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SECRETARY PROBLEMS AS A SOURCE OF BENCHMARK BOUNDS

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Secretary problems are those sequential selection problems in which the payoff (or cost) depends on the observations only through their ranks. A subclass of such problems allows only selection rules based on relative ranks. The performance of such rules provides readily accessible lower bounds for procedures based on more information. Included here are familiar bounds, like $1/e$; well-known bounds, like 3.8695; and brand-new bounds, like 2.6003.

1. Googol

Although there is a *pre-history* associated with Secretary Problems—which some have traced back into the 19th century—the generally agreed-upon "big bang" took place with Martin Gardner's presentation of the following problem in his Mathematical Games column in the February, 1960 *Scientific American.*

> Ask someone to take as many slips of paper as he pleases, and on each slip write a different positive number. The numbers may range from small fractions of one to a number the size of a *googol* (1 followed by a hundred zeros) or even larger. These slips are turned face-down and shuffled over the top of a table. One at a time you turn the slips face up. The aim is to stop turning when you come to the number that you guess to be the largest of the series. You cannot go back and pick a previously turned slip. If you turn over all the slips, then of course you must pick the last one turned.

The "solution" in the March, 1960 column, treated googol as though it were the classical best-choice problem. That is to say, it was taken for granted that only stopping rules based on the relative ranks of the numbers need be considered. (That was, after all, the point of calling it "googol," wasn't it?)

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The optimal rule based on relative ranks is well-known to be among rules of the form: let $k-1$ go by and then select the first relatively best one (if any). For such stopping rules, the probability of best choice is

$$\phi_n(k) = \sum_{i=k}^{n} \frac{1}{n} \cdot \frac{k-1}{i-1}. \tag{1}$$

where n is the total number of slips of paper. A well-known elementary argument (which need not be repeated here) shows that, for k equal to half of n, this probability is at least $1/4$, no matter how large n is. If $k(n)/n \to x$ as $n \to \infty$, then

$$\phi_n(k(n)) \to -x \log x, \tag{2}$$

which is about .35 if $x = .5$, but is maximized at $x = 1/e \approx .3679$; and the celebrated maximum is also $1/e$.

Here we have the most famous secretary problem benchmark bound. How sharp is it?

1.1. *Full Information Problem*

Suppose the numbers are known to be a random sample (i.e., i.i.d.) from some specified continuous distribution. Whatever the distribution, the optimal probability of selecting the largest number, call it w_n, decreases with n to

$$\begin{aligned} \lim_{n\to\infty} w_n &= e^{-c} - (e^c - c - 1)\int_1^{\infty} x^{-1}e^{-cx}\,dx. \\ &\approx .5802 \end{aligned} \tag{3}$$

where $c \approx .804$ is the solution to

$$\sum_{j=1}^{\infty} c^j/j!j = 1. \tag{4}$$

See Gilbert and Mosteller (1966, Section 3) and Samuels (1982).

1.2. *Partial Information Problems from the Minimax Point of View*

Now suppose that only some parametric family (e.g. normal) is specified, perhaps together with a prior distribution on the parameters.

For location and scale parameter families, Petruccelli (1978) found sufficient conditions for the existence of a sequence of invariant stopping rules for which the probability of best choice converges to the full-information limiting value as $n \to \infty$. The normal family satisfies his conditions, but the uniforms do not.

For the family of uniforms on $(\theta-\frac{1}{2}, \theta+\frac{1}{2})$, Petruccelli (1980) showed that the limiting probability of best choice for the best invariant rules (which are minimax by a version of the Hunt-Stein theorem) is .4352, a value intermediate between the full-information value of .5802 and the "no-information" value of $1/e = .3679$.

For the family of all uniforms, Samuels (1981) showed that the best-choice problem solution is minimax, so, at least from the minimax point of view, knowing that the distribution is uniform is "no information." A simplification of the original proof was given by Ferguson (1989), and can also be found in Samuels (1991).

1.3. *Partial Information Problems from a Bayesian Point of View*

This minimax result still begs the question of whether there is *any* exchangeable distribution for which it is optimal to consider only the relative ranks. (It is implicit in the statement of Googol that the numbers are exchangeable.) Samuels (1989) posed this question and called it Ferguson's Secretary Problem because Ferguson (1989) showed that for any $\varepsilon > 0$ and for any n, there is a two-parameter Pareto prior distribution on θ such that, when sequentially observing n uniform r.v.'s on $[0, \theta]$, the best rule based only on relative ranks comes within ϵ of being optimal. Recently, Hill and Krengel (1991b) have extended Ferguson's result to the case where the number of items, n, is unknown with a known upper bound. In this context, "optimal" is replaced by "minimax-optimal." The same authors had already found the minimax rules based on relative ranks in Hill and Krengel (1991a).

