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CONTINUITY PROPERTIES OF PEANO DERIVATIVES IN SEVERAL VARIABLES

Abstract

For a real-valued function of several real variables that is n-times Peano Differentiable, a sufficient condition is given for the Peano derivatives of order n to be Baire*1. An immediate consequence will be that the order n Peano derivatives of an $(n+1)$-times Peano differentiable function are Baire*1.

In 1935, A. Denjoy [1] showed that if a real function f is $(n+1)$-times Peano differentiable, then the nth Peano derivative f_n has the following property. For every nonempty closed set C there is an open interval (a,b) with $(a,b) \cap C \neq \phi$ so that f_n restricted to C, $f_n|_C$, is continuous on $(a,b) \cap C$. In 1976, R. J. O'Malley [3] named this property Baire*1. Approximate Peano derivatives were shown to be Baire*1 by M. J. Evans [2] in 1985. The notion of Baire*1 extends to functions from m-dimensional space, $\mathbb{R}^m$, to $\mathbb{R}$ in the obvious manner, replacing the one-dimensional interval (a,b) with an m -dimensional interval. In this work we obtain the result that if $f : \mathbb{R}^m \to \mathbb{R}$ is $(n+1)$-times Peano differentiable, then all the order n Peano derivatives of f are Baire*1. This is done by developing a condition that is implied by $(n+1)$-times Peano differentiability that is sufficient to guarantee that the order n Peano derivatives are Baire*1. We also show that the order n Peano derivatives of an n-times Peano differentiable function are Baire 1.

Throughout this paper we consider real valued functions defined on a subset of $\mathbb{R}^m$. For a vector $h = (h_1, h_2, \ldots, h_m) \in \mathbb{R}^m$, we define $||h|| = \max_i\{h_i\}$. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_m)$, we define $|\alpha| = \sum_{i=1}^{m} \alpha_i$, and $\binom{i}{\alpha_1,\ldots,\alpha_m} = \dfrac{i!}{\alpha_1! \cdots \alpha_m!}$.

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We say that a function f, defined in a neighborhood of a point x is n-times Peano differentiable at x if there is a set of numbers $f_\alpha(x)$, $1 \le |\alpha| \le n$, such that

$$\lim_{||h||\to 0} \frac{f(x+h) - \sum_{i=0}^{n} \sum_{|\alpha|=i} \binom{i}{\alpha_1,\ldots,\alpha_m} h_1^{\alpha_1}\cdots h_m^{\alpha_m} \dfrac{f_\alpha(x)}{i!}}{||h||^n} = 0 \tag{1}$$

where $f_{[0,\ldots,0]} = f(x)$. Equivalently

$$f(x+h) = \sum_{i=0}^{n} \sum_{|\alpha|=i} \binom{i}{\alpha_1,\ldots,\alpha_m} h_1^{\alpha_1}\cdots h_m^{\alpha_m} \frac{f_\alpha}{i!} + ||h||^n \cdot \epsilon_x(h)$$

where $\epsilon_x(h) \to 0$ as $\|h\| \to 0$.

It is easy to check that if f is n-times Peano differentiable at a point x, then the numbers $f_\alpha(x)$, are unique. Therefore the functions f_α, defined to be the $f_\alpha(x)$ from (1), are well defined, and each f_α is called a Peano derivative of f, of order $|\alpha|$

It is also easy to verify that if f is $(n+1)$-times Peano differentiable at x, it is also k-times Peano differentiable for $1 \le k \le n$. However, $(n+1)$-times Peano differentiable gives more than n-times Peano differentiable in that we actually have

$$\lim_{||h||\to 0} \frac{f(x+h) - \sum_{i=0}^{n} \sum_{|\alpha|=i} \binom{i}{\alpha_1,\ldots,\alpha_m} h_1^{\alpha_1}\cdots h_m^{\alpha_m} \frac{f_\alpha(x)}{i!}}{||h||^{n+s}} = 0. \tag{2}$$

for any $0 \le s < 1$.

