE CARTAN DECOMPOSITION

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Abstract. With the aim of uniform treatment of multiplicity-free representations of Lie groups, T. Kobayashi introduced the theory of visible actions on complex manifolds.

Our main results give a classification of triples (G, H, L) for a compact Lie group G and its Levi subgroups H, L, which satisfy $G = HBL$. Here, B is a subset of a Chevalley–Weyl involution σ-fixed points subgroup G^σ of G. The point here is that one decomposition $G = LBH$ produces three strongly visible actions on generalized flag varieties, and thus three finite-dimensional multiplicity-free representations (Kobayashi's triunity principle).

Furthermore, we can also prove that the visibility of actions of compact Lie groups, the existence of a decomposition $G = LBH$ and the multiplicity-freeness property of finite-dimensional tensor product representations are all equivalent.

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1. Introduction

Let G be a connected compact Lie group and L, H Levi subgroups of G. Then, the homogeneous spaces G/H and G/L are generalized flag varieties. In this article, we give a classification of triples (G, L, H) such that the following three group-actions are strongly visible

$$L \curvearrowright G/H, \qquad H \curvearrowright G/L, \qquad \mathrm{diag}(G) \curvearrowright (G \times G)/(H \times L). \tag{1}$$

Here $\mathrm{diag}(G)$ is the diagonal subgroup of the direct product group $G \times G$.

Definition 1 ([5]). *A holomorphic action of a group G on a complex manifold X is called strongly visible if the following two conditions are satisfied*

- *There exists a real submanifold S (slice) such that $X' := G \cdot S$ is an open subset of X.*
- *There exists an anti-holomorphic diffeomorphism σ of X' such that $\sigma|_S = \mathrm{id}_S$ and $\sigma(G \cdot x) = G \cdot x$ for any $x \in S$.*

Classification problem of visible actions was discussed previously in some other settings, see [5,7,8,13,14] (and Sections 3.1–3.4 in this article) for example.

This classification problem is closely related to the multiplicity-freeness property of finite-dimensional representations. Various examples of multiplicity-free representations have been obtained by many people. For finite-dimensional cases, typical approaches are

1. (Sphericity) Verify the existence of an open orbit of a Borel subgroup.
2. (Combinatorics) Using character formulas.

A new approach has been introduced by T. Kobayashi, that is, the propagation theorem of multiplicity-freeness property [6] (Theorem 1 in this article) using the notion of visible actions on complex manifolds [4]. The advantage of this approach is that not only finite-dimensional cases but also infinite-dimensional (both discrete and continuous spectra) cases can be applied by this method (c.f. [5,9]).

Our classification problem is also related to a theory of normal forms. A theory of normal forms is often connected with a decomposition theory of groups. A prototype is the diagonalization of symmetric matrices by orthogonal groups, which is equivalent to the Cartan decomposition $G = KAK$ for $G = \mathrm{GL}(n, \mathbb{R})$. A similar type of the decomposition theorem of the form $G = KBH$ or its variants has been well-established by the work of É. Cartan, M. Flensted-Jensen [2], B. Hoogenboom [3] and T. Matsuki [12] under the assumption that both (G, H) and (G, L) are symmetric pairs. As explained below, we find an analogous decomposition in the strongly visible setting where (G, H) and (G, L) are not necessarily symmetric pairs.

We explain the main results of this article. Suppose that Levi subgroups L and H contain the same maximal torus T of G, and let σ be a Chevalley–Weyl involution of G with respect to T in the sense that $\sigma(t) = t^{-1}$ for any $t \in T$. Our main theorem (Theorem 8) gives an answer to the following problem:

Classify triples (G, L, H) such that the multiplication mapping

$$L \times G^\sigma \times H \to G \tag{2}$$

is surjective.