For the case $n = 2$, the answer to the above question is NO; there is *no* exchangeable distribution for which it is optimal to consider only the relative ranks. This is easily seen by the following simple and well-known argument:

> Let X_1, X_2 be the first and second numbers examined, respectively. Now, pick any number, x, between the inf and the sup of the support of the X's, and choose X_1 if $X_1 > x$; otherwise choose X_2. If both X_1 and X_2 turn out to be bigger than x, or if both are smaller than x, then (by exchangeability) this rule selects the larger of the two with probability 1/2, while, if one random variable is larger than x while the other is smaller, the larger one is sure to be chosen. Thus, setting the unknown $P\{\min(X_1, X_2) < x < \max(X_1, X_2)\}$ equal to c, say, we have
>
> $$P\{X_\tau = \max(X_1, X_2)\} = c + (1-c)/2 = (1/2)(1+c),$$
>
> which is strictly greater than 1/2. This beats rules based only on relative ranks, which, for $n = 2$, are necessarily constants, so have probability 1/2 of success.

Recently Silverman and Nádas (1991) have shown, to the surprise of many, that for $n = 3$, there *are* such distributions. They began by giving the following necessary and sufficient conditions for achieving optimality with a rule based on relative ranks:

$$\begin{aligned} &P(X_1 = \max(X_1, X_2, X_3)|X_1) \\ &\qquad +(1/4)P(X_1 = \min(X_1, X_2, X_3)|X_1) \leq 1/2 \quad a.s. \end{aligned} \tag{5}$$

and

$$P(\max(X_1, X_2) = \max(X_1, X_2, X_3)|X_1, X_2) \geq 1/2 \quad a.s. \tag{6}$$

(This corrected an error in Samuels (1989) which had (6) all right, but omitted the second term of (5).) Then they let X_1, X_2, X_3, given θ, be conditionally i.i.d., uniform on $[0, \theta]$, with prior density on θ of the form:

$$g(\theta) = tI_{\{0<\theta\leq 1\}} + (1-t)\frac{\alpha}{\theta^{1+\alpha}}I_{\{\theta>1\}} \qquad \alpha > 0,\ 0 \leq t \leq 1. \tag{7}$$

This family includes—for $t = 0$—the Pareto priors used in Ferguson (1989) to "come within ε;" see above. By enlarging the class of available priors, Silverman and Nádas (1991) were able to find a subclass for which (5) and (6) are both satisfied; namely those with $3t/(2-3t) \leq \alpha \leq 2t/(1-2t)$.

If this result can be extended to all n, then it can truly be said that the famous $1/e$ benchmark bound is sharp.

2. A General Class of Problems

Problems involving arbitrary loss functions, a random number of arrivals, or a sampling cost—generally considered separately in the literature—can be combined into a single model, as follows:

Let N denote the number of rankable items which appear in random order, $X_1, X_2, \ldots, X_N$ be their ranks, and $Y_1, Y_2, \ldots, Y_N$ be the corresponding relative ranks. Conditional on $\{N = n\}$, $X_1, X_2, \ldots, X_n$ are a random permutation of $\{1, 2, \ldots, n\}$, hence the Y_i's are independent, uniform on $\{1, 2, \ldots, i\}$. All stopping rules, τ, are based on the Y's but suitably modified to contend with the possible randomness of N. And there are risks, $A(i, j)$, for stopping at time i with an item of relative rank j, of the form

$$A(i, j) = H(i) + K(i, j). \tag{8}$$

(The use of two terms, where one would do, is for clarity in what follows.) An optimal rule is one which minimizes $EA(\tau, Y_\tau)$.

For example, here are three problems, all of which have

$$
\begin{aligned}
H^{(n^*)}(i) &= (i-1)/n^*, \\
K^{(n^*)}(i,1) &= \sum_{m=i+1}^{n^*} (1 - i/m)(1/n^*), \\
K^{(n^*)}(i,j) &\equiv 1 \qquad\qquad\qquad j > 1. \qquad (9)
\end{aligned}
$$

• Best-choice problem with N uniform on 1 to n^*: subject to the "boundary condition" that, when the N-th item is best, we only get it by actually selecting it (Presman and Sonin (1972)).