This observation motivates considering the quotient

$$\epsilon_x(h) = \frac{f(x+h) - \sum_{i=0}^{n} \sum_{|\alpha|=i} \binom{i}{\alpha_1,\ldots,\alpha_m} h_1^{\alpha_1}\cdots h_m^{\alpha_m} \frac{f_\alpha(x)}{i!}}{||h||^{r}}. \tag{3}$$

for r a positive real number. If $\epsilon_x(h) = O(1)$, we say f is r-times Peano bounded at x. By $(n+s)$-times Peano bounded on $\mathbb{R}^m$ we mean there is a function $s : \mathbb{R}^m \to [0,1)$ so that f is $(n+s(x))$-times Peano bounded at each x. Obviously $(n+s)$-times Peano bounded on $\mathbb{R}^m$ implies n-times Peano differentiable. It turns out that the condition $(n+s)$-times Peano bounded on $\mathbb{R}^m$ is sufficient to gain information about continuity properties of the nth order Peano derivatives of f. In Theorem 5 below we show that the Baire*1 property is obtained.

The following useful lemma is easily proved by induction.

Lemma 1 *For $k \in \mathbb{N}$ we have*

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^i = \begin{cases} 0 & \text{if } i = 0, 1, ..., k-1, \\ k! & \text{if } i = k, \end{cases}$$

Let $D_n(u,h) = \sum\limits_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(u+jh)$, an nth forward difference applied to f.

Lemma 2 *Let f be $(n+s(u))$-times Peano bounded at u. Then $D_n(u,h) =$*

$$\sum_{|\alpha|=n}\binom{n}{\alpha_1, \ldots, \alpha_m}h_1^{\alpha_1}\cdots h_m^{\alpha_m}f_\alpha(u) + \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}||jh||^{n+s(u)}\cdot\epsilon_u(jh).$$

Proof. Writing each

$$f(u+jh) = \sum_{i=0}^{n}\sum_{|\alpha|=i}j^i\binom{i}{\alpha_1, \ldots, \alpha_m}h_1^{\alpha_1}\cdots h_m^{\alpha_m}\frac{f_\alpha(u)}{i!} + ||jh||^{n+s(u)}\cdot\epsilon_u(jh),$$

we get $D_n(u,h) = \sum\limits_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(u+jh) =$

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\left(\sum_{i=0}^{n}\sum_{|\alpha|=i}j^i\binom{i}{\alpha_1, \ldots, \alpha_m}h_1^{\alpha_1}\cdots h_m^{\alpha_m}\frac{f_\alpha(u)}{i!}\right.$$
$$\left. + ||jh||^{n+s(u)}\cdot\epsilon_u(jh)\right)$$

$$= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\left(\sum_{i=0}^{n}\sum_{|\alpha|=i}j^i\binom{i}{\alpha_1, \ldots, \alpha_m}h_1^{\alpha_1}\cdots h_m^{\alpha_m}\frac{f_\alpha(u)}{i!}\right)$$
$$+ \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}||jh||^{n+s}\cdot\epsilon_u(jh).$$

The error term is of the desired form and we rearrange the triple sum to get

$$\sum_{i=0}^{n}\sum_{|\alpha|=i}\left(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}j^i\right)\binom{i}{\alpha_1, \ldots, \alpha_m}h_1^{\alpha_1}\cdots h_m^{\alpha_m}\frac{f_\alpha(u)}{i!}$$
$$= \sum_{|\alpha|=n}\binom{n}{\alpha_1, \ldots, \alpha_m}h_1^{\alpha_1}\cdots h_m^{\alpha_m}f_\alpha(u)$$

by Lemma 1. □

Theorem 3 *If $f : \mathbb{R}^m \to \mathbb{R}$ is n-times Peano differentiable, then all nth order Peano derivatives are Baire 1.*

