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HÖLDER CONTINUITY AND L_p ESTIMATES FOR ELLIPTIC EQUATIONS UNDER GENERAL HÖRMANDER'S CONDITION

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Dedicated to Olga Ladyzhenskaya

Solutions of the Dirichlet problem for elliptic equations satisfying general Hörmander's condition are considered. It is proved that the C^α norm of solutions can be estimated through the L_p norm of right-hand sides.

1. Introduction

In a smooth bounded domain $D \subset \mathbb{R}^d$ we consider the operator

$$L_0 u(x) := \tfrac{1}{2}\sigma^{ik}(x)\big(\sigma^{jk}(x)u_{x^j}(x)\big)_{x^i} + b^i(x)u_{x^i}(x),$$

where $\sigma^k = (\sigma^{ik})$, $k = 1,\dots,d_1$, and $b = (b^i)$ are smooth (of class C^∞) vector fields given on $\mathbb{R}^d$ and d_1 is an integer. We assume that the Lie algebra generated by the family $\{b, \sigma^k : k = 1,\dots,d_1\}$ of vector fields has dimension d at all points in the closure $\overline{D}_0$ of a neighborhood D_0 of $\overline{D}$. Our main goal is to prove that for solutions of the problem $L_0 u - u = f$ in D with zero boundary data one can estimate the C^α norm in any subdomain through the L_p norm of f, where $\alpha \in (0,1)$ and $p \in (1,\infty)$ are independent of f.

We recall the classical result by Hörmander [2] which says that if $f \in C^\infty_{\mathrm{loc}}(D)$, then $u \in C^\infty_{\mathrm{loc}}(D)$. However, in some applications (see, for instance, [3]) one has

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to deal with the right-hand sides f which are only measurable and bounded and estimate at least the maximum of solutions in terms of L_p norms of f.

Of course, the solution can be written as

$$u(x) = -\int_D g(x,y)f(y)\,dy,$$

where g is the Green's function of the problem. Therefore the results needed can be obtained by referring to a very detailed information on g available in the literature (see, for instance [1]). However, this way of getting our main result may create a false impression that to understand it one needs to learn some quite sophisticated and advanced theories. In addition, usually only the case $d \geq 3$ is considered. Therefore, even for the case of one variable and the operator $L_0 u(x) = u'(x)$ some additional work needs to be done (like, say, adding dummy variables). Therefore here we present short proofs only based on old and well-known results and methods.

The probabilistic counterparts of our results may be found in [3].

2. The main result

Fix an $\varepsilon \in (0,1)$ and define $L = L_0 + \varepsilon\Delta$, where Δ is the Laplace operator in $\mathbb{R}^d$. One knows that for any $p \in (1,\infty)$ and $f \in L_p(D)$ there exists a unique solution $u =: Rf \in W^2_p(D)$ of the equation $Lu - u = f$ in D with zero boundary condition.

THEOREM 2.1. *There exist a (large) $p_0 \in (1,\infty)$ and a (small) $\alpha \in (0,1)$ both independent of ε and such that for any $p \geq p_0$, subdomain $D_1 \subset \overline{D}_1 \subset D$, and $f \in L_p(D)$ we have*

$$\sup_D |Rf| \leq N\|f\|_{L_p(D)}, \tag{2.1}$$

$$|Rf(x) - Rf(y)| \leq N|x-y|^\alpha \|f\|_{L_p(D)} \quad \forall x,y \in D_1, \tag{2.2}$$

where the constants N are independent of x, y, f, and ε.

By letting $\varepsilon \downarrow 0$ along a subsequence, this theorem allows one to define a generalized solution of the equation $Lu - u = f$ in D with zero boundary data. Observe that this solution satisfies the equation in the sense of distributions, is a locally Hölder continuous function in D but in general need not be continuous up to the boundary.

To prove Theorem 2.1 we need two lemmas the first of which is proved in Sec. 3 and the second one in Sec. 4. For a smooth domain $G \subset \mathbb{R}^d$ and $\lambda > 0$, we denote by $R_\lambda(G)f$ the solution of $\lambda u - Lu = f$ in G with zero boundary condition. If $G = D$, we write $R_\lambda f$ instead of $R_\lambda(D)f$ and if $\lambda = 1$, we drop the subscript λ.

