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CONGRUENCES ON $\mathfrak{N}$-SEMIGROUPS

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The study of a semigroup in terms of its congruence relations has been used many times in the past. In the case of $\mathfrak{N}$-semigroups Tamura initiated this study with a paper [9] determining the $\mathfrak{N}$-congruences of an $\mathfrak{N}$-semigroup. Recently, Dickinson [4] has determined the congruences which correspond to homomorphic images having no idempotents as refinements of $\mathfrak{N}$-congruences. Group congruences on a commutative semigroup have been determined by Tamura and the author [11], but here they are determined for an $\mathfrak{N}$-semigroup from the group of quotients of the $\mathfrak{N}$-semigroup. $\mathfrak{N}$-congruences and group congruences on $\mathfrak{N}$-semigroups are of fundamental importance in characterizing $\mathfrak{N}$-semigroups. In this paper we make a study of all types of congruences on $\mathfrak{N}$-semigroups.

1. Introduction. All semigroups in this paper are commutative semigroups, and all undefined terms may be found in [3].

If S is a semigroup then we will let $\mathscr{L}(S)$ denote the lattice of congruence relations on S. The universal relation and the equality relation on a semigroup S will be denoted by ω and i, respectively, or by ω_S and i_S if we wish to indicate the semigroup S.

The following is a well known theorem concerning congruences.

THEOREM 1.1. *Let $\rho \in \mathscr{L}(S)$. Then the set of congruences on S containing ρ form a sublattice of $\mathscr{L}(S)$ which is isomorphic onto $\mathscr{L}(S/\rho)$.*

Let $\mathscr{P}$ be some property or condition on semigroups. If $\rho \in \mathscr{L}(S)$ is such that S/ρ has property $\mathscr{P}$ then ρ is called a $\mathscr{P}$-congruence on S, or a congruence on S of type $\mathscr{P}$.

If A is a subsemigroup of a semigroup S and $\rho \in \mathscr{L}(S)$ then we call $\rho \cap (A \times A)$ (denoted $\rho \mid A$) the restriction of ρ to A. Also, a congruence ρ is said to extend a congruence σ on A if $\rho \mid A = \sigma$.

We, also, have the following well known correspondence between homomorphisms and congruences.

(A) If $f: S \to T$ is a surjective homomorphism then $\rho_f = \{(x, y) \in S \times S: f(x) = f(y)\}$ is a congruence on S.

(B) If ρ is a congruence on S then $f_\rho(x) = x_\rho$ is a homomorphism of S onto S/ρ.

Then $\rho_{f_\rho} = \rho$ and if we identify S/ρ_f and T we have $f_{\rho_f} = f$.

The following is a useful theorem from [11] concerning the restriction of a group-congruence to an ideal.

THEOREM 1.2. *If ρ is a group-congruence on S and J is an ideal of S then $\sigma = \rho \mid J$ is a group-congruence on J and $S/\rho \cong J/\sigma$. Furthermore, ρ is the unique extension of σ to a group-congruence on S.*

COROLLARY 1.3. *The join semilattice of group-congruences on a semigroup S is isomorphic onto the join semilattice of group-congruences on any ideal J of S. The isomorphism is just the restriction of a group-congruence on S to the ideal J.*

We will use Z ($Z^+, Z^{+,0}, Q, Q^+, R, R^+$) to denote the additive semigroup of the integers (positive integers, nonnegative integers, rationals, positive rationals, reals, postive reals). Also, $Z^- = Z \backslash Z^{+,0}$ and $Z^{-,0} = Z \backslash Z^+$. We will also use $\Box$ to denote the empty set.

DEFINITION 1.1. A commutative semigroup S is said to be an archimedean semigroup if for each x and y in S there is $m \in Z^+$ and $z \in S$ such that $x^m = yz$.

A fundamental theorem due to Tamura and Kimura **[12]** in 1954 states that every commutative semigroup is a semilattice or archimedean semigroups. The fact that a commutative archimedean semigroup has at most one idempotent lead Tamura **[6]** to the following classification of them:

TYPE 1. Archimedean semigroups with an idempotent which is zero.

TYPE 2. Archimedean semigroups with a nonzero idempotent.

TYPE 3. Cancellative archimedean semigroups with no idempotents.

TYPE 4. Noncancellative archimedean semigroups with no idempotents.

A Type 1 semigroup is called a nil semigroup, and a Type 3 semigroup is called an $\mathfrak{N}$-semigroup. A semigroup of Type 2 has been shown **[6]** to be an ideal extension of the group of units by a nil semigroup. Thus we will call such a semigroup a GN-semigroup, and we will call a semigroup of Type 4 a $T4$-semigroup after Dickinson **[4]**.