PROOF. Pick m distinct primes $q_1 > ... > q_m$. There are $L = \binom{m+n-1}{n}$ Peano derivatives of order n and for each natural number N we generate L vectors $\{h[k]\}_{k=0}^{L-1}$ by setting each $h[k]_i = \dfrac{q_i^k}{\sqrt[n]{N}}$, $0 \le k \le L-1$ and $1 \le i \le m$.. The formula for $D_n(u, h)$ in Lemma 2 with $s = 0$ gives L equations of the form

$$N\left(D_n(u,h[k]) - \sum_{j=1}^{n}(-1)^{n-j}\binom{n}{j}j^n\frac{q_1^{kn}}{N}\cdot\epsilon_u(jh[k])\right)$$
$$= \sum_{|\alpha|=n}\binom{n}{\alpha_1,\dots,\alpha_m}q_1^{k\alpha_1}\cdots q_m^{k\alpha_m}f_\alpha(u).$$

The coefficient matrix for this system, thinking of the $\binom{n}{\alpha_1,\dots,\alpha_m}f_\alpha(u)$ as the unknowns, is the Vandermonde matrix constructed using $\{q_1^{\alpha_1}\cdots q_m^{\alpha_m} \mid 0 \le k \le L-1\}$, that is, the entry in the ith row jth column is $q_1^{(i-1)\alpha_1}\cdots q_m^{(i-1)\alpha_m}$ where α is the jth index with $|\alpha| = n$. Since $\alpha \ne \alpha'$ implies $q_1^{\alpha_1}\cdots q_m^{\alpha_m} - q_1^{\alpha'_1}\cdots q_m^{\alpha'_m} \ne 0$, the determinant Δ of this matrix will be nonzero. By Cramer's Rule each $\binom{n}{\alpha_1,\dots,\alpha_m}f_\alpha(u)$ is of the form

$$\frac{1}{\Delta}\sum_{k=0}^{L-1}N\left(D_n(u,h[k]) - \sum_{j=1}^{n}(-1)^{n-j}\binom{n}{j}j^n\frac{q_1^{kn}}{N}\cdot\epsilon_u(jh[k])\right)\Delta_k$$

where each Δ_k is the appropriate cofactor in the expansion of Δ about the $(k+1)st$ column. As $N \to \infty$, $\|h[k]\| \to 0$ so $\sum\limits_{j=1}^{n}(-1)^{n-j}\binom{n}{j}j^n q_1^{kn}\cdot\epsilon_u(jh[k]) \to 0$. Therefore each $f_\alpha(u)$ is a pointwise limit of the sequence of continuous functions $\left\{\frac{1}{\Delta}\sum\limits_{k=0}^{L-1}ND_n(u,h[k])\Delta_k\right\}$ and is thus Baire 1. $\square$

We will also need the following form of $D_n(u, h)$ involving a second point x.

Lemma 4 *Let f be $(n + s(u))$-times Peano bounded at u. Then*

$$D_n(u,h) = \sum_{|\alpha|=n}\binom{n}{\alpha_1,\dots,\alpha_m}h_1^{\alpha_1}\cdots h_m^{\alpha_m}f_\alpha(x)$$
$$+ \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}||u-x+jh||^{n+s(x)}\cdot\epsilon_x(u-x+jh).$$

Proof. Expanding as we did in the proof of Lemma 2, we get

$$D_n(u,h) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(u+jh) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(x+u-x+jh) =$$

$$\sum_{j=0}^{n}(-1)^{n-j}\tbinom{n}{j}\left(\sum_{i=0}^{n}\sum_{|\alpha|=i}\tbinom{i}{\alpha_1,\ldots,\alpha_m}\prod_{\ell=1}^{m}(u_\ell - x_\ell + jh_\ell)^{\alpha_\ell}\tfrac{f_\alpha(x)}{i!}\right)+$$

$$\sum_{j=0}^{n}(-1)^{n-j}\tbinom{n}{j}||u-x+jh||^{n+s(x)}\cdot\epsilon_x(u-x+jh) = T_1+T_2.$$

The term T_2 is as desired. We rearrange the triple sum T_1 to get $T_1 =$

$$\sum_{i=0}^{n}\sum_{|\alpha|=i}\left(\sum_{j=0}^{n}(-1)^{n-j}\tbinom{n}{j}\tbinom{i}{\alpha_1,\ldots,\alpha_m}\prod_{\ell=1}^{m}(u_\ell - x_\ell + jh_\ell)^{\alpha_\ell}\tfrac{f_\alpha(x)}{i!}\right).$$