Lemma 2.1. *There exist* $\alpha \in (0,1)$, $\lambda \geq 1$, *and* $n \geq \alpha$ *such that for any* $r \in (0,1)$, $x \in D$ *with* $\operatorname{dist}(x, \partial D) \geq 2r$, *and* $f \in L_2(D)$ *vanishing in the ball* $B_{2r}(x)$ *of radius* $2r$ *centered at* x *we have*

$$|R_\lambda f(x)| \leq N r^{-n} \|f\|_{L_2(D)}, \tag{2.3}$$

$$|R_\lambda f(z) - R_\lambda f(y)| \leq N r^{-n} |z-y|^\alpha \|f\|_{L_2(D)} \quad \forall z, y \in B_r(x), \tag{2.4}$$

where N *is independent of* r, x, y, z, *and* f. *Furthermore* N, α, λ, *and* n *are independent of* ε.

Lemma 2.2. *There exists a constant* N (*independent of* ε) *such that for any ball* $B \subset D$ *we have*

$$\sup_D |R(D_0) I_B| \leq N |B|^{1/(3d)}. \tag{2.5}$$

Proof of Theorem 2.1. By Hölder's inequality if (2.1) and (2.2) hold for a p, they also hold for any $p_1 \geq p$. Therefore we only need to prove (2.1) and (2.2) for a $p \geq 1$. Take $\lambda \geq 1$ from Lemma 2.1 and notice that $Rf = R_\lambda f + (\lambda - 1) R R_\lambda f$ and $R1 \leq 1$. Therefore since $\lambda \geq 1$, we have

$$\sup_D |Rf| \leq \lambda \sup_D |R_\lambda f|$$

and to prove (2.1) it suffices to prove that

$$\sup_D |R_\lambda f| \leq N \|f\|_{L_p(D)}. \tag{2.6}$$

First we prove (2.6) for f being indicator functions. Take a Borel set $\Gamma \subset D$. We use (2.3) with D_0 in place of D, (2.5), and the fact that $R_\lambda f \leq Rf$ for $f \geq 0$ by the maximum principle. Then for any $r \leq \delta_0 := \operatorname{dist}(\partial D, \partial D_0)$ and $x \in D$ we have

$$\begin{aligned} R_\lambda I_\Gamma(x) &\leq R_\lambda(D_0) I_{\Gamma \setminus B_r(x)}(x) + R(D_0) I_{\Gamma \cap B_r(x)}(x) \\ &\leq N r^{-n} |\Gamma \setminus B_r(x)|^{1/2} + R(D_0) I_{B_r(x)}(x) \\ &\leq N r^{-n} |\Gamma|^{1/2} + N r^{1/3}. \end{aligned}$$

Upon minimizing the last expression with respect to $r \leq \delta_0$ we get $R_\lambda I_\Gamma \leq N |\Gamma|^\theta$ with $\theta = (6n+2)^{-1}$.

Now for $p = \theta^{-1} + 1$ and $F := \|f\|_{L_p(D)}$ we have

$$\begin{aligned} R_\lambda f &= \int_0^\infty R_\lambda I_{\{f > c\}} \, dc \leq N \int_0^\infty |\{f > c\}|^\theta \, dc \\ &\leq N \int_0^F dc + N \|f\|_{L_p(D)}^{\theta p} \int_F^\infty \frac{1}{c^{\theta p}} \, dc = N \|f\|_{L_p(D)} \end{aligned}$$

as asserted in (2.6). This proves (2.1).

Take p from (2.1) and α, λ, and n from (2.4). We prove (2.2) with $2p$ in place of p and with $\beta := \alpha d/(d+2pn)$ in place of α. To do so we first notice that $R = R_\lambda + (\lambda - 1)R_\lambda R$. Hence it suffices to prove that for any $\delta \in (0,1)$ there exists N such that

$$|R_\lambda f(x) - R_\lambda f(y)| \le N|x-y|^\beta \|f\|_{L_{2p}(D)} \tag{2.7}$$

for all $x, y \in D$ for which the distances of x, y to ∂D are greater than δ. In addition, by virtue of (2.1), one only needs to consider x, y which are close to each other, say such that $|x-y| \le (\delta/2)^{(d+2pn)/(2p\alpha)}$. Take such x, y and let $r = |x-y|^{2p\alpha/(d+2pn)}$. Then $2r \le \delta$, $|x-y| \le r$ (since $\alpha \le n$), and by (2.1) and (2.4) we have