Here G^σ is the σ-fixed points subgroup of G. The Chevalley–Weyl involution σ induces an anti-holomorphic involution on the flag variety G/H, and $G^\sigma/(G^\sigma \cap H)$ is a totally real submanifold of the complex manifold G/H. Therefore if the multiplication mapping (2) is surjective then every L-orbits on G/H meets the totally real submanifold $G^\sigma/(G^\sigma \cap H)$, and hence the L-action on G/H is strongly visible. Likewise the other two group-actions $H \curvearrowright G/L$ and $\mathrm{diag}(G) \curvearrowright (G \times G)/(H \times L)$ are strongly visible [4]. It is noteworthy that our classification shows that the converse is also true in the setting here. Indeed by our main results we can see that the surjectivity of the map (2), the visibility of compact Lie group-actions on generalized flag varieties and the multiplicity-freeness property of finite-dimensional representations are all equivalent (see Corollary 10). Therefore we can also obtain a classification of visible actions on generalized flag varieties (1) as stated in the beginning of this section.

2. Multiplicity-Free Representations

Our main theorem (Theorem 8) is a classification of a generalized Cartan decomposition for compact Lie groups (Definition 5), which produces strongly visible actions on generalized flag varieties and thus finite-dimensional multiplicity-free representations. Here, we interpret the multiplicity-freeness property of representations as follows.

Definition 2. *Let G be a locally compact group and V a unitary representation of G. We say V is multiplicity-free if the ring $\mathrm{End}_G(V)$ of G-intertwining operators on V is commutative.*

We note that if the dimension of V is finite, then V is multiplicity-free if and only if any irreducible unitary representation of G appears at most once in the irreducible decomposition of V.

Kobayashi introduced the notion of visible actions on complex manifolds with the aim of uniform treatment of multiplicity-free representations of Lie groups, and indeed, we can obtain multiplicity-free representations from a visible action by the following theorem called the propagation theorem of multiplicity-freeness property.

Theorem 1 ([6])**.** *Let G be a Lie group and $\mathcal{W}$ a G-equivariant Hermitian holomorphic vector bundle on a connected complex manifold X. Let V be a unitary representation of G. If the following conditions from (0) to (3) are satisfied, then V is multiplicity-free as a representation of G.*

0) *There exists a continuous and injective G-intertwining operator from V to the space $\mathcal{O}(X, \mathcal{W})$ of holomorphic sections of $\mathcal{W}$.*

1) *The action of G on X is S-visible. That is, there exist a subset $S \subset X$ and an anti-holomorphic diffeomorphism σ of X' satisfying the conditions given in Definition 1. Further, there exists an automorphism $\hat{\sigma}$ of G such that $\sigma(g \cdot x) = \hat{\sigma}(g) \cdot \sigma(x)$ for any $g \in G$ and $x \in X'$.*
2) *For any $x \in S$, the fiber $\mathcal{W}_x$ at x decomposes as the multiplicity-free sum of irreducible unitary representations of the isotropy subgroup G_x. Let $\mathcal{W}_x = \bigoplus_{1 \le i \le n(x)} \mathcal{W}_x^{(i)}$ denote the irreducible decomposition of $\mathcal{W}_x$.*
3) *σ lifts to an anti-holomorphic automorphism $\tilde{\sigma}$ of $\mathcal{W}$ and satisfies $\tilde{\sigma}(\mathcal{W}_x^{(i)}) = \mathcal{W}_x^{(i)}$ for each $x \in S$ $(1 \le i \le n(x))$.*

We can see in the statement of the above theorem that we do not need to assume that

G is compact, reductive

V is of finite-dimensional, discretely decomposable, or

X is compact.

In the following, we give a few examples of applications of Theorem 1.

Example 1. Let G be a semisimple Lie group and K a maximal compact subgroup of G. Then it is well-known that the space $L^2(G/K)$ of square integrable functions on the Riemannian symmetric space G/K is multiplicity-free (see [25] for example). We can also prove the multiplicity-freeness property by combining Theorem 1 with the following facts.

- The G-action on the complexification $G_{\mathbb{C}}/K_{\mathbb{C}}$ is strongly visible [8].
- There exists a G-embedding $L^2(G/K) \hookrightarrow \mathcal{O}(U)$ [10], where U is the complex crown of G/K [1].