If we write each $(u_l - x_l + jh_l)^{\alpha_l}$ as $\sum\limits_{p_l=0}^{\alpha_l}\binom{\alpha_l}{p_l}(u_l-x_l)^{\alpha_l-p_l}j^{p_l}h_i^{p_l}$, Then $T_1 =$

$$\sum_{i=0}^{n}\sum_{|\alpha|=i}\left(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{i}{\alpha_1,\ldots,\alpha_m}\right.$$

$$\left.\prod_{\ell=1}^{m}\sum_{p_\ell=0}^{\alpha_\ell}\binom{\alpha_\ell}{p_\ell}(u_\ell-x_\ell)^{\alpha_\ell-p_\ell}j^{p_\ell}h_i^{p_\ell}\right)\frac{f_\alpha(x)}{i!}.$$

When the product inside is expanded and then summed over $j = 0,...,n$ the only nonzero term, by Lemma 1, will be the term containing j^n. This happens exactly when $\sum\limits_{l=1}^{m} p_l = n$, that is, when each $p_l = \alpha_l$, and $|\alpha| = n$. We then obtain $T_1 =$

$$\sum_{|\alpha|=n}\left(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{n}{\alpha_1,\ldots,\alpha_m}\prod_{l=1}^{m}j^{\alpha_l}h_i^{\alpha_l}\right)\frac{f_\alpha(x)}{n!} =$$

$$\sum_{|\alpha|=n}\left(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}j^n\right)\binom{n}{\alpha_1,\ldots,\alpha_m}h_1^{\alpha_1}\cdots h_m^{\alpha_m}\frac{f_\alpha(x)}{n!} =$$

$$\sum_{|\alpha|=n}\binom{n}{\alpha_1,\ldots,\alpha_m}h_1^{\alpha_1}\cdots h_m^{\alpha_m}f_\alpha(x)$$

by Lemma 1. □

The main result of the paper is the following Theorem.

Theorem 5 *Let $f:\mathbb{R}^m\to\mathbb{R}$ be $(n+s)$-times Peano bounded on $\mathbb{R}^m$ and let $A_N=\{x|\ |\epsilon_x(h)|\le N \text{for all } 0<\|h\|<\frac{1}{N} \text{ and } s(x)\ge\frac{1}{N}\}$. Then there is a constant K so that whenever $u,x\in\overline{A_N}$ and $\|u-x\|<\frac{1}{(1+n)N}$ we have $|f_\alpha(u)-f_\alpha(x)|\le\|u-x\|^{\frac{1}{N}}K$ for all $|\alpha|=n$.*

Proof. Let u, $x\in A_N$ with $\|u-x\|<\frac{1}{(1+n)N}$. By Lemma 2, $D_n(u,h)=$

$$\sum_{|\alpha|=n}\binom{n}{\alpha_1,\ldots,\alpha_m}h_1^{\alpha_1}\cdots h_m^{\alpha_m}f_\alpha(u)+\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}||jh||^{n+s(u)}\cdot\epsilon_u(jh). \tag{4}$$

By Lemma 4 we also have

$$D_n(u,h)=\sum_{|\alpha|=n}\binom{n}{\alpha_1,\ldots,\alpha_m}h_1^{\alpha_1}\cdots h_m^{\alpha_m}f_\alpha(x) \tag{5}$$

$$+\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}||u-x+jh||^{n+s(x)}\cdot\epsilon_x(u-x+jh). \tag{6}$$

For $\|u-x\|\le\|h\|<\frac{1}{(1+n)N}$ the error terms in (4) and (6) are uniformly bounded on A_N and

