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A FIRST RETURN EXAMINATION OF THE LEBESGUE INTEGRAL*

Abstract

It is shown that a Lebesgue integrable function comes equipped with a sequence of points which one can use in conjunction with a simple "first return – Riemann" integration procedure to compute the integral.

First return limiting processes have yielded interesting insights into generalized derivatives [2, 5, 9, 11, 13] and have given rise to new characterizations of the class of Baire one ($\mathcal{B}_1$) functions [1, 8, 12], as well as several standard subclasses of $\mathcal{B}_1$ [3, 4, 6, 7, 10]. Thus, it seems natural to investigate whether a first return technique might be available for computing Lebesgue integrals. The goal of this paper is to prove the following theorem, which shows that such a procedure is, indeed, available and is closely akin to that of Riemann integration.

Theorem 1. *Suppose $f : \mathbb{I}^n \to \mathbb{R}$ is a Lebesgue-integrable function. Then there is a countable dense set D in $\mathbb{I}^n$ and an enumeration $(x_p : p \in \mathbb{N})$ of D such that for each $\epsilon > 0$ there is a $\delta > 0$ such that if $\mathcal{P}$ is a partition of $\mathbb{I}^n$ having norm less than δ, then*

$$\left| \sum_{J \in \mathcal{P}} f(r(J))|J| - \int_{\mathbb{I}^n} f \right| < \epsilon,$$

where $r(J)$ denotes the first element of the sequence (x_p) that belongs to J.

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Before proving this result, we need to establish some notation and verify an elementary lemma which will be used repeatedly in the proof of the theorem. Throughout this work the dimension n of our Euclidean space $\mathbb{R}^n$ is fixed and $\mathbb{I}^n$ denotes the unit "square" in $\mathbb{R}^n$; that is, $\mathbb{I}^n$ is the cartesian product of n copies of the unit interval $[0, 1]$. We shall use $\lambda(A)$ to denote the Lebesgue n-dimensional measure of a measurable set $A \subseteq \mathbb{R}^n$ and shall use ∂S and S^o to denote the boundary and interior, respectively, of a set in $S \subseteq \mathbb{R}^n$. By a "rectangle" we mean a set J of the form

$$J = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

where each $a_i < b_i$; we call each $[a_i, b_i]$ a "side" of J.

A partition $\mathcal{P}$ of $\mathbb{I}^n$ is a finite collection of non-overlapping rectangles whose union is $\mathbb{I}^n$. (By non-overlapping, we mean that if $J_1 \neq J_2$ belong to $\mathcal{P}$, then $\lambda(J_1 \cap J_2) = 0$.) An elementary fact that we shall use in the proof of the lemma is that no point of $\mathbb{I}^n$ belongs to more than 2^n rectangles $J \in \mathcal{P}$. The norm of $\mathcal{P}$, $\|\mathcal{P}\|$, is the maximum of the lengths of the sides of all of the $J \in \mathcal{P}$.

Let $i \in \mathbb{N}$ and for each $j = 0, 1, \ldots, 2^i$, let $c_j = \frac{j}{2^i}$. The uniform i-partition of $\mathbb{I}^n$, $\mathcal{Q}_i$, is the collection of all rectangles of the form

$$[c_{j_1}, c_{j_1+1}] \times [c_{j_2}, c_{j_2+1}] \times \cdots \times [c_{j_n}, c_{j_n+1}],$$

where each integer j_k satisfies $0 \leq j_k < 2^i$. If $A \subseteq B \subset \mathbb{I}^n$, we say that A is i-fine in B provided that for each $J \in \mathcal{Q}_i$ for which $J^o \cap B \neq \emptyset$, it follows that $J^o \cap A \neq \emptyset$.

We shall let $B(n)$ denote the number of $(n-1)$-dimensional rectangles of $(n-1)$-dimensional measure one which form the boundary of $\mathbb{I}^n$. In proving the lemma we shall make use of the elementary fact that if $J \subseteq \mathbb{I}^n$ is any rectangle, then the number of elements of $\mathcal{Q}_i$ which intersect the boundary of J is at most $B(n) \cdot (2^i)^{n-1}$.