Hence the study of commutative semigroups is reduced to the study of these four fundamental types of commutative semigroups and the study of how they are put together to form larger semigroups. Here we concern ourselves only with $\mathfrak{N}$-semigroups.

The following fundamental facts about commutative archimedean semigroups will be used later (see **[7]** for a proof).

Fact 1.1. If S is a commutative archimedean semigroup and $a \in S$ then $\bigcap_{n=1}^{\infty} a^n S$ is either empty or a kernel of S which is a group.

Fact 1.2. If S is a commutative archimedean semigroup without idempotent then for all $x, y \in S$ $x \neq xy$.

In 1957 Tamura gave the following characterization of $\mathfrak{R}$-semigroups [7].

THEOREM 1.4. *Let G be an abelian group and let I: $G \times G \to Z^{+,0}$ be a function such that for all α, β, and γ in G*

$$I(\alpha, \beta) = I(\beta, \alpha) \tag{1.1}$$

$$I(\alpha, \beta) + I(\alpha\beta, \gamma) = I(\alpha, \beta\gamma) + I(\beta, \gamma) \tag{1.2}$$

$$I(\epsilon, \alpha) = 1 \text{ where } \epsilon \text{ is the identity element of } G, \tag{1.3}$$

and

$$\text{for each } \alpha \in G \text{ there exists } m \in Z^+ \text{ such that } I(\alpha, \alpha^m) > 0. \tag{1.4}$$

Let $S = Z^{+,0} \times G$ and define an operation on S by

$$(m, \alpha)(n, \beta) = (m + n + I(\alpha, \beta), \alpha\beta)$$

where $m, n \in Z^{+,0}$ and $\alpha, \beta \in G$.

Then under this operation S becomes an $\mathfrak{R}$-semigroup. Conversely, every $\mathfrak{R}$-semigroup is isomorphic to an $\mathfrak{R}$-semigroup obtained in this way.

We use the notation $S = (G, I)$ to denote that S is the $\mathfrak{R}$-semigroup determined as above from the abelian group G and the function I: $G \times G \to Z^{+,0}$. The function I is sometimes called an I-function or an index-function. The group G is sometimes called a structure group of S.

More recently **[10]**, Tamura has given another characterization of an $\mathfrak{R}$-semigroup as a subdirect product of a positive real number additive semigroup and an abelian group. That is,

THEOREM 1.5. *Let G be an abelian group and let φ: $G \to R^+$ be a function such that*

$$\varphi(\epsilon) = 1 \text{ where } \epsilon \text{ is the identity element of } G. \tag{1.5}$$

$$\varphi(\alpha) + \varphi(\beta) - \varphi(\alpha\beta) \text{ is in } Z^{+,0} \text{ for all } \alpha, \beta \in G. \tag{1.6}$$

(1.7) *For all $\alpha \in G$ there exists $m \in Z^+$ such that*

$$\varphi(\alpha)+\varphi(\alpha^m)-\varphi(\alpha^{m-1})>0.$$

Let $S=\{(x,\alpha)\in R^+\times G\colon x-\varphi(\alpha)\in Z^{+,0}\}$. Then S is an $\mathfrak{N}$-semigroup under coordinatewise multiplication, and every $\mathfrak{N}$-semigroup is isomorphic onto an $\mathfrak{N}$-semigroup determined by some G and some φ, as above.

When we are using this characterization of an $\mathfrak{N}$-semigroup S we will write $S=(G,\varphi)$. If we are given $\varphi\colon G\to R^+$ satisfying conditions (1.5) thru (1.7) then we get an index-function $I\colon G\times G\to Z^{+,0}$ by defining $I(\alpha,\beta)=\varphi(\alpha)+\varphi(\beta)-\varphi(\alpha\beta)$ for all $\alpha,\beta\in G$. In this case we have the $\mathfrak{N}$-semigroup (G,I) is isomorphic to the $\mathfrak{N}$-semigroup (G,φ). We will sometimes abuse the notation and write $(G,I)=(G,\varphi)$, and we will work with the two characterizations simultaneously.