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\Big(||jh||^{n+s(u)}\cdot\epsilon_u(jh)-||u-x+jh||^{n+s(x)}\cdot\epsilon_x(u-x+jh)\Big)$$

may be written as $||h||^{n+\frac{1}{N}}\cdot\varepsilon(h)$ where $|\varepsilon(h)|\le H$ for some constant H depending only on N. Equating (4) and (6) gives

$$\sum_{|\alpha|=n}\binom{n}{\alpha_1,\ldots,\alpha_m}h_1^{\alpha_1}\cdots h_m^{\alpha_m}(f_\alpha(u)-f_\alpha(x))=||h||^{n+\frac{1}{N}}\varepsilon(h). \tag{7}$$

As in the proof of Theorem 3 pick m distinct primes $q_1>\ldots>q_m$ and generate L vectors $\{h[k]\}_{k=0}^{L-1}$ by setting each $h[k]_i=\dfrac{||u-x||q_i^k}{q_1^k}$, $0\le k\le L-1$. Substitution in equation (7) gives a system of L equations in the L unknowns $f_\alpha(u)-f_\alpha(x)$. By Cramer's Rule each $f_\alpha(u)-f_\alpha(x)=\|u-x\|^{\frac{1}{N}}\Delta'\Delta$ where Δ is the Vandermonde determinant as in Theorem 3 and Δ' is bounded and the bound depends only on N. Thus $|f_\alpha(u)-f_\alpha(x)|\le\|u-x\|^{\frac{1}{N}}K$ where $K=\frac{\Delta'}{\Delta}$. It remains to show that the constant K holds for $u,x\in\overline{A}_N$ with $\|u-x\|<\frac{1}{(1+n)N}$. To see this, let u and x be in $\overline{A}_N$ and pick $u_n,x_n\in A_N$ approaching u and x respectively. The calculation above shows that $|f_\alpha(x_n)-f_\alpha(x)|\le$

$\|x_n - x\|^{s_x} H_x$ for x_n sufficiently close to x, where $s_x = \min\{s(x), \frac{1}{N}\}$ but H_x depends on x as well as N. A similar inequality holds for u_n and u. Then

$$\begin{aligned}|f_\alpha(u) - f_\alpha(x)| &\le |f_\alpha(u) - f_\alpha(u_n)| + |f_\alpha(u_n) - f_\alpha(x_n)| + |f_\alpha(x_n) - f_\alpha(x)| \\ &\le \|u - u_n\|^{s_u} H_u + \|u_n - x_n\|^{\frac{1}{N}} K + \|x_n - x\|^{s_x} H_x.\end{aligned}$$

Letting $n \to \infty$ we obtain $|f_\alpha(u) - f_\alpha(x)| \le \|u - x\|^{\frac{1}{N}} K$ as desired. □

Corollary 6 *Let $f : \mathbb{R}^m \to \mathbb{R}$ be $(n+s)$-times Peano bounded on $\mathbb{R}^m$. Then the nth order Peano derivatives of f are Baire*1.*

Proof. Let C be a closed subset of $\mathbb{R}^m$. Since $\bigcup_{N=1}^{\infty} \overline{A}_N = \mathbb{R}^m$, by the Baire Category Theorem there is an open interval $I \in \mathbb{R}^m$ and an integer N such that $C \cap I$ is nonempty and contained in $\overline{A}_N$. By Theorem 5, the nth order Peano derivatives of f restricted to $\overline{A}_N$ are continuous. Therefore, the restriction of the derivatives to the set $C \cap I$ are also continuous. □

The next corollary follows easily from the Baire*1 property.

Corollary 7 *Let $f : \mathbb{R}^m \to \mathbb{R}$ be $(n+s)$-times Peano bounded on $\mathbb{R}^m$. Then there is a dense open set $G \subset \mathbb{R}^m$ such that the nth order Peano derivatives of f are continuous on G.*

Lastly, we summarize these results for $(n+1)$-times Peano differentiable functions.

Corollary 8 *Let $f : \mathbb{R}^m \to \mathbb{R}$ be $(n+1)$-times Peano differentiable on $\mathbb{R}^m$. Then the nth order Peano derivatives of f are Baire*1 and are thus continuous on some dense open set $G \subset \mathbb{R}^m$.*

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