Lemma 1. [The Blocking Lemma] *Let $A \subset \mathbb{I}^n$ be measurable, let F be a finite subset of $\mathbb{I}^n \setminus A$ and let $\eta > 0$. Then there is a finite subset $S_A \subset A$ such that if $\mathcal{P}$ is any partition of $\mathbb{I}^n$ and*

$$\mathcal{G} = \{J \in \mathcal{P} : J \cap F \neq \emptyset \text{ and } J \cap S_A = \emptyset\},$$

then $\lambda\left(A \cap \bigcup_{J \in \mathcal{G}} J\right) < \eta$.

PROOF: Let $F = \{y_1, y_2, \ldots, y_K\}$, A, and η be as described. We may assume that A has positive measure. Choose $i \in \mathbb{N}$ so large that $\frac{K \cdot B(n) \cdot 2^n}{2^i} < \eta$. Let $\mathcal{Q}_i$ denote the uniform i-partition of $\mathbb{I}^n$. For each $I \in \mathcal{Q}_i$ which intersects A, select a point $s_I \in I \cap A$. Let S_A denote the collection of all such selected points.

Now let $\mathcal{P}$ be a partition of $\mathbb{I}^n$ and let

$$\mathcal{G} = \{J \in \mathcal{P} : J \cap F \neq \emptyset \text{ and } J \cap S_A = \emptyset\}.$$

Fix a $y_t \in F$ and fix a $J \in \mathcal{G}$ containing y_t, if such a J exists. There can be at most 2^n such J's containing y_t. Without loss of generality suppose that $A \cap J$ has positive measure. Since $J \cap S_A = \emptyset$, if $I \in \mathcal{Q}_i$ and $I \subseteq J$, then $I \cap A = \emptyset$. Thus, if $I \in \mathcal{Q}_i$ satisfies $I \cap (J \cap A) \neq \emptyset$, then $I \cap \partial J \neq \emptyset$. However, there are at most $B(n) \cdot (2^i)^{n-1}$ such $I \in \mathcal{Q}_i$. Hence,

$$\lambda\left(A \cap \bigcup_{J \in \mathcal{G}} J\right) < 2^n \cdot K \cdot B(n) \cdot (2^i)^{n-1} \frac{1}{(2^i)^n} = \frac{K \cdot B(n) \cdot 2^n}{2^i} < \eta,$$

completing the proof. □

Proof of Theorem. For each $j \in \mathbb{N}$ we set

$$A_j = \{x : j-1 \leq |f(x)| < j\},$$

and note that since f is integrable, the series $\sum_{j=1}^{\infty} j\lambda(A_j)$ converges. It will be convenient to denote the tails of this series by $\zeta_j = \sum_{k=j+1}^{\infty} k\lambda(A_k)$.

For each j we use Lusin's Theorem repeatedly to obtain a sequence, $\{A_j^i\}$, of pairwise disjoint, perfect subsets of A_j such that $\lambda(A_j^i) = \frac{\lambda(A_j)}{2^i}$ and the restriction of f to A_j^i, $f|A_j^i$, is continuous. Thus, for each j we have

$$\lambda(A_j) = \sum_{i=1}^{\infty} \lambda(A_j^i).$$

Also, for each j we set

$$B_j = \bigcup_{k=1}^{j} \bigcup_{i=1}^{j} A_k^i, \quad C_j = \bigcup_{k=j+1}^{\infty} A_k, \quad \text{and} \quad D_j = \bigcup_{k=1}^{j} \bigcup_{i=j+1}^{\infty} A_k^i,$$