Let S be an $\mathfrak{N}$-semigroup then since S is commutative and cancellative it has a group of quotients $\mathcal{G}=\mathcal{G}(S)$ which is the smallest group into which we can embed S, in the following sense: If S can be embedded into a group G then $\mathcal{G}$ can be embedded in G. That is, $\mathcal{G}$ is the Grothendieck group (see [5]) of S. We can view $\mathcal{G}$ as $S\times S/\equiv$ where for x, y, u, and v in S $(x,y)\equiv(u,v)$ if and only if $xv=yu$. If $[x,y]_\equiv$ denotes the $\equiv$ – class of (x,y) we think of $[x,y]_\equiv$ as xy^{-1}. Then $xy^{-1}uv^{-1}=(xu)(yv)^{-1}$. In [8] Tamura shows that if $S=(G,I)$ is an $\mathfrak{N}$-semigroup then we can obtain $\mathcal{G}$ as the abelian group extension $(Z,G;f)$ of Z by G with respect to the factor system $f\colon G\times G\to Z$ defined by $f(\alpha,\beta)=I(\alpha,\beta)-1$. That is, $\mathcal{G}=Z\times G$ with the following operation: For $((m,\alpha)),((n,\beta))\in Z\times G$ let $((m,\alpha))((n,\beta))=((m+n+f(\alpha,\beta),\alpha\beta))$.

REMARK 1.1. We use the double parentheses $((\,,))$ to denote elements of $\mathcal{G}$ and single parentheses $(\,,)$ to denote elements of S.

$S=(G,I)$ is embedded in $\mathcal{G}=(Z,G\colon f)$ when $f=I-1$ by taking $(m,\alpha)\in S$ to $((m+1,\alpha))\in\mathcal{G}$. Thus S can be identified with its image, $\{((m,\alpha))\in\mathcal{G}\colon m>0\}$, in $\mathcal{G}$.

REMARK 1.2. The identity element of $\mathcal{G}$ is $((0,\epsilon))$ where ϵ is the identity element of $\mathcal{G}$, and the inverse of $((m,\alpha))$ is $((-m-f(\alpha,\alpha^{-1}),\alpha^{-1}))$.

In this paper we will be concerned with the study of $\mathfrak{N}$-semigroups in terms of their congruence realtions, equivalently in terms of their homomorphic images. Note that every homomorphic image of an archimedean semigroup is again an archimedean semigroup. Hence every factor semigroup of an $\mathfrak{N}$-semigroup is one of the four types listed earlier.

We now introduce some notation to denote certain subsets of $\mathcal{L}(S)$ when S is an $\mathfrak{N}$-semigroup.

$\mathscr{L}_C(S)$ = the set of cancellative congruences on S.
$\mathscr{L}_G(S)$ = the set of group-congruences on S.
$\mathscr{L}_{\mathfrak{N}}(S)$ = the set of $\mathfrak{N}$-congruences on S.
$\mathscr{L}_N(S)$ = the set of nil-congruences on S.
$\mathscr{L}_R(S)$ = the set of Rees-congruences on S.
$\mathscr{L}_{GN}(S)$ = the set of GN-congruences on S.
$\mathscr{L}_{T4}(S)$ = the set of $T4$-congruences on S.

2. Cancellative congruences. In this section we study $\mathscr{L}_C(S)$ where S is an $\mathfrak{N}$-semigroup and show that it is a complete modular sublattice of $\mathscr{L}(S)$. We also give three characterizations of the congruences in $\mathscr{L}_C(S)$. Here we also study the problem of when $\mathscr{L}_G(S)$ and $\mathscr{L}_{\mathfrak{N}}(S)$ are sublattices of $\mathscr{L}_C(S)$. This leads to a characterization of the power-joined $\mathfrak{N}$-semigroups.

As an immediate extension of Lemma 1.1 in **[9]** we have the following lemma.

LEMMA 2.1. *Let S and T be commutative cancellative semigroups and let h be a homomorphism of S onto T. Then h extends uniquely to a homomorphism of $\mathscr{G}(S)$ onto $\mathscr{G}(T)$. Furthermore, if $\bar{h}$ is a homomorphism of $\mathscr{G}(S)$ onto a group G then $\mathscr{G}(\bar{h}(S)) = G$.*

Proof. The first part of the lemma is proved exactly as Lemma 1.1 of **[9]**. That is, if $h: S \to T$ is a homomorphism then the unique extension of h to a homomorphism $\bar{h}$ from $\mathscr{G}(S)$ to $\mathscr{G}(T)$ is defined by $\bar{h}(ab^{-1}) = h(a)h(b)^{-1}$ for $a, b \in S$. To see the last statement of the theorem let $\bar{h}$: $\mathscr{G}(S) \to G$ be a surjective homomorphism. Let $g \in G$ then there exists $x, y \in S$ such that $\bar{h}(xy^{-1}) = g$. That is, $g = \bar{h}(x)\bar{h}(y)^{-1} \in \mathscr{G}(\bar{h}(S))$. Thus $G = \mathscr{G}(\bar{h}(S))$ and we are done.

Actually, if we remove the requirement of ontoness of the homomorphisms in Lemma 2.1 we see that the map $(S, h) \to (\mathscr{G}(S), \bar{h})$ is a covariant functor from the category of commutative cancellative semigroups into the category of abelian groups.