• Payoff equals proportion of time holding the relatively best: $N \equiv n^*$; and if we "stop" with the τ-th item, a relatively best one, and the next relatively best item is the σ-th item, our reward is $(\sigma - \tau)/n^*$; or, if there is no subsequent relatively best item, the reward is $(n^* - \tau)/n^*$ (Ferguson, Hardwick and Tamaki (1991)).

• Linear sampling cost plus oddball risk term: $N \equiv n^*$; $H^{(n^*)}(i)$ is the linear sampling cost, and $K^{(n^*)}(i,1)$ is the hard-to-interpret risk when selecting a relatively best item at stage i.

2.1. *General Loss Function*

Let us now specialize. Let $N \equiv n$ be fixed, and prescribe a non-decreasing loss function, $q(\cdot)$, where $q(i)$ is the loss for selecting the item which turns out to be i-th best overall. Then we can take $H(i) \equiv 0$ and $K(i,j) = R^{(n)}(i,j)$, where

$$
\begin{aligned}
R^{(n)}(i,j) &= E_n[q(X_i) \mid Y_i = j] \\
&= \sum_{k=j}^{n} \frac{\binom{k-1}{j-1}\binom{n-k}{i-j}}{\binom{n}{i}} q(k). \qquad (10)
\end{aligned}
$$

Using backward induction plus the independence of the sequence of relative ranks, we can conclude that the quantities

$$c_i^{(n)} \equiv \inf_{\tau > i} E_n(R^{(n)}(\tau, Y_\tau) \mid Y_1, \ldots, Y_i) \qquad (11)$$

are constants, and that the formula

$$c_{i-1}^{(n)} = \frac{1}{i} \sum_{j=1}^{i} \min[R^{(n)}(i,j),\, c_i^{(n)}] \qquad i = n-1,\, n-2, \ldots, 1, \qquad (12)$$

holds, with boundary condition

$$c_{n-1}^{(n)} = \frac{1}{n} \sum_{j=1}^{n} [R^{(n)}(n,j)] \qquad (13)$$

and minimal risk, over all stopping rules,

$$v_n = c_0^{(n)} \tag{14}$$

Equation (12) can be rewritten in difference equation form as

$$\frac{c_i^{(n)} - c_{i-1}^{(n)}}{1/n} = \frac{1}{i/n} \sum_{j=1}^{i} \left[c_i^{(n)} - R^{(n)}(i,j)\right]^+ . \tag{15}$$

For fixed j, the risk (10) is decreasing in i to $q(j)$, so, from (12), if we let

$$i_k^{(n)} = \begin{cases} \min\{i : R^{(n)}(i,k) \le c_i^{(n)}\} \\ n \quad \text{if no such } i \end{cases} , \tag{16}$$

then an optimal rule stops at the first i for which, for some k, $i_k \le i < i_{k+1}$ and $Y_i \le k$. Its risk is

$$v_n = c_0^{(n)} = c_1^{(n)} = \ldots = c_{i_1-1}^{(n)}. \tag{17}$$

In addition, the risks have a limit as $n \to \infty$, namely

$$i/n \to t \Rightarrow R^{(n)}(i,j) \to R_j(t) = \sum_{k=1}^{\infty} q(k) \binom{k-1}{j-1} t^j (1-t)^{k-j}, \tag{18}$$

and, as Mucci (1973a and 1973b) showed for a large class of $q(\cdot)$'s, $c_i^{(n)} \approx f(i/n)$, where

$$f'(t) = \frac{1}{t} \sum_{j=1}^{\infty} [f(t) - R_j(t)]^+ \quad 0 \le t < 1 \tag{19}$$

with boundary condition, $f(1) = \sup q(\cdot)$, and a non-decreasing sequence of *thresholds,*

$$t_k : \ f(t_k) = R_k(t_k) \qquad k = 1, 2, \ldots. \tag{20}$$

$f(\cdot)$ is constant on $[0, t_1]$, so the limiting optimal risk is

$$v = \lim v_n = f(0) = f(t_1), \tag{21}$$

and (19) can be rewritten as the piecewise differential equation,

$$\left(\frac{f(t)}{t^k}\right)' = -\frac{1}{t^{k+1}} \sum_{j=1}^{k} R_j(t) \qquad t_k \le t \le t_{k+1}. \tag{22}$$