and note that $\lambda(B_j) + \lambda(C_j) + \lambda(D_j) = 1$. Furthermore, we set $B_j^* = B_j \setminus B_{j-1}$, where we take $B_0 = \emptyset$. Note that for each j, $f|B_j$ is continuous and is in absolute value less than j. For each $j \in \mathbb{N}$, apply Tietze's extension theorem to obtain f_j as a continuous extension of $f|B_j$ to all of $\mathbb{I}^n$ with $|f_j(x)| < j$ for all $x \in \mathbb{I}^n$. For each $j \in \mathbb{N}$ let $\epsilon_j = \frac{1}{2^j}$ and let δ_j be a positive number such that δ_j witnesses the Riemann integrability of f_j over $\mathbb{I}^n$ with respect to ϵ_j; that is, if $\mathcal{P}$ is a partition of $\mathbb{I}^n$ with norm less than δ_j, and for each $J \in \mathcal{P}$, $s(J)$ denotes any point in J, then

$$\left| \sum_{J \in \mathcal{P}} f_j(s(J))|J| - \int_{\mathbb{I}^n} f_j \right| < \epsilon_j. \tag{1}$$

Our next goal is to inductively by stages define the sequence $(x_p : p \in \mathbb{N})$. At stage 1, we choose a finite set $S \subset B_1$ so that S is 1-fine in B_1. We list these points in any order as $x_1, x_2, \ldots, x_{p_1}$. Now, suppose stage j has been completed with $x_1, x_2, \ldots, x_{p_j}$ having been selected and ordered. We proceed to stage $j+1$. First, select a finite subset $S_{j+1} \subset B^*_{j+1}$ such that S_{j+1} is $(j+1)$-fine in B^*_{j+1}. We are going to apply the blocking lemma j times, each time taking $\eta = \frac{1}{(j+1)^2 2^{j+1}}$. Initially, apply the blocking lemma with $F = S_{j+1}$ and $A = B^*_j$ to determine a finite subset $S_j \subset B^*_j$ which satisfies the conclusion of that lemma. We may clearly assume that S_j is $(j+1)$-fine in B^*_j and contains no $x_p, p \leq p_j$, since all of the sets A^i_k are perfect. Next, assume that

$$S_j \subset B^*_j,\ S_{j-1} \subset B^*_{j-1}, \ldots, S_{j-k} \subset B^*_{j-k}$$

have been selected for some $0 \leq k \leq j-2$. Apply the blocking lemma with $F = \bigcup_{i=-1}^{k} S_{j-i}$, $A = B^*_{j-k-1}$, to yield a finite set $S_{j-k-1} \subset B^*_{j-k-1}$. Again, we may assume that S_{j-k-1} is $(j+1)$-fine in B^*_{j-k-1} and contains no $x_p, p \leq p_j$. We do this for each $0 \leq k \leq j-2$. We now complete stage $j+1$ by appending the points from $\bigcup_{k=-1}^{j-1} S_{j-k}$ to $(x_1, x_2, \ldots, x_{p_j})$, first appending those from S_1 (in any order), then those from S_2 (in any order), $\ldots$, and finally those from S_{j+1}. This completes stage $j+1$ and we have defined $x_1, x_2, \ldots, x_{p_j}, x_{p_j+1}, \ldots, x_{p_{j+1}}$.

Once all stages have been carried out, the sequence $(x_p : p \in \mathbb{N})$ has been completely specified and it remains to show that this sequence accomplishes what the theorem claims. First, note that if $D = \{x_p : p \in \mathbb{N}\}$, then D is clearly dense in $\mathbb{I}^n$.