Lemma 2.1 tells us that $\mathscr{L}(\mathscr{G})$ is isomorphic onto $\mathscr{L}_C(S)$. That is, if $\bar{\rho} \in \mathscr{L}(\mathscr{G})$ corresponds to $\bar{h}$ and $\rho \in \mathscr{L}_C(S)$ corresponds to h then $\bar{\rho} \to \rho$ is an order preserving one-to-one map of $\mathscr{L}(\mathscr{G})$ onto $\mathscr{L}_C(S)$ whose inverse is also order preserving. Thus since $\mathscr{L}(\mathscr{G})$ is a lattice so is $\mathscr{L}_C(S)$. Recall that the lattice of congruences on a group is isomorphic onto the lattice of normal subgroups of the group and is therefore a modular lattice **[1]**. We, therefore, have the following theorem.

THEOREM 2.2. *If S is a commutative cancellative semigroup then $\mathscr{L}_C(S)$ is a complete modular sublattice of $\mathscr{L}(S)$ which is isomorphic onto $\mathscr{L}(\mathscr{G}(S))$.*

Since the only commutative cancellative archimedean semigroups are groups and $\mathfrak{N}$-semigroups, we have

$$\mathcal{L}_C(S) = \mathcal{L}_G(S) \cup \mathcal{L}_{\mathfrak{N}}(S)$$

if S is an $\mathfrak{N}$-semigroup. Tamura has shown **[9]** that $\mathcal{L}_{\mathfrak{N}}(S)$ corresponds to those subgroups of $\mathcal{G}(S)$ which do not intersect S. Such subgroups are called $\mathfrak{N}$-kernels and (**[9]** Theorem 1.13) they are characterized as follows.

LEMMA 2.3. *Let $S = (G, I)$ be an $\mathfrak{N}$-semigroup. Let H be a subgroup of G and let $h: H \to Z^{+,0}$ be a map such that $h(\alpha) + h(\beta) - h(\alpha\beta) = I(\alpha, \beta) - 1$ for all $\alpha, \beta \in H$. Then $K = \{((-h(\alpha), \alpha)) \in \mathcal{G}(S)\colon \alpha \in H\}$ is an $\mathfrak{N}$-kernel and all $\mathfrak{N}$-kernels are obtained in this way.*

REMARK 2.1. Notice that in Lemma 2.3 $K \cong H \subseteq G$ and so every structure group G of S contains an isomorphic copy of every $\mathfrak{N}$-kernel of S. That is, the $\mathfrak{N}$-kernels of S are invariants of the structure groups of S.

The congruence associated with H and h (i.e. with K) is given **[9]** as follows

$$(m, \alpha)\rho(n, \beta) \text{ if and only if } \begin{cases} \alpha\beta^{-1} \in H, \quad \text{and} \\ m - n = I(\beta, \beta^{-1}) - I(\alpha, \beta^{-1}) - h(\alpha\beta^{-1}). \end{cases} \tag{2.0}$$

The next theorem characterizes all subgroups of $\mathcal{G} = \mathcal{G}(S)$ when $S = (G, I)$ and hence all cancellative congruences on S.

THEOREM 2.4. *Let $S = (G, I)$ be an $\mathfrak{N}$-semigroup. Let H be a subgroup of G. Let $A \in Z^{+,0}$, and let $h: H \to Z$ be a function with the following property:*

$$h(\alpha) + h(\beta) - h(\alpha\beta) \equiv 1 - I(\alpha, \beta) \bmod A \text{ for all } \alpha, \beta \in H. \tag{2.2}$$

Let $K = \{((x, \alpha)) \in \mathcal{G}\colon \alpha \in H \text{ and } x \equiv h(\alpha) \bmod A\}$. Then K is a subgroup of $\mathcal{G}$ and every subgroup of $\mathcal{G}$ is obtained in this way.

Proof. Assume that we have H, h and A given and let K be as defined from H, h and A in the statement of the theorem. Then choose $((x, \alpha))$ and $((y, \beta))$ in K. We have

$$\begin{aligned} ((x, \alpha))((y, \beta))^{-1} &= ((x, \alpha))((-y - f(\beta, \beta^{-1}), \beta^{-1})) \\ &= ((x - y - f(\beta, \beta^{-1}) + f(\alpha, \beta^{-1}), \alpha\beta^{-1})). \end{aligned}$$