$$\begin{aligned}
|R_\lambda f(x) - R_\lambda f(y)| &\le |R_\lambda f I_{B_{2r}(x)}(x) - R_\lambda f I_{B_{2r}(x)}(y)| \\
&\quad + |R_\lambda f I_{B^c_{2r}(x)}(x) - R_\lambda f I_{B^c_{2r}(x)}(y)| \\
&\le N\|f I_{B_{2r}(x)}\|_{L_p(D)} + N r^{-n}|x-y|^\alpha \|f\|_{L_2(D)} \\
&\le N(r^{d/(2p)} + r^{-n}|x-y|^\alpha)\|f\|_{L_{2p}(D)}.
\end{aligned}$$

The last expression is equal to the right-hand side of (2.7) due to our choice of r. The theorem is proved.

3. The proof of Lemma 2.1

The following lemma recalls a well-known result from [4] or [2]. More precisely, it is part of the assertions of Lemma 22.2.4 of [2].

LEMMA 3.1. *There exists $\beta \in (0,1)$ such that for any real s there exists a constant N such that for any C^∞-function v with support in D, we have*

$$\|v\|_{s+2\beta} + \sum_k \|v_{(\sigma^k)}\|_{s+\beta} \le N(\|Lv\|_s + \|v\|_s), \tag{3.1}$$

where $u_{(\xi)} := u_x \cdot \xi$ and $\|f\|_r$ is the norm of f in the Hilbert space H^r_2 (put otherwise, $\|f\|_r = \|(1+|\xi|)^r \tilde f\|_{L_2}$ where $\tilde f$ is the Fourier transform of f). Furthermore, β and N are independent of ε.

By virtue of Sobolev's embedding theorems Lemma 2.1 is a consequence of the following result.

LEMMA 3.2. *There exist $\lambda \ge 1$ and $n \ge 1$ such that for any $r \in (0,1)$, $x \in D$ with $\operatorname{dist}(x, \partial D) \ge 2r$, and $f \in L_2(D)$ vanishing in the ball $B_{2r}(x)$ of radius $2r$ centered at x we have*

$$\|R_\lambda f\|_{d+2\beta, B_r(x)} \le N r^{-n} \|f\|_{L_2(D)}, \tag{3.2}$$

where $\|\cdot\|_{r,B}$ is the norm in the Sobolev space $H_2^r(B)$ and N is independent of r, x, and f. Furthermore, λ, n, and N are independent of ε.

Proof. For brevity let us write

$$\|\cdot\|_r := \|\cdot\|_{r,D}.$$

Observe that we only need to prove (3.2) for smooth f.

By noticing that $\|Lv\| \le \|Lv - \lambda v\| + \lambda\|v\|$ and by iterating (3.1) one finds that

$$\|v\|_{d+2\beta} + \sum_k \|v_{(\sigma^k)}\|_{d+\beta} \le N(\|Lv - \lambda v\|_d + \|v\|_0). \tag{3.3}$$

We now use a standard procedure to get the interior estimate (3.2) from (3.3) for the case when $v = u := R_\lambda f$ so that $Lv - \lambda v = 0$ in $B_{2r}(x)$.

Without loss of generality assume $x = 0$ and fix an $r > 0$ such that $B_{2r} := B_{2r}(0) \subset D$. Define $r_m = r\sum_{i=0}^m 2^{-i}$. We need some functions $\zeta_m \in C_0^\infty(\mathbb{R}^d)$ such that $\zeta_m(x) = 1$ in B_{r_m}, $\zeta_m(x) = 0$ outside $B_{r_{m+1}}$ and

$$\max_{|\alpha|\le d+2,x} |D^\alpha \zeta_m| \le Nr^{-(d+2)}\theta^{-m}, \tag{3.4}$$

where $\theta = 2^{-(d+2)} < 1$ and N depends only on d. To construct them take an infinitely differentiable function $h(t)$, $t \in (-\infty, \infty)$, such that $h(t) = 1$ for $t \le 1$, $h(t) = 0$ for $t \ge 2$ and $0 \le h \le 1$. Next, define

$$\zeta_m(x) = h(2^{m+1}(|x| - r_m + r2^{-(m+1)})/r).$$

Now we put $u\zeta_m$ in (3.3), remember that $Lu - \lambda u = 0$ in $B_{2r} \subset D$, and we get

$$\begin{aligned} \|u\zeta_m\|_{d+2\beta} + \sum_k \|u\zeta_{m(\sigma^k)} + \zeta_m u_{(\sigma^k)}\|_{d+\beta} \\ \le N\Big(\Big\|\sum_k u_{(\sigma^k)}\zeta_{m(\sigma^k)} + uL\zeta_m\Big\|_d + \|u\zeta_m\|_0\Big). \end{aligned} \tag{3.5}$$