2.1.1. Some special cases

BEST-CHOICE PROBLEM: $q(1) = 0$ and $q(i) \equiv 1$ for all $i > 1$; $R^{(n)}(i,1) = 1 - i/n$, $R_1(t) = 1 - t$,

$$f(t) = \begin{cases} 1 - e^{-1} & 0 \leq t \leq e^{-1} \\ 1 - |t \ln t| & e^{-1} \leq t \leq 1 \end{cases} . \tag{23}$$

POLYNOMIAL-IN-RANKS PROBLEM: If $q(i) \equiv i(i+1)\cdots(i+a-1)$ for some integer $a \geq 1$, then (10) becomes

$$R^{(n)}(i,j) = j(j+1)\cdots(j+a-1) \cdot \left(\frac{(n+1)\cdots(n+a)}{(i+1)\cdots(i+a)} \right), \tag{24}$$

so

$$R_j(t) = j(j+1)\cdots(j+a-1)t^{-a} \tag{25}$$

and $v = a!/t_1^a = a!B_a^a$, where

$$B_a = \prod_{j=1}^{\infty} \left(\frac{j+a+1}{j} \right)^{1/(j+a)} \to \exp(\pi^2/6) \quad \text{as } a \to \infty. \tag{26}$$

Chow *et al* (1964) derived $B_1 \simeq 3.8695$ directly from the difference equation, without using the piecewise differential equation, and, recently, Robbins (1989) reported the above generalization. This remarkable result of a finite risk (less than four, in fact, for $a = 1$) despite an unbounded loss function can also be obtained in a more transparent way using memory-length one rules (see Section 3.1).

2.1.2. Monotonicity of the optimal risk, v_n

For any non-decreasing loss function, $q(\cdot)$, the optimal risk, v_n, is non-decreasing in n, the number of items to be observed. And v_n is strictly increasing whenever $q(\cdot)$ is. One way to prove this is to consider an $n+1$-arrival problem in which we are told when the current item is *worst* of all $n+1$. Since an optimal rule never selects a relatively worst item unless it is the last one, we can use the optimal $n+1$-arrival rule to select one of the other n arrivals. This is easily seen to be simply a *randomized* n-arrival rule, hence suboptimal for n arrivals, but possibly super-optimal for $n+1$ since it never selects the worst one. See, e.g., Chow *et al* (1964).

2.1.3. A risk 'paradox'

Since, for *fixed* sample size, the optimal risk, v_n, is increasing, one might naively expect the optimal risk for a bounded (by n^*) arrival distribution to be less than v_{n^*}. That this is quite false was demonstrated in Gianini-Pettitt (1979) for the arrival distributions defined by

$$P(N = i \mid N \geq i) = (n^* - i + 1)^{-\alpha} \qquad i = 1, 2, \ldots, n^* \tag{27}$$

(which includes, for $\alpha = 1$, the uniform distributions). For loss equal to the rank, the limiting optimal risk is infinite if $\alpha < 2$ and equal to the fixed sample size limit (3.8695) if $\alpha > 2$. The "paradox" disappears when we realize (as shown at the beginning of this section) that uncertainty about the number of arrivals is like imposing a sampling cost.

2.1.4. Minimax quantile problem

Take a random sample of size n from a member of some family, $\{F_\theta\}$, of distributions and try to minimize $\max_\theta EF_\theta(X_\tau)$. This is not a secretary problem but, in the special case where the F_θ's are uniform, the solution to the rank problem is relevant, because the expectations of the successive order statistics are proportional to the ranks. Thus, from the solution to the Rank Problem (26), the minimax risk is at most $\approx 3.8695/n$—for a rule based only on the relative ranks!—which is not far from the asymptotically optimal value of $(3 + 2\sqrt{2})^{1/\sqrt{2}}/n \simeq 3.4780/n$ (Samuels (1981)).

2.1.5. Full information rank problem

A rank problem which *is* a secretary problem is the one where we sample from a known continuous distribution and wish to maximize the expected rank of an item selected by a stopping rule. This problem is currently under study by Assaf and Samuel-Cahn (1991) and by Bruss and Ferguson (1991). The optimal expected rank, as $n \to \infty$ has, so far, been shown to be somewhere between 1.85 and 2.33.

