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# THE EXISTENCE OF EVOLUTION OF CLOSED TYPE

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*Dedicated to Olga Ladyzhenskaya*

The principal concern of the paper is the existence of an admissible solution of the first initial boundary value problem for fully nonlinear second-order differential equations. We consider equations nonlinear in the time derivative as well as in the space derivatives up to the second order.

## 1. The evolving functions

The notion of evolution of closed type was introduced by the author in [6] in the course of investigation of fully nonlinear second-order parabolic equations. The principal differential operator in these equations was described in terms of an evolving nonlinear function $G = G(s, S)$, $(s, S) \in D_0 \subset \mathbb{R}^1 \times \mathrm{Sym}(n)$, where $\mathrm{Sym}(n)$ is the set of symmetric $n \times n$ matrices. *Evolution of closed type* relates to functions $G$ independent of the scalar argument $s$, i.e., $G = G(S)$, $S \in D_0 \subset \mathrm{Sym}(n)$.

Denote by $D_1$ the set of positive monotonicity of $G$:

$$D_1 = \{S \in D_0 : G(S + \eta) \geq G(S) \text{ for all } \eta \in \mathrm{Sym}(n),\ \eta \geq \mathbf{0}\},$$

by $D_2$ the set of concavity of $G$, and finally by $D$ a connected component of $D_1 \cap D_2$.

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We always assume $D$ to be a convex cone with vertex $\mathbf{0}$ and with $I \in D$, $-I \notin D$, and relate to the pair $(G, D)$ the numbers

$$\underline{g} = \sup_{S\in D} \overline{\lim_{S\to\partial D}} G(S), \quad \overline{g} = \inf_{S\in D} \underline{\lim_{R\to\infty}} G(RS).$$

Either of $\underline{g}$, $\overline{g}$ can be infinite and the case of interest is $\underline{g} < \overline{g}$. Without loss of generality we assume $0 < G(I) < \overline{g}$.

The monotonicity of $G$ implies the inequality

$$G^{ij}(S)\xi^i\xi^j > 0, \quad |\xi| = 1,\ S \in D_1, \tag{1.1}$$

where $G^{ij}(S) = \partial G(S)/\partial S_{ij}$. Moreover, due to concavity of $G$ the inequality

$$\operatorname{tr}(G^{ij}(S)) \geq \nu(\delta) > 0 \tag{1.2}$$

holds for $S \in D^\delta = \{S \in D : \underline{g} < G(S) \leq \overline{g} - \delta\}$, where $\delta > 0$. It is worth noting that if $G$ is a one-homogeneous function then $\nu_{1.2}$ is independent of $\delta$ and $\nu_{1.2} = G(I)$. Here and below we index the constants by the numbers of the formulas where they first appear.

We also assume $G$ to be invariant under orthogonal transformations, i.e., $G(S) = G(\widehat{S})$ if $\widehat{S} = BSB'$ and $B' = B^{-1}$. This requirement ensures $G^{ii}(S^0) = G^{jj}(S^0)$, $i, j = 1, \ldots, n$, if $S^0 = sI$, $s \in \mathbb{R}^+$. Such invariance together with concavity of $G$ also leads to the inequality

$$\operatorname{tr} S > 0 \tag{1.3}$$

for any $S \in D$ if $G(S) > \underline{g}$. Indeed, the following holds for any $\varepsilon > 0$:

$$\begin{gathered} G(S) - G(\varepsilon I) \leq \frac{1}{n}(\operatorname{tr} S - \varepsilon)\operatorname{tr}(G^{ij}(\varepsilon I)), \\ \operatorname{tr} S > n(G(S) - G(\varepsilon I))/\operatorname{tr}(G^{ij}(\varepsilon I)). \end{gathered}$$

Inequality (1.3) follows from the latter and the requirement $\mathbf{0} \in \partial D$.

We now describe functions $G$ which are uniformly monotone over $D$.

Definition 1.1. The function $G$ is *uniformly positively monotone* over $D$ iff there exist constants $\nu, \mu > 0$ such that

$$\nu \operatorname{tr}(G^{ij}(S))\xi^2 \leq G^{ij}\xi^i\xi^j \leq \mu \operatorname{tr}(G^{ij}(S))\xi^2, \quad |\xi| = 1, \tag{1.4}$$

for any $S \in D$.

Following the ideas of N. Krylov [10] and N. Trudinger [13] we can associate with a pair $(G, D)$ the pair $(G^\varepsilon, D^\varepsilon)$, $\varepsilon > 0$, where $(G^\varepsilon, D^\varepsilon)$ satisfies all the above requirements and also $G^\varepsilon$ is uniformly monotone over $D^\varepsilon$. Let

$$S^\varepsilon = S + \varepsilon \operatorname{tr} S, \quad G^\varepsilon(S) = G(S^\varepsilon), \quad D^\varepsilon = \{S : S^\varepsilon \in D\}.$$

Inequality (1.3) implies the inclusion $D^{\varepsilon_1} \subset D^{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2$. Moreover, since

$$G^{\varepsilon ij}(S) = \frac{\partial G^\varepsilon(S)}{\partial S_{ij}} = \frac{\partial G(S^\varepsilon)}{\partial S^\varepsilon_{ij}} + \varepsilon\delta^{ij}\,\mathrm{tr}(\partial G(S^\varepsilon)/\partial S^\varepsilon_{kl}), \tag{1.4$'$}$$

inequality (1.4) holds with $G = G^\varepsilon$, $D = D^\varepsilon$, $\mu_{1.4} = 1$, $\nu_{1.4} = \varepsilon$, i.e., $G^\varepsilon$ is indeed uniformly monotone over $D^\varepsilon$.