Here

$$\begin{gathered} \|uL\zeta_m\|_d = \|u\zeta_{m+1}L\zeta_m\|_d \le Nr^{-(d+2)}\theta^{-m}\|u\zeta_{m+1}\|_d, \\ \|u_{(\sigma^k)}\zeta_{m(\sigma^k)}\|_d = \|\zeta_{m+1}u_{(\sigma^k)}\zeta_{m(\sigma^k)}\|_d \le Nr^{-(d+2)}\theta^{-m}\|\zeta_{m+1}u_{(\sigma^k)}\|_d, \\ \|u\zeta_{m(\sigma^k)} + \zeta_m u_{(\sigma^k)}\|_{d+\beta} \ge \|\zeta_m u_{(\sigma^k)}\|_{d+\beta} - Nr^{-(d+2)}\theta^{-m}\|u\zeta_{m+1}\|_{d+\beta}. \end{gathered}$$

Hence (3.5) implies that

$$\begin{aligned} &\|u\zeta_m\|_{d+2\beta} + \sum_k \|\zeta_m u_{(\sigma^k)}\|_{d+\beta} \\ &\qquad \le N\|u\zeta_m\|_0 + Nr^{-(d+2)}\theta^{-m}\Big(\|u\zeta_{m+1}\|_{d+\beta} + \sum_k \|\zeta_{m+1}u_{(\sigma^k)}\|_d\Big). \end{aligned}$$

Next, we use the interpolation inequality $\|v\|_k \le \gamma^{l-k}\|v\|_l + \gamma^{p-k}\|v\|_p$ for any $\gamma > 0$ if k is between l and p (which immediately follows from the inequality $a^{2k} \le a^{2l} + a^{2p}$). Then for $\gamma \in (0,1)$,

$$Nr^{-(d+2)}\theta^{-m}\|\zeta_{m+1}u_{(\sigma^k)}\|_d \le \gamma\|\zeta_{m+1}u_{(\sigma^k)}\|_{d+\beta} + N\gamma^{-d/\beta}r^{-n}\theta_1^{-m}\|u_{(\sigma^k)}\|_0,$$
$$Nr^{-(d+2)}\theta^{-m}\|u\zeta_{m+1}\|_{d+\beta} \le \gamma\|u\zeta_{m+1}\|_{d+2\beta} + N\gamma^{-2d/\beta}r^{-n}\theta_1^{-m}\|u\|_0,$$

where $n = (d+2)(2+d/\beta)$ and $\theta_1 = \theta^{2+d/\beta}$. By letting $\gamma = \theta_1/2$, we find that

$$\|u\zeta_m\|_{d+2\beta} + \sum_k \|\zeta_m u_{(\sigma^k)}\|_{d+\beta} \le \gamma\Big(\|u\zeta_{m+1}\|_{d+2\beta} + \sum_k \|\zeta_{m+1}u_{(\sigma^k)}\|_{d+\beta}\Big) + Nr^{-n}\theta_1^{-m}\Big(\sum_k \|u_{(\sigma^k)}\|_0 + \|u\|_0\Big).$$

We multiply both sides of the last inequality by γ^m, sum up for $m = 0,1,2,\dots$, and observe that $\gamma^m\theta_1^{-m} = (1/2)^m$ and that

$$S := \sum_{m=1}^{\infty} \gamma^m\Big(\|u\zeta_m\|_{d+2\beta} + \sum_k \|\zeta_m u_{(\sigma^k)}\|_{d+\beta}\Big) < \infty$$

by virtue of (3.4) and the fact that $u \in C^{d+10}(D)$. Then we get

$$\|u\zeta_0\|_{d+2\beta} + \sum_k \|\zeta_0 u_{(\sigma^k)}\|_{d+\beta} + S \le S + Nr^{-n}\Big(\sum_k \|u_{(\sigma^k)}\|_0 + \|u\|_0\Big),$$
$$\|u\|_{d+2\beta,B_r(x)} \le N\|u\zeta_0\|_{d+2\beta} \le Nr^{-n}\Big(\sum_k \|u_{(\sigma^k)}\|_0 + \|u\|_0\Big),$$