3. An Infinite Model

Let the best, second best, etc., of an *infinite* sequence of rankable items arrive at times $U_1, U_2, \ldots$, which are i.i.d., uniformly distributed on the unit interval, $[0, 1]$. For each t in this interval, let $V_i(t)$ be the arrival time of the item which is i-th best among all those which arrive before time t. Let $\mathcal{F}_t$ be the sigma field generated by $V_1(t), V_2(t), \ldots$, and consider the class of all stopping rules, τ, adapted to the $\mathcal{F}_t$'s and taking values in the set $\{U_i\} \cup \{0, 1\}$; the values 0 and 1 are included to allow for the possibility of not starting or not stopping.

As in Section 2, let $A(t, j)$ be the prescribed loss for stopping at time t with an item of relative rank j; the goal is to minimize $EA(\tau, Y_\tau)$, among all stopping rules, τ, where Y_τ is the relative rank, at its arrival time, of the item which arrives at time τ. For example, as in Section 2.1, a non-decreasing loss function, $q(\cdot)$, may be given, and

$$A(t,j) = R_j(t) \equiv E[q(X_t) \mid Y_t = j] \tag{28}$$

where, in a slight abuse of notation, we are letting X_t and Y_t denote the absolute and relative ranks at time t, of an arrival at time t. This can easily be made rigorous and the result is that $R_j(t)$ is precisely the limit of $R^{(n)}(i,j)$ given in (18).

This model was first proposed in an abstract by Rubin (1966) and worked out in detail in Gianini and Samuels (1976), Gianini (1977), and Lorenzen (1979). It is appealing for a number of reasons, among them

- it is consistent with the "finite model" of the previous section, because if the ranks of the n successive arrivals are a random permutation of 1 to n, then the arrival times of the best, second best, etc., are also a random permutation of 1 to n and *vice versa;*
- it yields upper bounds for finite problem risks in an elementary way;
- it is, in several ways, the limit of the finite problem—in particular, backward induction yields the differential equation (19) directly;
- it is the natural setting for a number of important applications.

3.1. *Easy Upper Bounds with Memory-Length One Rules*

If the stopping risk is given by (28)—i.e. there is no sampling cost—then the risk using any stopping rule in the infinite problem is an upper bound for the optimal risks for all n in the corresponding (i.e. same $q(\cdot)$ function) finite problem of Section 2.1. This is because, if we augment the sigma-fields, $\mathcal{F}_t$ to include information about which arrivals by time t are among the n best overall, and modify an arbitrary infinite problem stopping rule by having it select the last arrival among the n best whenever the original rule does not select one of the n best, then the modified rule has reduced risk (because $q(\cdot)$ is non-decreasing); but it is also a *randomized* rule (hence suboptimal) for the n-arrival problem, because, regardless of their arrival times, the successive ranks of the n best arrivals are a random permutation of 1 to n.

The infinite model includes stopping rules which are much more tractable than any in the finite problem. For example, suppose we choose an infinite sequence of numbers

$$0 = R_0 < A_1 < R_1 < A_2 < \cdots < A_k < R_k < \cdots < 1 \tag{29}$$

(that's R as in *Remember* and A as in *Accept*), and stop at τ equal to the first time we have an arrival in an interval of the form $[A_i, R_i)$ which is better than the best arrival in the previous interval, $[R_{i-1}, A_i)$. Let the A's and R's be chosen so that for all $i = 0, 1, \dots,$

(30) $$(R_{i+1} - R_i) = R_1(1 - R_1)^i$$

and

(31) $$(A_{i+1} - R_i)/(R_{i+1} - R_i) \equiv p.$$

(The idea here is to make the problem recursive.) Then $P(\tau > R_i) = p^i$, so $\tau < 1$ a.s. Moreover, since the k-th best arrival in $(R_1, 1)$ has expected rank $k/(1 - R_1)$, we easily conclude that

(32) $$EX_\tau = EX_\tau I_{\{\tau < R_1\}} + P(\tau > R_1)[EX_\tau/(1 - R_1)],$$

which is finite if and only if

(33) $$P(\tau > R_1) = p < 1 - R_1.$$

Thus, for the Rank Problem of Section 2.1, the infinite model provides an easy demonstration of the finiteness of the limiting risk. Rubin and Samuels (1977) studied these memory-length one rules and showed that, for the optimal rule of this type, the expected rank is about 7.4. This same argument can be used for any polynomial loss. In particular, for losses of the form $q(i) = i(i+1)\cdots(i+m)$, the above rules can be shown to have finite risk if and only if $p < (1 - R_1)^m$.