The above description of the pairs $(G, D)$ assembles relevant pieces from the theory of second-order fully nonlinear differential equations (see for instance [1], [4], [10], [13]). One can also extract from there many examples of such pairs. The most common example is

$$\begin{aligned} &G(S) = F_m(S) \equiv (\mathrm{tr}_m\, S)^{1/m}, \quad 1 \le m \le n, \\ &D = K_m \equiv \{S \in \mathrm{Sym}(n) : \mathrm{tr}_l\, S > 0,\ l = 1, \dots, m\}, \end{aligned} \tag{1.5}$$

where $\mathrm{tr}_l\, S$ is the sum of all principal $l$-minors of the matrix $S$.

## 2. Evolving operators and the first initial boundary value problem for evolutionary equations

Let $A = A(p) \in \mathrm{Sym}(n)$, $p \in \mathbb{R}^n$, be a smooth positive definite matrix and $u = u(z)$, $z = (x, t) \in Q = \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^n$, be a $C^{2,1}$-function. Define

$$u^A_{(xx)} = A^{1/2}(u_x) u_{xx} A^{1/2}(u_x), \tag{2.1}$$

where $u_x$ and $u_{xx}$ are the gradient and the Hesse matrix of $u$ respectively. In the sequel the matrix $A$ will always be fixed in some way and we will omit the upper index $A$ in (2.1) for simplicity. We also fix a function $a = a(p) > 0$ and a matrix $W = W(x, p)$ and define the matrix operator $S[u]$ as

$$S[u] = u_{(xx)} - a(u_x) u_t I + W. \tag{2.2}$$

By definition an *evolving operator* $G$ looks as

$$G[u] = G(S[u]). \tag{2.3}$$

We qualify functions $u \in C^{2,1}(Q)$ with respect to $G$ as follows.

Definition 2.1. A function $u$ is a *subfunction* for the operator (2.1)–(2.3) iff $S[u](z) \in \overline{D}_1$ for all $z \in Q$, and a *superfunction* iff $S[u](z) \notin D_1$ for all $z \in Q$.

Definition 2.2. The function $u$ is an *admissible function* for the operator (2.1)–(2.3) iff $S[u](z) \in \overline{D}$ for all $z \in Q$.

It is obvious that the operator (2.1)–(2.3) is parabolic on the set of subfunctions in the usual sense, i.e.,

$$-\frac{\partial G[u]}{\partial u_t}(\xi^0)^2 + \frac{\partial G[u]}{\partial u_{ij}}\xi^i\xi^j > 0, \quad (\xi^0)^2 + \xi^2 = 1,$$

for any subfunction $u \in C^{2,1}(Q)$. Here and further on $u_i = du/dx^i$, $u_{ij} = d^2u/dx^i dx^j$.

If $G$ is uniformly monotone over $D$ and

$$a(p) > \nu \operatorname{tr} A(p), \tag{2.4}$$

and the eigenvalues of $A(p)$ are of order $\operatorname{tr} A(p)$, then the operator (2.1)–(2.3) is called *uniformly parabolic* and we have the inequalities

$$\begin{gathered} -\frac{\partial G}{\partial u_t} > \nu_{2.4} H(S[u]) \operatorname{tr} A(u_x), \\ \nu H(S[u]) \operatorname{tr} A(u_x)\xi^2 \le \frac{\partial G[u]}{\partial u_{ij}} \xi^i \xi^j \le \mu H(S[u]) \operatorname{tr} A(u_x)\xi^2 \end{gathered} \tag{2.5}$$

for any subfunction $u$ with $H(S[u]) = \operatorname{tr}(G^{ij}(S[u]))$. The above holds all the more for admissible functions.

As a source of model matrices $A$ we take $\{A(\sigma; p) : p \ge 1\}$, where

$$A(\sigma; p) = (\partial^2 v^\sigma(p)/\partial p_i \partial p_j) = (1+p^2)^{\sigma/2-1}\left(\delta^{ij} + (\sigma - 2)\frac{p_i p_j}{1+p^2}\right), \tag{2.6}$$

$v = (1+p^2)^{\sigma/2}/\sigma$. The simplest case $\sigma = 2$ ($A(2) = I$) gives Hessian operators. The curvature operators correspond to $\sigma = 1$,

$$A(1; p) = \frac{1}{\sqrt{1+p^2}}\left(\delta^{ij} - \frac{p_i p_j}{1+p^2}\right).$$

In contrast to $\sigma > 1$ the curvature operators are nonuniformly parabolic for any function $G$. Perhaps it would be reasonable to consider evolving operators on the base of

$$\widetilde{S}[u] = u_{(xx)} - A(u_x)u_t + W \tag{2.7}$$

at least for $A = A(1)$.

The evolutionary equation of our concern is

$$G[u] = g \tag{2.8}$$

and we set up initial boundary values as

$$u(x, 0) = \psi(x), \quad u(x, t)|_{\partial'' Q} = \Phi(x, t), \tag{2.9}$$

where $\partial'' Q = \partial\Omega \times [0, T]$.