where N are independent of u and r. Now to get (3.2) it suffices to prove that

$$\sum_k \|u_{(\sigma^k)}\|_0 + \|u\|_0 \le N\|f\|_0, \tag{3.6}$$

where $f = \lambda u - Lu$. By the way, observe that until this point we did not use the fact that we can choose λ as large as we like. By multiplying $f = \lambda u - Lu$ by u and integrating by parts, it is proved in [4] that we indeed have (3.6) for λ large enough. The lemma is proved.

4. The proof of Lemma 2.2

We need one more lemma in which Hörmander's condition is not used. Recall that $B_r = \{x : |x| < r\}$ and define

$$a^{ij} = \tfrac{1}{2}\sigma^{ik}\sigma^{jk} + \varepsilon\delta^{ij}, \quad \widetilde{b} = b + \tfrac{1}{2}\sigma^k_{x^j}\sigma^{jk}.$$

Lemma 4.1. *Let $M_b \geq m_b > 0$, $M_a \geq m_a > 0$, and $\varrho > r > 0$ be some constants. Let $T_r(x) = R(B_\varrho)I_{C_r}(x)$, where $C_r = \{x : |x^1| < r\}$. Then*

$$a^{11}(x)I_{(r,\varrho)}(x^1) \leq M_a, \quad \widetilde{b}^1(x)I_{(-r,\varrho)}(x^1) \geq m_b I_{(-r,\varrho)}(x^1) \quad \forall x \in B_\varrho \\ \Rightarrow \sup_{B_\varrho} T_r \leq \frac{M_a + 2rm_b}{m_b^2}, \tag{4.1}$$

$$a^{11}(x)I_{(-r,r)}(x^1) \geq m_a I_{(-r,r)}(x^1), \quad \widetilde{b}^1(x)I_{(-r,\varrho)}(x^1) \geq 0 \quad \forall x \in B_\varrho \\ \Rightarrow \sup_{B_\varrho} T_r \leq \frac{2r\varrho}{m_a}, \tag{4.2}$$

$$a^{11}(x) \geq m_a, \quad |\widetilde{b}^1(x)| \leq M_b \quad \forall x \in B_\varrho \quad \Rightarrow \quad \sup_{B_\varrho} T_r \leq \frac{r\varrho}{m_a} e^{M_b\varrho/m_a}. \tag{4.3}$$

Proof. We start with proving (4.1). This estimate does not look good because the right-hand side does not go to zero as $r \downarrow 0$. However, the most surprising fact is that in the class of operators satisfying the conditions in (4.1) the estimate is sharp. To prove (4.1) define

$$\delta = \varrho - r, \quad \lambda = \frac{m_b}{M_a}, \quad C_1 = \frac{1}{\lambda m_b}(1 - e^{-\lambda\delta}), \quad C_2 = \frac{1}{\lambda m_b} e^{\lambda r},$$

and define a function on $[-\varrho, \varrho]$ by

$$u(t) = \begin{cases} 2rm_b^{-1} + C_1 & \text{for } t \in [-\varrho, -r], \\ (r - t)m_b^{-1} + C_1 & \text{for } t \in [-r, r], \\ C_2(e^{-\lambda t} - e^{-\lambda\varrho}) & \text{for } t \in [r, \varrho]. \end{cases}$$

It is easy to check that $u \geq 0$, u is continuous and piecewise twice continuously differentiable on $[-\varrho, \varrho]$, u' has a discontinuity only at $t = -r$, $u' \leq 0$ for $t \neq -r$, $u'' = 0$ on $(-\varrho, r)$ apart from $t = -r$, and $u'' \geq 0$ on (r, ϱ).

These properties of derivatives of u and its explicit representation yield that

$$au''(t) + bu'(t) + I_{(-r,r)}(t) \leq 0 \tag{4.4}$$

if

- $t \in (-\varrho, -r)$ and a, b are any numbers, or
- $t \in (-r, r)$, a is any number, and $b \geq m_b$, or
- $t \in (r, \varrho)$ and a, b are such that $0 \leq a \leq M_a, b \geq m_b$.