Since H is a subgroup of G, $\alpha\beta^{-1} \in H$. Thus to show that K is a subgroup of $\mathcal{G}$ we need to show that $x - y - f(\beta, \beta^{-1}) + f(\alpha, \beta^{-1}) = h(\alpha\beta^{-1}) \bmod A$. By the definition of K we have $x \equiv h(\alpha) \bmod A$ and $y \equiv h(\beta) \bmod A$, and by (2.2) $h(\alpha) + h(\beta) - h(\alpha\beta) + f(\alpha, \beta) \equiv 0 \bmod A$. In particular, $h(\alpha\beta^{-1}) + h(\beta) - h(\alpha) + f(\alpha\beta^{-1}, \beta) = 0 \bmod A$. Therefore, from (1.2) with I replaced by f, we have $x - y - f(\beta, \beta^{-1}) + f(\alpha, \beta^{-1}) = h(\alpha\beta^{-1}) \bmod A$. This completes the proof that K is a subgroup of $\mathcal{G}$.

Conversely, let K be a subgroup of $\mathcal{G}$. Let $\pi: K \to G$ be the map taking $((x, \alpha)) \in K$ to α. then π is a homomorphism of K onto a subgroup $H = \pi(K)$ of G. Now for each $\alpha \in H$ let $K_\alpha = \{((m, \alpha)) \in K\}$. Note that since K is a group $K_\alpha((x, \beta)) = K_{\alpha\beta}$ whenever $((x, \beta)) \in K$. More precisely, the map $((m, \alpha)) \mapsto ((m, \alpha))((x, \beta)) = ((m + x + f(\alpha, \beta), \alpha\beta))$ is a bijection of K_α onto $K_{\alpha\beta}$. If $\langle((1, \epsilon))\rangle$ denotes the subgroup of $\mathcal{G}$ generated by $((1, \epsilon))$ then we have $K_\epsilon = \langle((1, \epsilon))\rangle \cap K$ is a subgroup of $\langle((1, \epsilon))\rangle \cong Z$. Hence $K_\epsilon \cong Z$ or $K_\epsilon = \{((0, \epsilon))\}$. Let $A \in Z^{+,0}$ be such that $((A, \epsilon))$ is the generator of K_ϵ. Next, for each $\alpha \in H$ choose an integer $h(\alpha)$ such that $((h(\alpha), \alpha)) \in K_\alpha$ then $\alpha \mapsto h(\alpha)$ is a map $h: H \to Z$ such that (2.2) holds. Property (2.2) follows from the fact that K is closed under the multiplication in $\mathcal{G}$ and the fact that $K_\alpha = K_\epsilon \cdot ((h(\alpha), \alpha))$. This completes the proof of the theorem.

The problem with this characterization of the subgroups of $\mathcal{G}$ is that it is not one-to-one. Note that K uniquely determines H and A but not h, unless $A = 0$. In the next theorem we remedy this situation by replacing h by a homomorphism $k: H \to R/(A)$ where $R/(A)$ is the additive group of real numbers factored by the subgroup of integral multiples of A, $A \in Z^{+,0}$.

We will denote the subgroup K of $\mathcal{G}$ determined by H, h and A with (2.2) as $K = (H, h, A)$.

THEOREM 2.5. *Let* $S = (G, I) = (G, \varphi)$ *with* $I(\alpha, \beta) = \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha\beta)$ *for all* $\alpha, \beta \in G$. *Let* $K_i = (H, h_i, A)$ *for* $i = 1, 2$ *be subgroups of* $\mathcal{G}$. *Then* h_i *induces a homomorphism* $k_i: H \to R/(A)$ *for each* $i = 1, 2$ *such that*

$$(2.3) \qquad -\varphi(\alpha) + \pi_A^{-1}(k_i(\alpha)) \subseteq Z \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad \alpha \in H.$$

Where $\pi_A: R \to R/(A)$ *is the natural map. Furthermore, if* $K_1 = K_2$ *then* $k_1 = k_2$. *Conversely given a homomorphism* $k: H \to R/(A)$ *satisfying* (2.3) *we can define* $h: H \to Z$ *satisfying* (2.2).

Proof. Let $K = (H, h, A)$ be a subgroup of $\mathcal{G}$. Define $h'(\alpha) = 1 - \varphi(\alpha)$ for all $\alpha \in H$. Then

$$h'(\alpha)+h'(\beta)-h'(\alpha\beta)=1-\varphi(\alpha)+1-\varphi(\beta)-1+\varphi(\alpha\beta)=1-I(\alpha,\beta).$$