In [6] the notion of proper data was defined as data which do not contradict the admissibility of possible solution of problem (2.8), (2.9). In the simplest case of closed evolution which is of interest here, the functions $g, \Phi$ have to satisfy the following relations:

$$\underline{g} < g < \overline{g}, \quad z \in \overline{Q}, \tag{2.10}$$

$$\Phi_t(x,0)|_{\partial\Omega} = v(x), \tag{2.11}$$

where $v$ is the unique solution to the equation

$$G(\psi_{(xx)} - a(\psi_x)vI + W(\cdot,\psi_x)) = g(\cdot,0), \quad x \in \overline{\Omega}, \tag{2.12}$$

satisfying the inclusion

$$\psi_{(xx)} - a(\psi_x)vI + W(\cdot,\psi_x) \in D, \quad x \in \overline{\Omega}. \tag{2.13}$$

In this setting we admit an arbitrary initial value $\psi$. Line (2.11) looks as a compatibility condition but here it has the additional task to ensure the admissibility of our closed evolution at the start.

The problem (2.6), (2.3), (2.8), (2.9) has to be supplemented by (2.11) with $v = \widetilde{v}$ being the unique solution to the equation

$$G(\psi_{(xx)} - A(\psi_x)vI + W(\cdot,\psi_x)) = g(\cdot,0),$$

satisfying the analog of condition (2.13).

In fact, one more factor could a priori hinder the admissibility of solution. It is the boundary $\partial\Omega$. In order to eliminate this possibility we impose restrictions on the principal curvatures $\widetilde{k} = (\widetilde{k}^1,\dots,\widetilde{k}^{n-1})$ of $\partial\Omega$. Let $R \in \mathbb{R}^+$, $p \in \mathbb{R}^n$ and

$$\mathcal{A}(R,p;\widetilde{k}) = A^{1/2}(p)\begin{pmatrix} \widetilde{k}^1 & & & 0 \\ & \ddots & & \\ & & \widetilde{k}^{n-1} & \\ 0 & & & R \end{pmatrix} A^{1/2}(p).$$

ASSUMPTION 2.3. *There exist $R_0 > 0$ and $\delta > 0$ such that*

$$\overline{\mathcal{A}}(R_0;\widetilde{k}(x)) = \lim_{|p|\to\infty} \mathcal{A}(R_0,p;\widetilde{k}(x))/\operatorname{tr} A(p) \in D, \tag{2.14}$$

$$\varliminf_{|p|\to\infty} G(|p|\mathcal{A}(R_0,p;\widetilde{k}(x))) > g \tag{2.15}$$

*for all $x \in \partial\Omega$, $t \in [0,T]$.*

To illustrate (2.14) and (2.15) consider $A = A(\sigma)$ (see (2.6)). Then

$$\overline{\mathcal{A}}(R;\widetilde{k}) = \begin{pmatrix} \widetilde{k}^1 & & & 0 \\ & \ddots & & \\ & & \widetilde{k}^{n-1} & \\ 0 & & & (\sigma-1)R \end{pmatrix}.$$

If $\sigma > 1$, then (2.14) reduces to

$$\begin{pmatrix} \widetilde{k}^1 & & & 0 \\ & \ddots & & \\ & & \widetilde{k}^{n-1} & \\ 0 & & & (\sigma-1)R_0 \end{pmatrix} \in D \tag{2.16}$$

for some $R_0 > 0$ and then (2.15) holds. The inclusion (2.16) is exactly the restriction discovered by the authors of [1] for Hessian elliptic equations, $\sigma = 2$. In the case of the curvature equations, $\sigma = 1$, (2.14) amounts to (2.16) with $R_0 = 0$, while (2.15) this time represents an independent restriction on $\partial\Omega$, $g$ and is very close to the corresponding requirement discovered by N. S. Trudinger [13] for curvature elliptic equations.

Remark 2.4. (2.15) implies the inequality

$$\operatorname{tr} A(p) \geq \nu |p|^{-1} \tag{2.17}$$

and if $A(p)$ is uniformly positive definite, only the existence of $R_0$ in (2.16) is sufficient for some $\nu$ under control in (2.17) to exist, that is, we do not need (2.15) in this case.

We also complement (2.11) by

$$\Phi(x,0)|_{\partial\Omega} = \psi(x), \quad \Phi_{\widetilde{x}}(x,0)|_{\partial\Omega} = \psi_{\widetilde{x}}(x), \quad \Phi_{\widetilde{x}\widetilde{x}}(x,0)|_{\partial\Omega} = \psi_{\widetilde{x}\widetilde{x}}(x), \tag{2.18}$$

where $\widetilde{x}$ is any direction tangent to $\partial\Omega$.

The function $a$ and matrix $W$ are required to satisfy the relations

$$\frac{\nu}{\sqrt{1+p^2}} \leq a(p), \quad |W_x(p,x)| \leq \mu a(p), \quad x \in \overline{\Omega}, \tag{2.19}$$

$$\lim_{|p|\to\infty} \frac{a(p) + W(x,p)}{|p| \operatorname{tr} A(p)} = 0, \quad x \in \partial\Omega, \tag{2.20}$$

where $|W|^2 = \operatorname{tr} W^2$.

## 3. The existence theorems

In order to exhibit the crucial role of the matrix $W$ in our reasoning we prove the existence theorems for equations (2.8), (2.3) with

$$S[u] = u_{(xx)} - a(u_x)u_t I. \tag{3.1}$$

Theorem 3.1. *Let $0 < \alpha < 1$. Assume the following:*

(a) *$G \in C^2(D)$ is uniformly monotone over $D$, $A \in C^2(\mathbb{R}^n)$;*
(b) *$\partial\Omega \in C^3$ satisfies Assumption* 2.3*;*
(c) *$\psi \in C^3(\overline{\Omega})$ is an arbitrary function;*
(d) *$\Phi \in C^{3,2}(\partial''Q)$ satisfies* (2.11), (2.18)*;*

(e) $g \in C^{2,1}(\overline{Q})$ *satisfies* (2.10), (2.15)*;*

(f) $a \in C^2(\mathbb{R}^n)$ *satisfies* (2.19), (2.20)*.*

*Then there exists a unique admissible solution* $u$ *to problem* (2.8), (2.3), (3.1), (2.9) *and* $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$.