However, the graph of u has a corner at $t = -r$. For any $\beta > 0$, one can change $u(t)$ for $t < -r$ to get a new function u_β so that $u_\beta(-r) = u(-r)$, $u'_\beta(-r) = u'(-r+)$, $u''_\beta(-r) = u''(-r+)$, u_β is smooth, decreasing, and concave on $(-\varrho, -r]$ and $u_\beta(-\varrho) \leq u(-r) + \beta$. For $t \geq -r$ we define $u_\beta(t) = u(t)$ and $v_\beta(x) = u_\beta(x^1)$ and we get

$$Lv_\beta(x) - v_\beta(x) + I_{C_r}(x) \\ = a^{11}(x)u''_\beta(x^1) + \widetilde{b}^1(x)u'_\beta(x^1) - u_\beta(x^1) + I_{(-r,r)}(x^1) \leq 0$$

almost everywhere in B_ϱ. By the maximum principle

$$T_r(x) \le v_\beta(x) = u_\beta(x^1) \le u(-r) + \beta = 2rm_b^{-1} + C_1 + \beta.$$

We let here $\beta \downarrow 0$ and immediately get (4.1).

We now pass to (4.2). Let

$$u(t) = \begin{cases} 2r\varrho m_a^{-1} & \text{for } t \in [-\varrho, -r], \\ (1/2)[4r\varrho - (t+r)^2]m_a^{-1} & \text{for } t \in [-r, r], \\ 2rm_a^{-1}(\varrho - t) & \text{for } t \in [r, \varrho]. \end{cases}$$

This function has properties similar to the previous one. In particular, it is nonnegative, decreasing, and (4.4) holds for $t \in (-\varrho, -r)$ with any a and b, for $t \in (-r, r)$ if $a \ge m_a$ and $b \ge 0$, and for $t \in (r, \varrho)$ with any $b \ge 0$ and any a. As above by the maximum principle we conclude that $T_r(x) \le u(x^1) \le u(-r)$, and this is (4.2).

To prove (4.3) we define

$$u(t) = \begin{cases} (\lambda M_b)^{-1}(1 - e^{-\lambda r})(e^{\lambda\varrho} - e^{\lambda|t|}) & \text{for } r \le |t| \le \varrho, \\ (\lambda M_b)^{-1}(1 + \lambda|t| - e^{\lambda|t|}) + C & \text{for } t \in [-r, r], \end{cases}$$

where

$$\lambda = \frac{M_b}{m_a}, \quad C = \frac{e^{\lambda\varrho}}{\lambda M_b}(1 - e^{-\lambda r} - \lambda r e^{-\lambda\varrho}).$$

This function decreases on $[0, \varrho]$, has negative second order derivative on $(-\varrho, \varrho)$ apart from $t = \pm r$ and satisfies (4.4) whenever $a \ge m_a$ and $|b| \le M_b$. Hence as above $T_r(x) \le u(0)$ and one gets (4.3) after observing that

$$u(0) = C \le \frac{e^{\lambda\varrho}}{\lambda M_b}(\lambda r - \lambda r e^{-\lambda\varrho}) \le \frac{e^{\lambda\varrho}}{\lambda M_b}\lambda^2 r\varrho.$$

The lemma is proved.

PROOF OF LEMMA 2.2. By the maximum principle we have $R(G)I_B \le 1$. Therefore we only need to prove (2.5) for sufficiently small balls.

From Hörmander's condition, it follows that the continuous function $|\widetilde{b}| + \sum_k |\sigma^k|$ is strictly positive in $\overline{D}_0$. It follows easily that there exist constants $m > 0$ and $\varrho_0 \in (0, 1)$ such that for any point $x \in D$ there exists a unit vector η such that

$$\text{either} \quad \widetilde{b} \cdot \eta \ge m \quad \text{in } B_{\varrho_0}(x), \quad \text{or} \quad \sum_{k=1}^{d_1} |\sigma^k \cdot \eta|^2 \ge m \quad \text{in } B_{\varrho_0}(x).$$

We will prove (2.5) for balls $B_r(x)$ with $r \le \varrho_0^3/8 \le 1/8$. Take such a ball and without loss of generality assume that $x = 0$ and that the corresponding vector η is the first coordinate vector. Then for $\varrho = r^{1/3}$ we have

$$\text{either} \quad \widetilde{b}^1 \ge m \quad \text{in } B_\varrho, \quad \text{or} \quad a^{11} \ge m \quad \text{in } B_\varrho. \tag{4.5}$$