Hence $h-h'\colon H\to R$ has the property

$$(h-h')(\alpha)+(h-h')(\beta)-(h-h')(\alpha\beta)=0 \bmod A \quad \text{for all} \quad \alpha,\beta\in H$$

so that $h-h'$ followed by π_A is a homomorphism from H into $R/(A)$. Let $k=\pi_A\circ(h-h')$. Then $\pi_A^{-1}(k(\alpha))=\{x\in R: x=(h-h')(\alpha) \bmod A\}$. Therefore, for $x\in\pi_A^{-1}(k(\alpha))$ we have for some $n\in Z$

$$\begin{aligned}-\varphi(\alpha)+x &= -\varphi(\alpha)+h(\alpha)-h'(\alpha)+nA\\ &= -\varphi(\alpha)+h(\alpha)+\varphi(\alpha)-1+nA\\ &= h(\alpha)-1+nA\in Z \quad \text{since} \quad h(\alpha)\in Z.\end{aligned}$$

Next, suppose that $K_1=(H,h_1,A)=(H,h_2,A)=K_2$ then by the proof of Theorem 2.4 we see that there exists a function $l\colon H\to Z$ such that $h_2(\alpha)=h_1(\alpha)+l(\alpha)A$. Then

$$\begin{aligned}k_2(\alpha)&=\pi_A(h_2(\alpha)+\varphi(\alpha)-1)=\pi_A(h_1(\alpha)+l(\alpha)A+\varphi(\alpha)-1)\\ &=\pi_A(h_1(\alpha)+\varphi(\alpha)-1)=k_1(\alpha) \quad \text{for all} \quad \alpha\in H.\end{aligned}$$

To see the converse statement let $k\colon H\to R/(A)$ be a homomorphism satisfying (2.3). For each $\alpha\in H$ choose $x_\alpha\in\pi_A^{-1}(k(\alpha))$ and define h on H by $h(\alpha)=x_\alpha-\varphi(\alpha)+1$. By (2.3) we have $h(\alpha)\in Z$. Also,

$$h(\alpha)+h(\beta)-h(\alpha\beta)=x_\alpha+x_\beta-x_{\alpha\beta}-I(\alpha,\beta)+1,$$

and $\pi_A(x_\alpha+x_\beta-x_{\alpha\beta})=0$ in $R/(A)$ and we have $h(\alpha)+h(\beta)-h(\alpha\beta)\equiv 1-I(\alpha,\beta) \bmod A$. Thus from k we have constructed a map $h\colon H\to Z$ satisfying (2.2) and the theorem is proved.

As an immediate consequence of Theorem 2.5 we have a one-to-one correspondence between subgroups K of $\mathcal{G}$ and triples $\{H,k,A\}$ where H is a subgroup of G, $A\in Z^{+,0}$ and k is a homomorphism of H into $R/(A)$ such that (2.3) holds. We write $K=[H,k,A]$ to denote that K is determined from (or determines) H, k, and A as above. We now describe the cancellative congruences on S determined by $K=(H,h,A)$ or $K=[H,k,A]$.

Recall that given $(H,h,A)=K$ we have $K=\{((x,\alpha))\in\mathcal{G}: \alpha\in H$ and $x\equiv h(\alpha) \bmod A\}$. The congruence $\rho\in\mathcal{L}_C(S)$ corresponding to K is just the restriction to S of the congruence on $\mathcal{G}$ determined by the cosets of K in $\mathcal{G}$. That is, if (m,α) and (n,β) are in S then $(m,\alpha)\rho(n,\beta)$

if and only if $((m+1,\alpha))((n+1,\beta))^{-1} \in K$. But

$$((m+1,\alpha))((n+1,\beta))^{-1} = ((m+1,\alpha))((-n-1-f(\beta,\beta^{-1}),\beta^{-1}))$$
$$= ((m-n-f(\beta,\beta^{-1})+f(\alpha,\beta^{-1}),\alpha\beta^{-1})) \in K$$

if and only if $\alpha\beta^{-1} \in H$ and $m-n-f(\beta,\beta^{-1})+f(\alpha,\beta^{-1}) \equiv h(\alpha\beta^{-1}) \bmod A$ or $\alpha\beta^{-1} \in H$ and

$$m-n \equiv I(\beta,\beta^{-1}) - I(\alpha,\beta^{-1}) + h(\alpha\beta^{-1}) \bmod A.$$

Thus we have

THEOREM 2.6. *The cancellative congruence ρ on $S=(G,I)$ corresponding to (H,h,A) is given by*

$$(2.4) \qquad (m,\alpha)\rho(n,\beta) \text{ if and only if } \begin{cases} \alpha\beta^{-1} \in H, \quad \text{and} \\ m-n \equiv I(\beta,\beta^{-1}) - I(\alpha,\beta^{-1}) \\ \qquad + h(\alpha\beta^{-1}) \bmod A. \end{cases}$$