3.2. *Infinite Problem as Limit of Finite Problem*

When the stopping loss satisfies (28), [and under mild conditions otherwise], if there is any stopping rule with finite risk, then, using backward induction, we can conclude, as in Section 2 that the quantities

(34) $$f(t) \equiv \inf_{\tau > t} E(A(\tau, Y_\tau) \mid \mathcal{F}_t)$$

are constants satisfying the differential equation (19), with boundary condition $f(1) = \sup q(\cdot)$, minimal risk, v, given by (21), and optimal rule which is the (appropriately scaled) limit of the finite problem optimal rules, namely, if we haven't stopped before time t_k, given by (20), then we stop with the first arrival in $[t_k, t_{k+1}]$, if any, which has relative rank $\leq k$.

In addition, (21) *always* holds, i.e., the infinite problem minimal risk, finite or infinite, is always the limit of the finite problem minimal risks. We have already seen, in Section 3.1, that v is finite whenever $q(\cdot)$ grows no faster than a polynomial. On the other hand, if $\sum[\log q(k)]/k^2 = \infty$, then v is infinite (Gianini (1977)).

3.3. *Easy Upper Bounds for Unknown Number of Arrivals*

Instead of prescribing a prior distribution of the number of arrivals, as in Section 2, and having them appear in discrete time, we may have the arrivals occur in continuous time as some kind of stochastic process. Suppose for example that the best, second best, etc., of N (a random variable) items arrive at times which are I.I.D. with some continuous distribution, F, on the time interval, $(0, \infty)$. A much-studied special case is the best-choice problem with a Poisson arrival process, which is equivalent to $N \sim$ Poisson and $F \sim$ uniform on $(0, T)$ for some T (Cowan and Zabczyk (1978) and Bruss (1987)).

Since, without loss of generality, the arrival times can be taken to be uniform on $(0, 1)$, the infinite model provides a unifying framework for such problems (Bruss (1984), Bruss and Samuels (1987 and 1990), and Sakaguchi (1989)). Suppose, in the infinite model, instead of observing the entire infinite collection of arrivals, we can only observe the N best, where N has distribution G. What effect does this "censoring" have on, say, an optimal rule, τ^*, for the general loss problem, (28)? Clearly censoring delays stopping, and—as long as the loss for not stopping, $Q(N)$, is no bigger than $q(N+1)$—censoring is guaranteed to reduce the risk. Thus, the particular G, be it Poisson or whatever, is nearly irrelevant: The optimal infinite problem stopping rule for the given loss function $q(\cdot)$ provides an upper bound for the optimal risk for *all* distributions of N. Moreover, it can easily be shown to be nearly optimal itself for all stochastically large N. Specifically, letting $v^{(N)}$ denote the minimal risk, we have

$$v^{(N)} \le E^{(N)} q(R_{\tau^*}) \le v \tag{35}$$

and

$$N \uparrow \infty \text{ in distribution} \Rightarrow v^{(N)} \uparrow v. \tag{36}$$

(The result applies whenever v is finite.)

For loss functions that are eventually constant, as in choosing one of the r best, a more logical loss for not stopping is $Q(N) \equiv c$ (a constant). If $c \le \max q(\cdot)$, the above results must be modified slightly; they apply only to N's for which

$$P\{q(N+1) = \max q(\cdot) | N > 0\} = 1. \tag{37}$$

This, by the way, is guaranteed in the *best choice problem*: $r = 1$.

3.4. *Best-Choice Problem with Recall*

Suppose we relax the stopping-rule requirement of no recall to allow "backward solicitation" of previously observed items. Specifically, let $\alpha \epsilon [0, 1)$ be a *recall time;* an item which arrives at time t can be held until time $t + \alpha$, when it must be either selected or discarded (Rocha (1988)). As one would expect, the optimal rule is of the form τ_t: ignore all arrivals up to time t, and thereafter select the first "candidate" which is still relatively best at the end of the recall period (or at time 1, whichever comes first). This rule selects the overall best item with probability

(38) $$\psi_\alpha(t) \equiv P[\tau_t = \min(U_1 + \alpha, 1)] \equiv P[\tau_t \text{ "picks best"}].$$

The goal is to find—for each $\alpha \in [0,1]$—the optimal t, say t_α, and the corresponding probability, $\psi_\alpha(t_\alpha) \equiv v(\alpha)$.