The uniqueness follows from a version of the comparison principle adapted to our case. We recall the notions of sub- and supersolutions to equation (2.2), (2.3), (2.8) (see [3, 6]). Namely, $\underline{u}$ is a *subsolution* iff $S[\underline{u}](z) \in D$ for all $z \in Q$ and $G[\underline{u}] \geq g$, and $\overline{u}$ is a *supersolution* iff $G[\overline{u}] \leq g$ at all points of $\{\overline{z} \in Q : S[\overline{u}](\overline{z}) \in \overline{D}\}$.

THEOREM 3.2. *Let* $\overline{u}, \underline{u} \in C(Q) \cap \operatorname{Lip} \Omega$ *be a super- and a subsolution to equation* (2.2), (2.3), (2.8)*. Then*

$$\underline{u} - \overline{u} \leq \max_{\partial' Q} (\underline{u} - \overline{u})^+. \tag{3.2}$$

For simplicity we restrict ourselves to $\overline{u}, \underline{u} \in C^{2,1}(Q)$. Assume there exists a point $z_0 \in \overline{Q} \setminus \partial' Q$ where $w(z) = (\underline{u} - \overline{u})(z) \exp(-\varepsilon t)$ attains its maximum $w(z_0) > 0$ with some $\varepsilon > 0$. Then $(\underline{u} - \overline{u})(z_0) > 0$, $\underline{u}_x(z_0) = \overline{u}_x(z_0)$,

$$(\underline{u}_{xx} - a(\underline{u}_x)(\underline{u}_t - \varepsilon(\underline{u} - \overline{u}))I)(z_0) \leq (\overline{u}_{xx} - a(\overline{u}_x)\overline{u}_t I)(z_0) \tag{3.3}$$

and $S[\overline{u}](z_0) \in D$. Inequalities (1.3), (3.3) lead to $G[\underline{u}](z_0) < G[\overline{u}](z_0)$, which is impossible under the assumption of Theorem 3.2. Hence there is no such $z_0$ for any $\varepsilon > 0$. This proves (3.2).

The existence of an admissible solution is obtained by the continuity method taking the relevant homotopy from [6]:

$$G^\tau[u^\tau] = g^\tau, \quad \tau \in [0,1], \tag{3.4}$$

$$u^\tau(x,0) = \psi(x), \quad u^\tau|_{\partial'' Q} = \Phi^\tau(x,t), \tag{3.5}$$

with

$$S^\tau[u^\tau] = u^\tau_{(xx)} - a(u^\tau_x)(u^\tau_t - (1-\tau)v)I, \tag{3.6}$$

where

$$G^\tau[u^\tau] = G(S^\tau[u^\tau]),$$
$$g^\tau = \tau g(x,t) + (1-\tau)g(x,0),$$
$$\Phi^\tau(x,t) = \tau\Phi(x,t) + (1-\tau)\Phi(x,0).$$

The matrix $W^\tau = (1-\tau)a(u^\tau_x)vI$ appears here with $v$ being the solution to problem (2.2), (2.3), where $W = \mathbf{0}$. By the choice of $W$, if the data (3.5) are compatible and proper for $\tau = 0$, then they are so for all $\tau$.

The operators (3.4) are locally invertible on $\mathbf{D}^\tau = \{S^\tau[u](z) \in D : z \in \overline{Q}\}$. We denote by $\{\widehat{\tau}\}$ the set of solvability of problems (3.4), (3.5) and note that $\{\widehat{\tau}\}$ is nonempty since $u^0 = \psi$ is an admissible solution relating to $\tau = 0$.

The closedness of $\{\widehat{\tau}\}$ validates Theorem 3.1 and it will follow from the statements below.

Theorem 3.3. *Let $G \in C^1(D)$, $a, A \in C^2(\mathbb{R}^n)$, $a > 0$, $A > \mathbf{0}$. Assume that* (2.14), (2.15), (2.17)–(2.20) *hold. Then any admissible solution $u \in C^{3,2}(Q) \cap C^{2,1}(\overline{Q})$ to problem* (2.8), (2.2), (2.3), (2.9) *satisfies the inequality*

$$\|u\|_{C^{1,1}(Q)} \le c(\nu_{1.2}, \|\partial\Omega, \psi\|_{C^3}, \|\Phi\|_{C^{2,1}}, \|g, W\|_{C^{1,1}}). \tag{3.7}$$

Theorem 3.4. *Let $G \in C^2(D)$, $a, A \in C^2(\mathbb{R}^n)$, $a > 0$, $A > \mathbf{0}$. Assume $G$ to be uniformly monotone. Then any admissible solution $u \in C^{4,2}(Q) \cap C^{2,1}(\overline{Q})$ satisfies the inequality*

$$\|u\|_{C^{2,1}(Q)} \le c(\nu_{1.4}, \mu_{1.4}, c_{3.7}, \alpha, \|\partial\Omega\|_{C^{2+\alpha}}, \|\Phi\|_{C^{2+\alpha,1+\alpha/2}}, \|g, w\|_{C^{2,1}}), \tag{3.8}$$

*where $\alpha \in (0,1)$.*

By well known results of N. Krylov and M. Safonov Theorems 3.3 and 3.4 imply the a priori boundedness of $\|u\|_{C^{2+\alpha,1+\alpha/2}(Q)}$. Hence the continuity method is completed, i.e., Theorem 3.1 follows from Theorems 3.3 and 3.4.