Next, for a unit vector l and $v(x) := \exp(-\lambda x \cdot l)$ with $\lambda > 0$ small enough, we have $Lv - v \le 0$ in D_0. Hence by the maximum principle,

$$R(D_0)I_{B_r}(x) \le e^{-\lambda(x\cdot l - r)} \sup_{D_0} R(D_0)I_{B_r},$$
$$R(D_0)I_{B_r}(x) \le e^{-\lambda(|x| - r)} \sup_{D_0} R(D_0)I_{B_r},$$
$$R(D_0)I_{B_r}(x) \le e^{-\lambda(|x| - r)} \sup_{B_r} R(D_0)I_{B_r}.$$

In particular, $R(D_0)I_{B_r}$ attains its maximum on $\overline{B}_r$ (which is obvious from the maximum principle). Also observe that for $x \in B_r$,

$$R(D_0)I_{B_r}(x) = R(B_\varrho)I_{B_r}(x) + u(x),$$

where u is a unique solution of $Lu - u = 0$ in B_ϱ with $u = R(D_0)I_{B_r}$ on ∂B_ϱ. Hence,

$$u \le \max_{\partial B_\varrho} R(D_0)I_{B_r} \le e^{-\lambda(\varrho - r)} \sup_{B_r} R(D_0)I_{B_r},$$
$$\sup_{B_r} R(D_0)I_{B_r} \le \sup_{B_r} R(B_\varrho)I_{B_r} + e^{-\lambda(\varrho - r)} \sup_{B_r} R(D_0)I_{B_r},$$
$$\begin{aligned}\sup_{D_0} R(D_0)I_{B_r} &= \sup_{B_r} R(D_0)I_{B_r} \le (1 - e^{-\lambda(\varrho - r)})^{-1} \sup_{B_r} R(B_\varrho)I_{B_r} \\ &\le e^{\lambda(\varrho - r)} \frac{1}{\lambda(\varrho - r)} \sup_{B_r} R(B_\varrho)I_{B_r} \le \frac{4e^\lambda}{3\lambda} r^{-1/3} \sup_{B_r} R(B_\varrho)I_{B_r},\end{aligned}$$

where we use $\varrho - r = r^{1/3} - r \ge 3r^{1/3}/4$ which is true due to the inequality $r \le 1/8$.

Therefore, to prove (2.5) it suffices to prove that given (4.5), we have

$$R(B_\varrho)I_{B_r} \le Nr^{2/3}, \tag{4.6}$$

where N is independent of r and $r \le 1$. We break the proof of (4.6) into three cases.

CASE 1. Assume the second inequality in (4.5) holds in B_ϱ. Then by (4.3) we get (4.6) with $r^{4/3}$ in place of $r^{2/3}$.

CASE 2. Assume the first inequality in (4.5) holds and $a^{11}(0) \ge N_1 r^{2/3}$, where N_1 is a constant to be specified later. Since σ^k are smooth functions, and $2a^{11}(0) = \sum_k |\sigma^{1k}(0)|^2 \ge N_1 r^{2/3}$, we have

$$\begin{aligned}4a^{11}(x) = 2\sum_k |\sigma^{1k}(x)|^2 &\ge \sum_k |\sigma^{1k}(0)|^2 - 2\sum_k |\sigma^{1k}(x) - \sigma^{1k}(0)|^2 \\ &\ge N_1 r^{2/3} - N\varrho^2 = (N_1 - N_2) r^{2/3}\end{aligned} \tag{4.7}$$

in B_ϱ, where the constant N_2 depends only on d, d_1, and uniform estimates of the first derivatives of σ^k. We take $N_1 = N_2 + 1$ and from (4.2) we see that the left hand side of (4.6) is less than $Nr\varrho r^{-2/3} = Nr^{2/3}$.

CASE 3. Assume the first inequality in (4.5) holds and $a^{11}(0) \leq N_1 r^{2/3}$. Then similarly to (4.7), $a^{11}(x) \leq N r^{2/3}$ in B_ϱ. In this case from (4.1) we see that the left-hand side of (4.6) is less than

$$N(r^{2/3} + r) \leq N r^{2/3}.$$

The lemma is proved.

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