For $t \geq 1 - \alpha$ it is trivial that

(39) $$\psi_\alpha(t) = P[U_1 > t] = 1 - t \qquad 1 - \alpha \leq t \leq 1,$$

and the classical no-recall case, $\alpha = 0$, is also immediate:

(40) $$\psi_0(t) = P[\{U_1 > t\} \cap \{V_1(U_1) < t\}] = \int_t^1 \frac{t}{z} dz = -t \ln t.$$

(Both U and V were defined at the beginning of this section.) But, the recall case, with $t < 1 - \alpha$ is more complicated. For $0 < \delta \leq \alpha$ and $t + \delta \leq 1 - \alpha$,

(41) $$\{\tau_t \text{ "picks best" but } \tau_{t+\delta} \text{ "doesn't"}\} = \{t < U_1 \leq t + \delta\}$$

while

(42) $$\begin{gathered} \{t < V_1(t+\alpha) < t+\delta\} \cap \{\tau_{t+\alpha} \text{ "picks best"}\} \\ \supset \{\tau_{t+\delta} \text{ "picks best" but } \tau_t \text{ "doesn't"}\} \supset \\ \{t < V_1(t+\alpha+\delta) < t+\delta\} \cap \{\tau_{t+\alpha+\delta} \text{ "picks best"}\}. \end{gathered}$$

This leads to the useful inequalities,

(43) $$\delta\left[1 - \frac{\psi_\alpha(t+\alpha)}{t+\alpha}\right] < \psi_\alpha(t) - \psi_\alpha(t+\delta) < \delta\left[1 - \frac{\psi_\alpha(t+\alpha+\delta)}{t+\alpha+\delta}\right].$$

which tell us that $\psi_\alpha(t)/t$ is decreasing; and that $\psi_\alpha(\cdot)$ is unimodal, with its maximum at

(44) $$t_\alpha \equiv t_\alpha^* - \alpha$$

where

(45) $$t_\alpha^* = \begin{cases} \alpha & \text{if } \alpha \geq \frac{1}{2} \\ \psi_\alpha(t_\alpha^*) & \text{if } \alpha \leq \frac{1}{2} \end{cases}$$

and satisfies the differential equation

(46) $$\psi_\alpha'(t) = \frac{\psi_\alpha(t+\alpha)}{t+\alpha} - 1 \qquad 0 < t < 1 - \alpha.$$

It's easy to solve the differential equation on $[(1-2\alpha)^+, 1-\alpha]$ by substituting the known value $\psi_\alpha(t+\alpha) = 1 - t - \alpha$, from (39), into the right side of (46). The solution is

$$\psi_\alpha(t) = 2 - 2t - \alpha + \ln(t+\alpha) \qquad (1-2\alpha)^+ \leq t \leq 1-\alpha. \tag{47}$$

Thus the complete solution for $\alpha \geq \frac{1}{2}$ is

$$\left.\begin{aligned} t_\alpha &= 0 \\ v(\alpha) &= 2 - \alpha + \ln\alpha \end{aligned}\right\} \quad \alpha \geq \tfrac{1}{2}. \tag{48}$$

And, for $\psi_\alpha(1-2\alpha) \geq (1-2\alpha)$—which holds for $\alpha \geq .260303$—we can solve for t_α^* from (47). But then the trouble starts. For $t < 1-2\alpha$ we face integrals of the form

$$\int \frac{\ln(t+2\alpha)}{t+\alpha} dt \tag{49}$$

which cannot be expressed in closed form. An attractive alternative to numerical integration (which works quite well) is to obtain, probabilistically, improved upper and lower bounds for $\psi(\cdot)$—better than those in (43)—and use them to get upper and lower bounds for t_α^* and for $\psi_\alpha(t_\alpha^*)$.

Equations (48) are the limiting solutions for the corresponding finite problem studied by Smith and Deely (1975).