Actually, Theorem 3.4 is a particular case of a theorem concerning evolutions of all types from the paper [9]. The proof of Theorem 3.3 is given in Section 4.

Theorem 3.1 and estimate (3.7) yield the existence of viscosity admissible solutions for problems with nonuniformly monotone functions $G$.

Theorem 3.5. *Let $G \in C^2(D)$, $a, A \in C^1(\mathbb{R}^n)$, $a > 0$, $A > \mathbf{0}$. Assume $\partial\Omega \in C^2$, $g \in C^{1,1}(\overline{Q})$ and* (2.14), (2.15), (2.18), (2.20) *hold. Assume also that the compatibility conditions up to the second order are satisfied and the data $\Phi$, $\psi$, $g$ are proper. Then there exists a unique viscosity admissible solution $u$ to problem* (3.1), (2.8), (2.9) *and $u \in \operatorname{Lip}\overline{Q}$.*

Similarly to [6] we regularize our problem as follows:

$$G^\varepsilon[u^\varepsilon] = g, \tag{3.9}$$

$$u^\varepsilon(x,0) = \psi(x), \quad u^\varepsilon|_{\partial''Q} = \Phi^\varepsilon(x,t), \tag{3.10}$$

where

$$\begin{gathered} G^\varepsilon[u^\varepsilon] = G(S^\varepsilon[u^\varepsilon]), \quad S^\varepsilon[u] = S[u] + \varepsilon \operatorname{tr} S[u], \\ \Phi^\varepsilon(x,t) = \Phi(x,t) + t\varphi^\varepsilon(x) \end{gathered} \tag{3.11}$$

with $\varphi^\varepsilon = v^\varepsilon - v$, $v$ is the solution to equation (2.12) and $v^\varepsilon$ solves the analog of (2.12) corresponding to $S^\varepsilon$. In view of (1.4$'$) Theorem 3.1 embraces problems (3.9)–(3.11) for all $\varepsilon > 0$, i.e. there always exists an admissible solution $u^\varepsilon \in$

$C^{2+\alpha,1+\alpha/2}(\overline{Q})$. Letting $\varepsilon$ tend to 0 we obtain Theorem 3.5 by the viscosity limit passage. This does not spoil inequality (3.7) and hence $u \in \operatorname{Lip}\overline{Q}$.

In this argument we adapted to our case the idea of N. Trudinger from [13]. Perhaps one could also try the Perron method which allows circumventing the concavity of $G$ (see [3], [11]).

Sometimes we can guarantee the existence of $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ for nonuniformly monotone functions $G$. Examples of such statements are the following theorems.

Theorem 3.6. *Let $G \in C^2(D)$, $A = A(2) = I$, $a \in C^2(\mathbb{R}^n)$, $g \in C^{2,1}(\overline{Q})$, $\psi \in C^4(\overline{\Omega})$, $\Phi \in C^{4,2}(\partial''Q)$. Assume that $g$, $a$, $\Phi$ satisfy* (2.10), (2.19), (2.20), (2.18) *respectively and that there exists $R_0 > 0$ such that*

$$\mathcal{A}_2(x) = \begin{pmatrix} \widetilde{k}^1 & & & 0 \\ & \ddots & & \\ & & \widetilde{k}^{n-1} & \\ 0 & & & R \end{pmatrix}(x) \in D, \quad x \in \partial\Omega.$$

*Then there exists a unique admissible solution $u$ to problem* (3.1), (2.8), (2.9) *and $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ for any $\alpha \in (0,1)$.*

Theorem 3.7. *Let*

$$G(S) = F_{m,l}(S) = (\operatorname{tr}_m S/\operatorname{tr}_l S)^{1/(m-l)}, \quad 0 \le l < m < n,$$

$A = A(1)$ (see (2.6)), *$a = 1/\sqrt{1+p^2}$. Assume that*

$$\mathcal{A}_1(x) = \begin{pmatrix} \widetilde{k}^1 & & & 0 \\ & \ddots & & \\ & & \widetilde{k}^{n-1} & \\ 0 & & & 0 \end{pmatrix}(x) \in K_m, \quad x \in \partial\Omega,$$

*where $K_m$ is defined by* (1.5), *and*

$$\max_t g(x,t) < F_{m,l}(\mathcal{A}_1)(x), \quad x \in \partial\Omega.$$

*Assume also that the smoothness requirements from Theorem* 3.6 *as well as compatibility conditions are satisfied. Then there exists a unique admissible solution $u$ to problem* (3.1), (2.8), (2.9) *and $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ for any $\alpha \in (0,1)$.*

The principal point in the proof of Theorems 3.6, 3.7 is of course the estimation of second spatial derivatives. But the relevant reasoning from the papers [1], [14], [5], [8], [12] devoted to Hessian and curvature fully nonlinear equations serves our cases as well. From such an estimate Theorems 3.6 and 3.7 can be deduced in the same way as Theorem 3.1.

We do not know if the estimate of second derivatives can be found for $A = A(\sigma)$, $\sigma \neq 1, 2$.