Now if $(H,h,A) = [H,k,A]$ and if $\rho \in \mathscr{L}_C(S)$ corresponds to (H,h,A) then ρ is defined by (2.4). What does this say in terms of ρ from k instead of h? Well, $h(\alpha) \equiv x_\alpha + 1 - \varphi(\alpha) \bmod A$ where $x_\alpha \in \pi_A^{-1}(k(\alpha))$ by the proof of Theorem 2.5. Thus

$$h(\alpha) - h(\beta) \equiv x_\alpha - \varphi(\alpha) - x_\beta + \varphi(\beta) \equiv x_{\alpha\beta} - 1 + \varphi(\beta) - \alpha(\alpha) \bmod A.$$

Therefore, $\pi_A(h(\alpha) - h(\beta)) = k(\alpha\beta^{-1}) + \pi_A(\varphi(\beta) - \varphi(\alpha))$. So in terms of k is given by

$$(2.5) \qquad (m,\alpha)\rho(n,\beta) \text{ if and only if } \begin{cases} \alpha\beta^{-1} \in H, \quad \text{and} \\ \pi_A(m-n) = k(\alpha\beta^{-1}) + \pi_A(\varphi(\beta) - \varphi(\alpha)). \end{cases}$$

We now give a group theoretical condition for a subgroup H of a structure group G of an $\mathfrak{R}$-semigroup $S=(G,I)$ to have a function h, satisfying (2.2) for some $A \in Z^{+,0}$, defined on it.

Suppose H is a subgroup of G and $h: H \to Z$ satisfies (2.2) for some nonnegative integer A. Then let $h_A = \pi_A \circ h$ where $\pi_A: Z \to Z/(A)$ is the natural map. Also, let $f_A: H \times H \to Z/(A)$ be the map given by $f_A = \pi_A \circ (-f) | (H \times H)$ where $f: G \times G \to Z$ is the factor set which determines the group of quotients of S.

That is, $f(\alpha,\beta)=I(\alpha,\beta)-1$ for all $\alpha,\beta\in G$. Since f is a a factor set and π_A is a homomorphism, we have that f_A is a factor set. Property (2.2) for h and the fact that π_A is a homomorphism imply that

$$h_A(\alpha)+h_A(\beta)-h_A(\alpha\beta)=f_A(\alpha,\beta) \text{ for all } \alpha,\beta\in H. \tag{2.6}$$

Thus f_A is a transformation set and thus it is equivalent to the trivial factor set Θ: $H\times H\to Z/(A)$ (i.e. $\Theta(\alpha,\beta)=0$ for all $\alpha,\beta\in H$). Thus the abelian group extension $(H,Z/(A);f_A)$ of H by $Z/(A)$ determined by the factor set f_A is equivalent to the direct product $H\times(Z/(A))$.

Conversely, if $(H,Z/(A);f_A)$ is equivalent to the direct product $H\times(Z/(A))$ then $f_A(\alpha,\beta)=c(\alpha)+c(\beta)-c(\alpha\beta)$ for all $\alpha,\beta\in H$ for some function $c\colon H\to Z/(A)$. We can get a function $h\colon H\to Z$ by simply choosing for each $\alpha\in H$ a representative $h(\alpha)$ from $\pi_A^{-1}(c(\alpha))$. From (2.6) with h_A replaced by c we have that h satisfies (2.2). We therefore have the following theorem.

THEOREM 2.7. *Let H be a subgroup of G and $A\in Z^{+,0}$. There exists a function $h\colon H\to Z$ satisfying* (2.2) *if and only if the abelian group extension $(H,Z/(A);f_A)$ is equivalent to the direct product $H\times(Z/(A))$, that is, if and only if f_A is a transformation set.*

Next we will separate $\mathscr{L}_C(S)$ into its two pieces $\mathscr{L}_G(S)$ and $\mathscr{L}_{\mathfrak{N}}(S)$.

LEMMA 2.8. *Let $\rho\in\mathscr{L}_C(S)$ for $S=(G,I)$ such that $(m,\alpha)\rho(n,\alpha)$ for some $\alpha\in\hat{G}$ and $m,n\in Z^{+,0}$ with $m\neq n$. Then $\rho\in\mathscr{L}_G(S)$ and ρ corresponds to a subgroup K of $\mathscr{G}$ such that $K=(H,h,A)$ with $A>0$ for some subgroup H of G and $h\colon H\to Z$ satisfying* (2.2).