Before proceeding to see that the rest of the conclusion holds, we wish to make an additional observation. Fix a $j \in \mathbb{N}$ and let $\mathcal{P}$ be any partition of $\mathbb{I}^n$. If $k \in \mathbb{N}$ and

$$\mathcal{G}_k = \{J \in \mathcal{P} : r(J) \notin B_j, \text{ and } r(J) \text{ was appended during stage } j+k\},$$

then

$$\lambda\left(B_j \cap \bigcup_{J\in\mathcal{G}_k} J\right) = \sum_{i=1}^{j} \lambda\left(B_i^* \cap \bigcup_{J\in\mathcal{G}_k} J\right) < j\cdot\frac{1}{(j+k)^2 2^{j+k}} < \frac{1}{(j+k)2^{j+k}}. \tag{2}$$

Now, let $\epsilon > 0$, choose j so large that $\frac{5j+12}{2^j}+4\zeta_j < \epsilon$, and set $\delta = \delta_j$. Let $\mathcal{P}$ be any partition of $\mathbb{I}^n$ having norm less than δ. Let $\mathcal{P}_1 = \{J \in \mathcal{P} : r(J) \in B_j\}$ and $\mathcal{P}_2 = \mathcal{P} \setminus \mathcal{P}_1$. Then, adopting the notation $\bigcup \mathcal{P}_1$ for the union of all the J's in $\mathcal{P}_1$, we have

$$\begin{aligned}\left|\sum_{J\in\mathcal{P}} f(r(J))|J| - \int_{\mathbb{I}^n} f\right| &\le \left|\sum_{J\in\mathcal{P}_1} f_j(r(J))|J| - \int_{\bigcup\mathcal{P}_1} f\right| \\ &\quad + \left|\sum_{J\in\mathcal{P}_2} f(r(J))|J| - \int_{\bigcup\mathcal{P}_2} f\right| \\ &\le \left|\sum_{J\in\mathcal{P}_1} f_j(r(J))|J| - \int_{\bigcup\mathcal{P}_1} f_j\right| \\ &\quad + \left|\int_{\bigcup\mathcal{P}_1} (f_j - f)\right| + \sum_{J\in\mathcal{P}_2} |f(r(J))||J| \\ &\quad + \int_{\bigcup\mathcal{P}_2} |f|.\end{aligned} \tag{3}$$

We shall obtain estimates on each of the four terms on the right hand side of the final inequality.

For each $J \in \mathcal{P}_2$, employ the mean value theorem to select a point $s_J \in J$ such that $f_j(s_J)|J| = \int_J f_j$. Also, for each $J \in \mathcal{P}_1$, set $s_J = r(J)$. Then

$$\left|\sum_{J\in\mathcal{P}_1} f_j(r(J))|J| - \int_{\bigcup\mathcal{P}_1} f_j\right| = \left|\sum_{J\in\mathcal{P}} f_j(s_J)|J| - \int_{\mathbb{I}^n} f_j\right| < \epsilon_j = \frac{1}{2^j}, \tag{4}$$

where the inequality follows from (1) and the fact that $\|\mathcal{P}\| < \delta_j$.

Next,

$$\begin{aligned}\left|\int_{\bigcup \mathcal{P}_1}(f_j - f)\right| \le & \left|\int_{B_j \cap \bigcup \mathcal{P}_1}(f_j - f)\right| + \left|\int_{C_j \cap \bigcup \mathcal{P}_1}(|f_j| + |f|)\right| \\ & + \left|\int_{D_j \cap \bigcup \mathcal{P}_1}(|f_j| + |f|)\right| \\ \le & 0 + \int_{C_j}(|f| + j) + \int_{D_j}(|f| + j).\end{aligned} \tag{5}$$

Now,

$$\begin{aligned}\int_{C_j}(|f| + j) = & \sum_{k=j+1}^{\infty}\int_{A_k}(|f| + j) \le \sum_{k=j+1}^{\infty}(k + j)\lambda(A_k) \\ \le & \sum_{k=j+1}^{\infty} 2k\lambda(A_k) = 2\zeta_j,\end{aligned} \tag{6}$$

and

$$\begin{aligned}\int_{D_j}(|f| + j) = & \sum_{k=1}^{j}\sum_{i=j+1}^{\infty}\int_{A_k^i}(|f| + j) \le \sum_{k=1}^{j}\sum_{i=j+1}^{\infty}(k + j)\lambda(A_k^i) \\ \le & \sum_{k=1}^{j}\sum_{i=j+1}^{\infty}(k + j)\frac{\lambda(A_k)}{2^i} \le 2j\sum_{k=1}^{j}\lambda(A_k)\sum_{i=j+1}^{\infty}\frac{1}{2^i} \\ = & \frac{2j}{2^j}\sum_{k=1}^{j}\lambda(A_k) \le \frac{2j}{2^j}.\end{aligned} \tag{7}$$