## 4. The estimate of $\|u\|_{C^{1,1}(Q)}$

We start the proof of (3.7) by estimating $u$ from above. Define $B_\delta(p^0) = \{p \in \mathbb{R}^n : |p - p^0| < \delta\}$. The following general assertion implies the boundedness of an admissible solution from above.

Lemma 4.1. *Let $G \in C(D_1)$, $a, A \in C(B_\delta(0))$, $a(0) > 0$, $A(0) > \mathbf{0}$, $W \in C(B_\delta(0) \times \Omega)$. Assume that*

$$S^- = \{S \in \mathrm{Sym}(n) : S < 0\} \notin D_1. \tag{4.1}$$

*Then any continuous subfunction for the operator* (2.2), (2.3) *satisfies the inequality*

$$\max_Q u < \sup\left\{0; \max_{\Omega, |\xi|=1} \frac{(W(0,x)\xi, \xi)}{\lambda a(0)}; \max_{\partial' Q} u\right\} \exp(\lambda T) \tag{4.2}$$

*for all $\lambda > 0$.*

For simplicity we consider a subfunction $u \in C^{2,1}(Q)$. Suppose $w = u\exp(-\lambda t)$ attains its positive maximum at a point $z_0 \in \overline{Q} \backslash \partial' Q$. Then $u(z_0) > 0$, $u_x(z_0) = w_x(z_0) = 0$, $u_t(z_0) = (\exp(\lambda t) w_t + \lambda u)(z_0) \geq \lambda u(z_0)$, $u_{(xx)}(z_0) = \exp(\lambda t) w_{(xx)}(z_0) \leq 0$ and

$$S[u](z_0) \leq -a(0)\lambda u(z_0) I + W(0, x_0). \tag{4.3}$$

Under condition (4.1) this yields inequality (4.2).

If $W \leq \mathbf{0}$ the following maximum principle for subfunctions is valid.

Corollary 4.2. *Under assumption* (4.1) *and the requirement*

$$W(0, x) \leq \mathbf{0}, \quad x \in \Omega, \tag{4.4}$$

*any subfunction for the operator* (2.2), (2.3) *attains its maximum on $\partial' Q$.*

Indeed, (4.3), (4.4) we can drop the second term on the right-hand side of (4.2) and then let $\lambda = 0$. Moreover, since $S[u]$, $G[u]$ are invariant under translations $\{u + C\}$, we can omit the first term there for $\underline{u} = u - \min_{\partial' Q} u$. But if $u + C$ attains its maximum on $\partial' Q$ then so does $u$.

We remark here that the minimum principle for superfunctions for the operator (2.2), (2.3) with $W \geq 0$ supplements Corollary 4.2 but it does not concern our equation (2.8) even with $W = \mathbf{0}$. Generally speaking, superfunctions can differ from supersolutions to equations (2.8), in particular, an admissible solution is not a superfunction under requirement (2.10). To bound an admissible solution from below we assume the function $w = u\exp(-t)$ attains its negative minimum at $z_0 \in \overline{Q} \setminus \partial' Q$. Now the relation

$$S[u](z_0) \geq -a(0) u(z_0) I + W(0, x_0) \tag{4.5}$$

replaces (4.3) and therefore equation (2.8) together with (4.5) leads to the inequality

$$g(z_0) \geq G(-a(0)u(z_0)I + W(0, x_0)).$$

Together with the right-hand inequality of (2.10) this ensures the estimate for $u$ from below, i.e., there exists

$$-M_0 = -M_0(\max_Q g, a(0), \max_{\Omega, |\xi|=1}(W(0,x)\xi, \xi)^-, 1/(\overline{g} - \max_Q g)), \quad M_0 \geq 0,$$

which bounds $u$ from below. Here $f^- = \max(-f, 0)$.

The above argument contributes to the proof of Theorem 3.3 the following statement.

Lemma 4.3. *Let $G \in C(D)$, $a, A \in C(B_\delta(0))$, $a(0) > 0$, $A(0) > \mathbf{0}$, $W \in C(B_\delta(0) \times \Omega)$ for some $\delta > 0$. Assume $g \in C(Q)$ satisfies the right-hand inequality of* (2.10). *Then any continuous admissible solution $u$ to equation* (2.8) *satisfies*

$$\begin{aligned} \text{(4.6)} \qquad -\sup\{M_0, \max_{\partial' Q} u^-\} &\leq u \exp(-T) \\ &\leq \sup\left\{\frac{1}{a(0)} \max_{\Omega, |\xi|=1}(W(0,x)\xi, \xi)^+, \max_{\partial' Q} u^+\right\}. \end{aligned}$$

We now deduce the estimate for the spatial gradient at interior points of $Q$. Assume function $|u_x|\exp(-\lambda t)$, $\lambda > 0$, attains its maximum at $z_0 \in \overline{Q} \setminus \partial' Q$. In view of invariance of $G$ under orthogonal transformations we can assume $|u_x(z_0)| = u_1(z_0)$, $u_1(z_0) > 0$. Then the function $w = u_1(z)\exp(-\lambda t)$ also attains its maximum at $z_0$ and hence

$$(u_1)_x(z_0) = 0, \quad (u_1)_{(xx)}(z_0) \leq 0, \quad w_t(z_0) = (u_{1t} - \lambda u_1)(z_0)\exp(-\lambda t_0) \geq 0$$

or

$$\text{(4.7)} \qquad -u_{1t}(z_0) \leq -\lambda u_1(z_0).$$

We differentiate equation (2.8) to obtain

$$\text{(4.8)} \qquad \frac{\partial g}{\partial x^1} \equiv g_1 = -\operatorname{tr}(G^{ij}(S[u]))a(u_x)u_{1t} + G^{ij}(S[u])(u_{1(ij)} + W_1^{ij}) + b^i u_{1i}$$

with some $b^i = b^i(u_t, u_x, u_{xx}, a, w, (G^{ij}))$, $i = 1, \ldots, n$.