4. Multiple Criteria

4.1. *Best Choice Problem with Independent Criteria*

Suppose each item's relative ranks with respect to m independent criteria are to be observed and we want to maximize the probability of choosing an item which is best in an at least one criterion. The finite problem was studied by Gnedin (1982). Asymptotically, an m-dimensional infinite model applies, leading to the differential equation

$$f'(t) = \frac{m}{t}[f(t) - (1-t)]^+ \qquad 0 \leq t < 1 \tag{50}$$

where $f(t)$ is, as usual, the minimal risk for rules which do not stop before t and the boundary condition is $f(1) = 1$. The solution for $m > 1$ is

$$\begin{aligned} f(t) &= 1 - \frac{m}{m-1}t(1-t^{m-1}) \qquad t > t^* \\ 1 - f(t^*) &= t^* = (\frac{1}{m})^{1/(m-1)}. \end{aligned} \tag{51}$$

In particular, for $m = 2$, an asymptotically optimal rule lets half of the items go by before being willing to stop, and has probability $1/2$ of success.

4.2. *Best Rank Problem with Independent Criteria*

Govindarajulu (1991) studied a version of the secretary problem in which there are two independent streams of candidates. At stage i, the relative ranks of the i-th item—relative to its predecessors in its own stream—are observed. Only one item, from one stream or the other, will be selected, and the goal is to minimize its rank in its own stream.

If selection is made after seeing both items, this problem is equivalent to a two independent criteria problem; if not, they are still asymptotically equivalent.

For m criteria, asymptotically, the same m-dimensional model as in Section 4.1 applies, leading to the piecewise differential equation

$$f'(t) = \frac{mk}{t}f(t) - \frac{m}{2}\frac{k(k+1)}{t^2} \qquad t_k \leq t \leq t_{k+1} \tag{52}$$

with boundary condition $f(1) = \infty$, where the thresholds, t_k, are increasing and satisfy $f(t_k) = k/t_k$. Dividing both sides of (52) by t^{mk} leads to the solution

$$f(t_1) = 1/t_1 = \prod_{k=1}^{\infty}\left(1 + \frac{2(mk+1)}{k(mk+2-m)}\right)^{1/(mk+1)} . \tag{53}$$

	Best Choice	Best Rank	
m	t^*	t_1	$f(t_1)$
1	.3679	.2584	3.8695
2	.5000	.3846	2.6003
3	.5774	.4670	2.1413
4	.6300	.5267	1.8987
5	.6687	.5724	1.7469
10	.7743	.7040	1.4205
20	.8541	.8088	1.2363
50	.9233	.9011	1.1097
100	.9545	.9425	1.0610

Table 1. Asymptotic Results for m-Criteria Problems.

4.3. *Sum of the Ranks Problem with Independent Criteria*

Suppose, as before, that we observe the relative ranks with respect to m independent criteria, but now want to minimize the sum of the ranks. This is a multivariate extension of the rank problem in Section 2.1; however, for $m \geq 2$, we cannot expect the risk to remain bounded as the number of items becomes infinite. Samuels and Chotlos (1986) give both the optimal stopping and extreme value results for m independent permutations,

$$\langle X_1^{(1)}, \ldots, X_n^{(1)}\rangle, \; \cdots \; , \langle X_1^{(m)}, \ldots, X_n^{(m)}\rangle, \tag{54}$$

of the integers 1 to n. As $n \to \infty$,

$$\frac{1}{n^{1-1/m}} E\left[\min_{1\le i\le n} \sum_{j=1}^{m} X_i^{(j)}\right] \to (m!)^{1/m}\Gamma(1+1/m) \tag{55}$$

and

$$\frac{1}{n^{1-1/m}} E\left[\min_{\tau\le n} \sum_{j=1}^{m} X_\tau^{(j)}\right] \to \left[\frac{(m+1)!}{m}\right]^{1/m}. \tag{56}$$

The ratio of these two limits—the right side of (56) divided by the right side of (55)—decreases to one as $m \uparrow \infty$.

4.3.1. Sums of i.i.d uniforms

It is worth noting that (55) and (56) also hold if all of the $X_i^{(j)}$'s are i.i.d., uniform on $(0,1)$. In fact the solution to the above secretary problem was obtained in Samuels and Chotlos (1986) by showing that it can be approximated by this non-secretary problem. The approximation works for $m > 1$ but not for $m = 1$. Lindley (1961), in effect, tried it for $m = 1$ and got an asymptotic risk of 2, which he noted was incorrect. The correct limit is 3.8695, as in (26).

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