Thus, from (5), (6), and (7) we have

$$\left|\int_{\bigcup \mathcal{P}_1}(f_j - f)\right| \le 2\zeta_j + \frac{2j}{2^j}. \tag{8}$$

Next we turn our attention to $\sum_{J \in \mathcal{P}_2} |f(r(J))||J|$. For each $i \in \mathbb{N}$, let

$$\mathcal{P}_{2,i} = \{J \in \mathcal{P}_2 : r(J) \in B_{j+i}^*\}.$$

Then $\sum_{J\in\mathcal{P}_2}|f(r(J))||J| = \sum_{i=1}^{\infty}\sum_{J\in\mathcal{P}_{2,i}}|f(r(J))||J|$. Now,

$$\begin{aligned}\sum_{J\in\mathcal{P}_{2,i}}|f(r(J))||J| = &\sum_{J\in\mathcal{P}_{2,i}}|f(r(J))|\lambda(J\cap B_{j+i-1})\\ &+\sum_{J\in\mathcal{P}_{2,i}}|f(r(J))|\lambda(J\cap D_{j+i-1})\\ &+\sum_{J\in\mathcal{P}_{2,i}}|f(r(J))|\lambda(J\cap C_{j+i-1})\\ \leq&(j+i)\lambda(\cup\mathcal{P}_{2,i}\cap B_{j+i-1})+(j+i)\lambda(\cup\mathcal{P}_{2,i}\cap D_{j+i-1})\\ &+(j+i)\lambda(\cup\mathcal{P}_{2,i}\cap C_{j+i-1}).\end{aligned}\tag{9}$$

Keeping in mind that $r(J)$ could have been appended to the (x_p) sequence during any stage $j+i+m$, $m=0,1,\ldots$, we have from (2) that

$$\lambda(\cup\mathcal{P}_{2,i}\cap B_{j+i-1})\leq\sum_{m=0}^{\infty}\frac{1}{(j+i+m)2^{j+i+m}}\leq\frac{2}{(j+i)2^{j+i}}.\tag{10}$$

Next,

$$\begin{aligned}\lambda(\cup\mathcal{P}_{2,i}\cap D_{j+i-1})\leq\lambda(D_{j+i-1})\leq&\sum_{k=1}^{j+i-1}\sum_{m=0}^{\infty}\lambda(A_k^{j+i+m})\\ \leq&\sum_{k=1}^{j+i-1}\frac{2\lambda(A_k)}{2^{j+i}}\leq\frac{2}{2^{j+i}}.\end{aligned}\tag{11}$$

From (9), (10), and (11), we obtain

$$\sum_{J\in\mathcal{P}_{2,i}}|f(r(J))||J|\leq\frac{2}{2^{j+i}}+(j+i)\frac{2}{2^{j+i}}+(j+i)\lambda(\cup\mathcal{P}_{2,i}\cap C_{j+i-1})$$

Consequently,

$$\begin{aligned}\sum_{J\in\mathcal{P}_2}|f(r(J))||J|\leq&\sum_{i=1}^{\infty}\frac{2}{2^{j+i}}+\sum_{i=1}^{\infty}(j+i)\frac{2}{2^{j+i}}\\ &+\sum_{i=1}^{\infty}(j+i)\lambda(\cup\mathcal{P}_{2,i}\cap C_{j+i-1})\\ =&\frac{2}{2^j}+\frac{2(j+4)}{2^j}+\sum_{i=1}^{\infty}(j+i)\lambda(\mathcal{P}_{2,i}\cap C_{j+i-1})\\ =&\frac{2j+10}{2^j}+\sum_{i=1}^{\infty}(j+i)\lambda(\cup\mathcal{P}_{2,i}\cap C_{j+i-1}).\end{aligned}$$