Now (4.7), (4.8) and also (1.2) yield the crucial inequality

$$\lambda a(u_x)u_1 \leq \max_\Omega |W_x| + \max_Q |g_x|/\nu_{12}.$$

Indeed, under conditions (2.19), (2.20) the latter implies the boundedness of $u_1(z_0)$ and consequently the estimate for $|u_x|$ in terms of $\max_{\partial' Q}|u_x|$. We remark that the idea of fixing $x^1$ the above way belongs to the authors of [2].

The estimate for $|u_t|$ can be obtained similarly. We thus proved the following statement.

LEMMA 4.4. *Let* $G \in C^1(D)$, $A, a \in C^1(\mathbb{R}^n)$, $A \geq \mathbf{0}$, $W \in C^1(\mathbb{R}^n \times \Omega)$. *Assume inequalities* (2.10), (2.19), (2.20) *to be satisfied. Then*

$$\max_Q(|u_x| + |u_t|) \leq c(\nu_{1.2}, \nu_{2.19}, \mu_{2.19}, \|g\|_{C^{1,1}(Q)}, \max_{\partial' Q}(|u_x| + |u_t|)) \tag{4.9}$$

*for every admissible solution* $u \in C^{3,2}(Q)$ *to equation* (2.8).

As to the estimate on $\partial'' Q$, Theorem 3.2 reduces the matter to the existence of local sub- and supersolutions to equation (2.8) attaining prescribed boundary data, i.e. to the existence of local barriers. Here we slightly modify the known barriers of elliptic theory (see for instance [5], [13]). Namely, relate to $z_0 \in \partial'' Q$ a coordinate system (called *primary*) such that the vector $(0, \ldots, 0, 1)$ is directed along the interior normal to $\partial\Omega$ at $x_0$, the surface $\partial\Omega \cap B_d(0)$ is the graph of a function $\omega$, $x^n = \omega(\widetilde{x})$, $\widetilde{x} = (x^1, \ldots, x^{n-1})$ and $\omega(0) = 0$, $\omega_x(0) = 0$, $\omega_{ij}(0) = \delta_{ij}\omega_{ii} = \widetilde{k}^i$, $i, j = 1, , \ldots, n-1$. Relate also to $z_0$ the domain

$$\widetilde{\Omega} = \{|\widetilde{x}| < d : \omega(\widetilde{x}) < x^n < \widetilde{\omega}(\widetilde{x}) + \varkappa d^2/2\},$$

where $\widetilde{\omega}(\widetilde{x}) = \omega(\widetilde{x}) - \varkappa\widetilde{x}^2/2$ and $\varkappa > 0$ is some number to be chosen. Our barrier $w$ looks as

$$w = \Phi(x,t) + h(\varrho), \tag{4.10}$$

where $\varrho = \widetilde{\omega}(\widetilde{x}) - x^n + \varkappa d^2/2$ and $h(\varrho) = \exp(R\varrho) - \exp(\varkappa R d^2/2)$, $R \gg 1$. It is obvious that

$$w|_{\partial'' Q \cap (B_d \times [0,T])} \leq \Phi, \quad w(0) = \Phi(0). \tag{4.11}$$

On the other hand, the function (4.10) with $R = R(\varkappa, d, M)$ satisfies the inequality

$$w|_{(\partial\widetilde{\Omega} \setminus (\partial\Omega \cap B_d) \times [0,T]} \leq -M \tag{4.12}$$

for large $M$ and small $\varkappa, d$.

In order to find the principal parts of $w_x$, $w_{(xx)}$ we introduce in $\widetilde{\Omega}$ an orthogonal smooth moving frame $\{b_1, \ldots, b_n\}$ with $b_n(x) = (-\widetilde{\omega}_x(x), 1)/\sqrt{1 + \widetilde{\omega}_x^2}$. Define $w_{\langle i \rangle} = w_k b_i^k$, $w_{\langle ij \rangle} = w_{kl} b_i^k b_j^l$. Then

$$\begin{aligned}
&w_{\langle i \rangle} = \Phi_{\langle i \rangle}, \quad w_{\langle n \rangle} = \Phi_{\langle n \rangle} - h'/\sqrt{1 + \widetilde{\omega}_x^2}, \\
&w_{\langle ij \rangle} = \Phi_{\langle ij \rangle} + h'\widetilde{\omega}_{\langle ij \rangle}, \quad i = 1, \ldots, n-1, \; j = 1, \ldots, n, \\
&w_{\langle nn \rangle} = \Phi_{\langle nn \rangle} + h'\widetilde{\omega}_{\langle nn \rangle} + h''/\sqrt{1 + \widetilde{\omega}_x^2}.
\end{aligned} \tag{4.13}$$