Example 2. Let G be a simple Lie group of Hermitian type, K a maximal compact subgroup and H a symmetric subgroup of G, i.e., H is an open subgroup of the τ-fixed points subgroup G^τ for an involution τ of G. Let π be a unitary highest weight representation of the scalar type of G. Then the restriction of π to H is multiplicity-free [8] by Theorem 1 and the following facts.

- π can be realized in the space $\mathcal{O}(G/K, \mathcal{L})$ of holomorphic sections of a G-equivariant holomorphic line bundle $\mathcal{L}$ on the Hermitian symmetric space G/K.
- The H-action on G/K is strongly visible [8] by the Cartan decomposition $G = HAK$ in the symmetric setting [2, 3, 12].

Example 3. Let G be a simple Lie group of Hermitian type, K a maximal compact subgroup and N a maximal unipotent subgroup of G. Let π be a unitary highest weight representation of the scalar type of G. Then the restriction of π to N is multiplicity-free by Theorem 1 and the following facts.

- π can be realized in $\mathcal{O}(G/K, \mathcal{L})$ for a G-equivariant holomorphic line bundle $\mathcal{L}$ on G/K.
- The action of N on G/K is strongly visible [5] by Iwasawa decomposition $G = NAK$.

3. Classification of Visible Actions (Known Results)

As we saw in the above examples, if we have a visible action of a Lie group, then we can obtain multiplicity-free theorems. So we want to find, or even more, to classify visible actions. In the following Sections 3.1–3.4 we show some known results for a classification of visible actions.

3.1. Hermitian Symmetric Space

Theorem 2 ([8]). *Let (G, K) be a Hermitian symmetric pair and (G, H) a symmetric pair. Then the action of H on the Hermitian symmetric space G/K is strongly visible.*

3.2. Linear Multiplicity-Free Space

Definition 3. *Let $G_{\mathbb{C}}$ be a connected complex reductive algebraic group and V a finite-dimensional representation of $G_{\mathbb{C}}$. We say V is a linear multiplicity-free space of $G_{\mathbb{C}}$ if the space $\mathbb{C}[V]$ of polynomials on V is multiplicity-free as a representation of $G_{\mathbb{C}}$.*

Theorem 3 ([13, 16]). *Let V be a linear multiplicity-free space of a connected complex reductive algebraic group $G_{\mathbb{C}}$. Then a compact real form of $G_{\mathbb{C}}$ acts on V strongly visibly.*

Remark 4. *Although the $G_{\mathbb{C}}$-action on $\mathbb{C}[V]$ is not unitary, we can apply our definition of the multiplicity-freeness property* (Definition 2) *to this case by using the Weyl's unitary trick.*

3.3. Affine Homogeneous Spherical Variety

Definition 4. *Let $G_{\mathbb{C}}$ be a complex reductive algebraic group and X a connected complex algebraic $G_{\mathbb{C}}$-variety. We call X a spherical variety of $G_{\mathbb{C}}$ if a Borel subgroup B of $G_{\mathbb{C}}$ (e.g. $G_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$ and B ={upper triangular matrices}) has an open orbit on X.*

Remark 5. • *Any complex symmetric space (e.g.* $\mathrm{SO}(n,\mathbb{C})/(\mathrm{SO}(m,\mathbb{C}) \times \mathrm{SO}(n-m,\mathbb{C})))$ *is spherical.*

- *Let $G_{\mathbb{C}}$ be a connected complex reductive algebraic group and $H_{\mathbb{C}}$ a complex reductive algebraic subgroup. Let G be a connected real form of $G_{\mathbb{C}}$ and define $H := G \cap H_{\mathbb{C}}$. Assume that H is a compact real form of $H_{\mathbb{C}}$. Then $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical if and only if (G,H) is a reductive Gelfand pair (e.g. $G = \mathrm{SO}(1,2n)_0$ and $H = \mathrm{U}(n)$, where the subscript 0 means the identity component), that is, $L^2(G/H)$ is multiplicity-free as a representation of G(see* [25] for example).