Keeping in mind that the sets $\cup\mathcal{P}_{2,i}$ and $\cup\mathcal{P}_{2,i'}$ are disjoint for $i \neq i'$, that $C_{j+i-1} = \cup_{k=j+i}^{\infty} A_k$, and hence that $A_{j+m} \cap C_{j+i-1} = \emptyset$ for $i > m$, it is readily seen that

$$\sum_{i=1}^{\infty}(j+i)\lambda(\cup\mathcal{P}_{2,i} \cap C_{j+i-1}) \leq \sum_{i=1}^{\infty}(j+i)\lambda(A_{j+i}) = \zeta_j.$$

Thus,

$$\sum_{J\in\mathcal{P}_2} |f(r(J))||J| \leq \frac{2j+10}{2^j} + \zeta_j. \tag{12}$$

Next, from the right hand side of (3) we consider the term

$$\begin{aligned}\int_{\cup\mathcal{P}_2} |f| &= \int_{\cup\mathcal{P}_2\cap B_j} |f| + \int_{\cup\mathcal{P}_2\cap C_j} |f| + \int_{\cup\mathcal{P}_2\cap D_j} |f| \\ &\leq j\lambda(\cup\mathcal{P}_2 \cap B_j) + \int_{C_j} |f| + \int_{D_j} |f|.\end{aligned} \tag{13}$$

Keeping in mind that for $J \in \mathcal{P}_2$ we know that $r(J) \notin B_j$ and hence was appended to the sequence (x_p) at some stage $(j+i)$, $i \in \mathbb{N}$, we observe from (2) that

$$j\lambda(\cup\mathcal{P}_2 \cap B_j) \leq j\sum_{i=1}^{\infty} \frac{1}{(j+i)2^{j+i}} < \sum_{i=1}^{\infty} \frac{1}{2^{j+i}} = \frac{1}{2^j} \tag{14}$$

Next,

$$\int_{C_j} |f| = \sum_{k=j+1}^{\infty} \int_{A_k} |f| \leq \sum_{k=j+1}^{\infty} k\lambda(A_k) = \zeta_j, \tag{15}$$

and

$$\begin{aligned}\int_{D_j} |f| &= \sum_{k=1}^{j} \sum_{i=j+1}^{\infty} \int_{A_k^i} |f| \leq \sum_{k=1}^{j} \sum_{i=j+1}^{\infty} k\lambda(A_k^i) \leq \sum_{k=1}^{j} \sum_{i=j+1}^{\infty} k\frac{\lambda(A_k)}{2^i} \\ &\leq j\sum_{k=1}^{j} \lambda(A_k) \sum_{i=j+1}^{\infty} \frac{1}{2^i} = \frac{j}{2^j}\sum_{k=1}^{j} \lambda(A_k) \leq \frac{j}{2^j}.\end{aligned} \tag{16}$$

Thus, from (13), (14), (15), and (16) we obtain

$$\int_{\cup\mathcal{P}_2} |f| \leq \frac{1}{2^j} + \zeta_j + \frac{j}{2^j} = \frac{j+1}{2^j} + \zeta_j. \tag{17}$$

Combining (3), (4), (8), (12), and (17), we obtain

$$\begin{aligned}\left|\sum_{J\in\mathcal{P}} f(r(J))|J| - \int_{\mathbb{I}^n} f\right| \le \left(\frac{1}{2^j}\right) + \left(2\zeta_j + \frac{2j}{2^j}\right) \\ + \left(\frac{2j+10}{2^j} + \zeta_j\right) + \left(\frac{j+1}{2^j} + \zeta_j\right) \\ = \frac{5j+12}{2^j} + 4\zeta_j < \epsilon,\end{aligned}$$

and this inequality completes the proof. □

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