In a primary coordinate system formulas (4.13) turn into

$$\frac{w_{\langle xx\rangle}}{h'} = \begin{pmatrix} \widetilde{k}_0^1 & & & 0 \\ & \ddots & & \\ & & \widetilde{k}_0^{n-1} & \\ 0 & & & R \end{pmatrix} + \alpha, \quad z \in \widetilde{\Omega} \times [0,T], \tag{4.14}$$

where $\widetilde{k}_0 = \widetilde{k}(0)$, $\alpha = (\alpha_{ij}(z, \varkappa, 1/h'))$ and $\alpha_{ij} = O(d, \varkappa, 1/h')$. Consider (4.14) as determining the matrix $V = V(P, R, \varkappa, z)$ so that

$$\frac{w_{\langle xx\rangle}}{h'} = V(h', R, \varkappa, z).$$

In further consideration matrices $A = A(p)$ are involved and for simplicity we confine ourselves to matrices (2.6). Define

$$\tau(\sigma; p) = (1 + p^2)^{(1-2\sigma)/4} A^{1/2}(\sigma; p).$$

Then

$$\tau_j^i = (\delta_j^i + (\sigma - 2)\frac{p_i p_j}{\sqrt{1+p^2}\sqrt{1+(\sigma-1)p^2}}$$

and for $P = p_n$,

$$\overline{\overline{\mathcal{A}}}(\sigma; R, \varkappa, z) = \lim_{P\to\infty} \tau V \tau = \begin{pmatrix} \widetilde{k}_0^1 & & & 0 \\ & \ddots & & \\ & & \widetilde{k}_0^{n-1} & \\ 0 & & & (\sigma-1)R \end{pmatrix} + \overline{\alpha} \tag{4.15}$$

where $\overline{\alpha}_{ij} = \sqrt{(1 - \delta_n^i \sigma)(1 - \delta_n^j \sigma)}\alpha_{ij}(z, \varkappa, 0)$.

The principal term in (4.15) is exactly the matrix $\overline{\mathcal{A}}$ from Assumption 2.3 and $\varkappa, d$ can be chosen so small that essentially $\overline{\overline{\mathcal{A}}}$ has the properties of $\overline{\mathcal{A}}$ for all $z \in \widetilde{\Omega} \times [0,T]$. In particular, $\overline{\overline{\mathcal{A}}} \in D$ if $\varkappa, d$ are controllably small and $R > R_0$. Then there is $R_1 \gg 1$, and therefore $h' = R_1 \exp(R_1 \varrho) \gg 1$ such that

$$\frac{w_{\langle xx\rangle}}{h'(1 + w_x^2)^{(1-2\sigma)/2}} = \overline{\mathcal{A}}(\sigma; R_1, \varkappa, z) + O(1/h') \in D$$

and by the requirements (2.19), (2.20),

$$\frac{S[w]}{h'(1 + w_x^2)^{(1-2\sigma)/2}} = \overline{\mathcal{A}}(\sigma; R_1, \varkappa, z) + O(1/(h')^{\varepsilon}) \in D.$$

Note that we have made the choice of $R_1$ after having fixed $\varkappa, d$.

We now separate the cases $\sigma > 1$, $\sigma = 1$. If the first holds then

$$\lim_{R\to\infty} h'(1 + w_x^2)^{(1-2\sigma)/2} = \infty, \tag{4.16}$$

which leads to $G[w] > \overline{g} - \delta$ for some $R = R(\delta) > R_1$, for any small $\delta > 0$. This ensures the inequality

$$G[w] > g, \quad z \in \widetilde{\Omega} \times [0, T]. \tag{4.17}$$

If $\sigma = 1$ the limit in (4.16) equals 1 and we can by no means approach $\overline{g}$ but can obtain (4.17) due to Assumption 2.3.

The above construction can certainly be extended to general matrices $A(p)$ satisfying (2.14), (2.17) and the following assertion is valid.

PROPOSITION 4.5. *Under Assumption* 2.3 *and requirement* (2.20) *there exist* $\varkappa, d, R$ *under control such that the function* (4.10) *is admissible, satisfies inequality* (4.17) *and boundary conditions* (4.11), (4.12) *for all large* $M$.

The Comparison Theorem applied in $\widetilde{\Omega} \times [0, T]$ to the barrier (4.10) and our admissible solution $u$ yields the estimation of $u_n|_{\partial'' Q}$ in the obvious way. To complete the proof of Theorem 3.3 we consider the solution $\overline{u}$ to the quasilinear second-order parabolic differential equation $\operatorname{tr} S[\overline{u}] = 0$, satisfying the initial boundary conditions (2.9). On the one hand, $\overline{u}$ does exist under our assumptions. On the other hand, $\overline{u}$ is a superfunction for the operator $G$ (see (1.3)). Hence $\overline{u} \geq u$ in $\overline{Q}$ and $\overline{u} = u$ on $\partial' Q$. This guarantees the boundedness of $u_n|_{\partial'' Q}$ from above.

We conclude the argument by stating

LEMMA 4.6. *Let* $u \in C^{1,0}(\overline{Q})$ *be an admissible solution to problem* (2.8), (2.3), (2.9) *with* $W = (1 - \tau) a(u_x) v I$, $\tau \in [0, 1]$. *Assume* $\partial\Omega \in C^3$, $g \in C(\overline{Q})$, $a, A \in C(\mathbb{R}^n)$, $a > 0$, $A \geq \mathbf{0}$ *satisfy Assumption* 2.3 *and requirement* (2.20) *respectively. Then*

$$|u_x|_{\partial'' Q} \leq c(\|\partial\Omega\|_{C^3}, \|\Phi\|_{C^3(\partial'' Q)}, \|\psi\|_{C^1(\Omega)}, \delta).$$

Lemma 4.6 and inequality (4.9) complete the proof of Theorem 3.3 and hence our existence theorems are established.

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