Theorem 6 ([14, 15, 17])**.** *Let $G_{\mathbb{C}}/H_{\mathbb{C}}$ be one of the following reductive homogeneous spherical varieties*

$$\begin{aligned}
&\mathrm{SL}(m+n,\mathbb{C})/(\mathrm{SL}(m,\mathbb{C}) \times \mathrm{SL}(n,\mathbb{C}))\ (m \neq n)\\
&\mathrm{Spin}(4n+2,\mathbb{C})/\mathrm{SL}(2n+1,\mathbb{C})\\
&\mathrm{SL}(2n+1,\mathbb{C})/\mathrm{Sp}(n,\mathbb{C})\\
&\mathrm{E}_6(\mathbb{C})/\mathrm{Spin}(10,\mathbb{C})\\
&\mathrm{SO}(8,\mathbb{C})/\mathrm{G}_2(\mathbb{C}).
\end{aligned}$$

Then the action of a compact real form of $G_{\mathbb{C}}$ on $G_{\mathbb{C}}/H_{\mathbb{C}}$ is strongly visible.

3.4. Generalized Flag Variety of Type A

Let $G = \mathrm{U}(n)$ and L, H Levi subgroups of G. Kobayashi [7] classified the triples (G, H, L) that are strongly visible.

Remark 7. *In fact, the above three actions are all strongly visible if and only if at least one of those is strongly visible* [5].

A classification of strongly visible actions on generalized flag varieties of type A is a prototype of the main theorem of this article (Theorem 8). Before stating the main theorem, we give the definition of a generalized Cartan decomposition.

Definition 5. *Let G be a connected compact Lie group, T a maximal torus and H, L Levi subgroups of G, which contain T. We take a Chevalley–Weyl involution of G with respect to T, that is, $\sigma(t) = t^{-1}$ for any $t \in T$ (e.g. $G = \mathrm{U}(n)$, $T = \{$diagonal matrices$\}$ and $\sigma =$the complex conjugation). If the multiplication mapping*

$$L \times B \times H \to G$$

is surjective for a subset B of the σ-fixed points subgroup G^{σ}, then we call the decomposition $G = LBH$ a generalized Cartan decomposition.

The above definition comes from that of visible action (Definition 1). Let us explain. We retain the setting of Definition 5. Then we note that σ acts on generalized flag varieties

$$G/H, \qquad G/L, \qquad (G \times G)/(H \times L)$$

as anti-holomorphic diffeomorphisms. Now suppose that $G = LBH$ holds for some $B \subset G^\sigma$. Then we obtain three strongly visible actions

$$L \curvearrowright G/H, \quad H \curvearrowright G/L, \quad \mathrm{diag}(G) \curvearrowright (G \times G)/(H \times L).$$

Furthermore, we can obtain three multiplicity-free theorems by using Theorem 1.

$$\mathrm{ind}_H^G \chi_H|_L, \quad \mathrm{ind}_L^G \chi_L|_H, \quad \mathrm{ind}_H^G \chi_H \otimes \mathrm{ind}_L^G \chi_L.$$

Here ind_H^G and ind_L^G denote the holomorphically induced representations from H and L, and $|_L$ and $|_H$ the restrictions to L and H, respectively. Also, χ_H and χ_L are unitary characters of H and L, respectively. As we saw, one generalized Cartan decomposition leads us to three strongly visible actions, and three multiplicity-free theorems (Kobayashi's triunity principle [4]).

4. Main Results

The following is the main theorem of this article.

Theorem 8 ([19–23])**.** *Let G be a connected compact Lie group, T a maximal torus, Π a simple system and L_1, L_2 Levi subgroups of G, whose simple systems are given by proper subsets Π_1, Π_2 of Π. Let σ be a Chevalley–Weyl involution of G with respect to T. Then the triples (G, L_1, L_2) listed in* Sections 4.1–4.7 *exhaust all the triples such that the multiplication mapping*

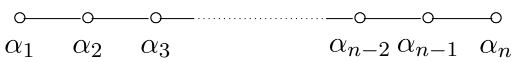
$$L_1 \times B \times L_2 \to G$$

is surjective for a subset B of G^σ.

Remark 9.
- *In the type A case $(G = \mathrm{U}(n))$, this theorem is proved in* [7].
- *The double coset decomposition $L \backslash G/H$ is well-studied in the symmetric setting (i.e., both H and L are symmetric subgroups of G) in* [2, 3, 12].

In the following, we specify only the type of G since our classification is independent of coverings, and list pairs (Π_1, Π_2) of proper subsets of Π instead of pairs (L_1, L_2) of Levi subgroups of G. Also, we put $(\Pi_j)^c := \Pi \setminus \Pi_j$ $(j = 1, 2)$.

4.1. Classification for Type A_n



Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\},\ (\Pi_2)^c = \{\alpha_j\}$.

Non-Hermitian type:

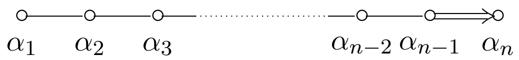
I. $(\Pi_1)^c = \{\alpha_i, \alpha_j\},\ \ (\Pi_2)^c = \{\alpha_k\},\ \ \min_{p=i,j}\{p, n+1-p\} = 1$ or $i = j \pm 1$.

II. $(\Pi_1)^c = \{\alpha_i, \alpha_j\},\ \ (\Pi_2)^c = \{\alpha_k\},\ \min\{k, n+1-k\} = 2$.

III. $(\Pi_1)^c = \{\alpha_l\},\ \Pi_2$: arbitrary, $\ l = 1$ or n.

Here i, j, k satisfy $1 \le i, j, k \le n$.

4.2. Classification for Type B$_n$



Hermitian type:

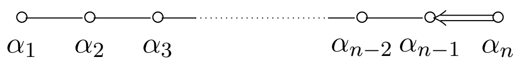
I. $(\Pi_1)^c = \{\alpha_1\},\ (\Pi_2)^c = \{\alpha_1\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_n\},\ (\Pi_2)^c = \{\alpha_n\}$.

II. $(\Pi_1)^c = \{\alpha_1\},\ \ (\Pi_2)^c = \{\alpha_i\},\ 2 \le i \le n$.

4.3. Classification for type C$_n$



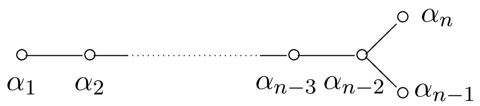
Hermitian type:

I. $(\Pi_1)^c = \{\alpha_n\},\ (\Pi_2)^c = \{\alpha_n\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_1\},\ (\Pi_2)^c = \{\alpha_i\},\ 1 \le i \le n$.

4.4. Classification for type D$_n$



Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\},\ \ (\Pi_2)^c = \{\alpha_j\},\ i, j \in \{1, n-1, n\}$.

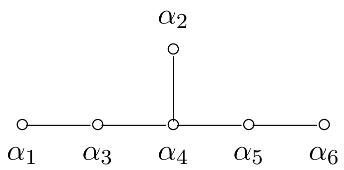
Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_1\},\ (\Pi_2)^c = \{\alpha_j\},\ j \ne 1, n-1, n$.

II. $(\Pi_1)^c = \{\alpha_i\},\ (\Pi_2)^c = \{\alpha_j\},\ i \in \{n-1, n\},\ j \in \{2, 3\}$.

III. $(\Pi_1)^c = \{\alpha_i\},\ (\Pi_2)^c = \{\alpha_j, \alpha_k\},\ i \in \{n-1, n\},\ j, k \in \{1, n-1, n\}.$
IV. $(\Pi_1)^c = \{\alpha_i\},\ (\Pi_2)^c = \{\alpha_1, \alpha_2\},\ i \in \{n-1, n\}.$
V. $(\Pi_1)^c = \{\alpha_1\},\ (\Pi_2)^c = \{\alpha_j, \alpha_k\},\ j$ or $k \in \{n-1, n\}.$
VI. $(\Pi_1)^c = \{\alpha_i\},\ (\Pi_2)^c = \{\alpha_2, \alpha_j\},\ n = 4,\ (i, j) = (3, 4)$ or $(4, 3).$

4.5. Classification for type $\mathbf{E}_6$



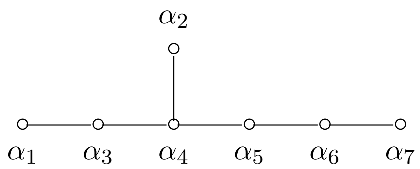
Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\},\ (\Pi_2)^c = \{\alpha_j\},\ i, j \in \{1, 6\}.$

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\},\ (\Pi_2)^c = \{\alpha_1, \alpha_6\},\ i = 1$ or 6.
II. $(\Pi_1)^c = \{\alpha_i\},\ (\Pi_2)^c = \{\alpha_j\},\ i = 1$ or $6,\ j \neq 1, 4, 6.$

4.6. Classification for type $\mathbf{E}_7$



Hermitian type:

I. $(\Pi_1)^c = \{\alpha_7\},\ (\Pi_2)^c = \{\alpha_7\}.$

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_7\},\ (\Pi_2)^c = \{\alpha_i\},\ i = 1$ or 2.

4.7. Classification for type $\mathbf{E}_8$, $\mathbf{F}_4$, $\mathbf{G}_2$

There is no pair (Π_1, Π_2) of proper subsets of Π such that $G = L_1 G^\sigma L_2$ holds.

As we explained before, one generalized Cartan decomposition leads us to three strongly visible actions. The following corollary shows that the converse is also true in our setting. Therefore we can obtain a classification of visible actions on generalized flag varieties from Theorem 8.

Corollary 10 ([19]). *We retain the setting of* Theorem 8. *We denote by $G_{\mathbb{C}}$ and $(L_j)_{\mathbb{C}}$ the complexifications of G and L_j, respectively ($j = 1, 2$). We let P_j be a* *parabolic subgroup of* $G_{\mathbb{C}}$ *with Levi subgroup* $(L_j)_{\mathbb{C}}$*, and put* $\mathcal{P}_j = G/P_j$ *(*$j = 1, 2$*). Then the following eleven conditions are equivalent.*

i) *The multiplication mapping* $L_1 \times G^{\sigma} \times L_2 \to G$ *is surjective.*

ii) *The natural action* $L_1 \curvearrowright \mathcal{P}_2$ *is strongly visible.*

iii) *The natural action* $L_2 \curvearrowright \mathcal{P}_1$ *is strongly visible.*

iv) *The diagonal action* $G \curvearrowright \mathcal{P}_1 \times \mathcal{P}_2$ *is strongly visible.*

v) *Any irreducible representation of G, which belongs to* $\mathcal{P}_2$*-series is multiplicity-free when restricted to* L_1*.*

vi) *Any irreducible representation of G, which belongs to* $\mathcal{P}_1$*-series is multiplicity-free when restricted to* L_2*.*

vii) *The tensor product of arbitrary two irreducible representations* π_1 *and* π_2 *of G, which belong to* $\mathcal{P}_1$ *and* $\mathcal{P}_2$*-series, respectively, is multiplicity-free.*

viii) $\mathcal{P}_2$ *is a spherical variety of* $(L_1)_{\mathbb{C}}$*.*

ix) $\mathcal{P}_1$ *is a spherical variety of* $(L_2)_{\mathbb{C}}$*.*

x) $\mathcal{P}_1 \times \mathcal{P}_2$ *is a spherical variety of* $G_{\mathbb{C}}$*.*

xi) *The pair* (Π_1, Π_2) *is one of the entries listed in* Sections 4.1–4.7 *up to switch of the factors.*

Here an irreducible representation of $G_{\mathbb{C}}$ *is in* $\mathcal{P}_j$*-series if it is a holomorphically induced representation from a character of the Levi part* $(L_j)_{\mathbb{C}}$ *of* P_j $(j = 1, 2)$.

Remark 11.

- *For the type A case* $(G = \mathrm{U}(n))$*, this corollary is obtained in* [5].
- *The equivalence between the multiplicity-freeness property and the sphericity is proved in* [24].
- *A classification of finite-dimensional multiplicity-free tensor product representations in the maximal parabolic setting is given in* [11].
- *A classification of finite-dimensional multiplicity-free tensor product representations in the general setting is completed in* [18] *by a combinatorial method.*

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