Proof. Assume $m>n$ and $(m,\alpha)\rho(n,\alpha)$ then

$$(m,\alpha)=(0,\alpha)(0,\epsilon)^m\rho(0,\alpha)(0,\epsilon)^n=(n,\alpha).$$

And since ρ is a cancellative congruence we have $(0,\epsilon)^{m-n+1}\rho(0,\epsilon)$; hence S/ρ has an idempotent. Thus $\rho\in\mathscr{L}_G(S)$. Now let $K=(H,h,A)$ correspond to ρ. Recall that $A=0$ if and only if $K\cap\langle((1,\epsilon))\rangle=\{((0,\epsilon))\}$. Well $(0,\epsilon)\rho(0,\epsilon)^{m-n+1}$ implies that $((1,\epsilon))^{m-n+1}((1,\epsilon))^{-1}\in K$ and $((1,\epsilon))^{m-n+1}((1,\epsilon))^{-1}=((m-n+1,\epsilon))((-1,\epsilon))=((m-n,\epsilon))\in K$ with $m-n>0$; therefore, $K\cap\langle((1,\epsilon))\rangle\neq\{((0,\epsilon))\}$ and so $A>0$.

Let $\mathscr{L}_{\mathscr{A}}(S)=\{\rho\in\mathscr{L}_C(S)\colon \rho \text{ corresponds to } (H,h,A) \text{ with } A>0\}$. Since a homomorphic image of a group is a group, it is immediate that $\mathscr{L}_G(S)$ is a join subsemilattice of $\mathscr{L}_C(S)$ and that $\mathscr{L}_{\mathfrak{N}}(S)$ is a meet subsemilattice of $\mathscr{L}_C(S)$.

THEOREM 2.9. *$\mathscr{L}_{\mathscr{A}}(S)$ is a sublattice of $\mathscr{L}_C(S)$.*

Proof. It is immediate from Lemma 2.8 and property (2.4) that $\mathcal{L}_{\mathcal{A}}(S)=\{\rho\in\mathcal{L}_C(S)$: there exists $m,n\in Z^{+,0}$ with $m\neq n$ and $\alpha\in G$ such that $(m,\alpha)\rho(n,\alpha)\}$. Thus, since $\mathcal{L}_C(S)$ is closed under joins, this characterization of $\mathcal{L}_{\mathcal{A}}(S)$ makes it clear that $\mathcal{L}_{\mathcal{A}}(S)$ is closed under joins. Let $\rho_i=(H_i,h_i,A_i)$ with $A_i>0$ for $i=1,2$. Then for each $\alpha\in G$ we have by (2.4) $(0,\alpha)\rho_i(A_1A_2,\alpha)$ for $i=1,2$. Hence $(0,\alpha)\rho_1\cap\rho_2(A_1A_2,\alpha)$ and so $\rho_1\cap\rho_2\in\mathcal{L}_{\mathcal{A}}(S)$.

An $\mathfrak{R}$-semigroup $S=(G,I)=(G,\varphi)$ is said to be power-joined if for all $x,y\in S$ there exists $m,n\in Z^+$ such that $x^m=y^n$. From [2] we see that S is power joined if and only if G is a periodic abelain group. Tamura [9] has shown that if S is power joined then $\mathcal{L}_{\mathfrak{R}}(S)$ is a sublattice of $\mathcal{L}_C(S)$ isomorphic to the lattice of subgroups of the subgroup H of G defined by $H=\{\alpha\in G:\varphi(\alpha)\in Z^+\}$. Hence if S is power joined then $\mathcal{L}_{\mathfrak{R}}(S)$ is a modular lattice.

THEOREM 2.10. *The following are equivalent for an $\mathfrak{R}$-semigroup S:*

(a) *S is power joined.*

(b) $\mathcal{L}_{\mathcal{A}}(S)=\mathcal{L}_G(S)$.

(c) *$\mathcal{L}_G(S)$ is a sublattice of $\mathcal{L}(S)$.*

Proof. (a) implies (b): Assume S is power joined. Let $\rho=(H,h,0)$. Let $k:H\to R$ be the homomorphism induced by h as in Theorem 2.5. Then since H is periodic we have k must be the zero map. But recall $k(\alpha)=h(\alpha)+\varphi(\alpha)-1$ for all $\alpha\in H$. Therefore $\varphi(\alpha)=1-h(\alpha)$ is in Z (since $h:H\to Z$), and $\varphi(\alpha)>0$ for all $\alpha\in H$. Hence $h(\alpha)<0$ for all $\alpha\in H$. And so $-h(\alpha)\geqq 0$ so $-h:H\to Z^{+,0}$ and $-h(\alpha)-h(\beta)+h(\alpha\beta)=I(\alpha,\beta)-1$ so that by Lemma 2.3 the subgroup of $\mathcal{G}(S)$ determined by H and $-h$ as in that lemma is an $\mathfrak{R}$-kernel. But that subgroup is just $K=(H,h,0)$. Thus ρ is an $\mathfrak{R}$-congruence and $\mathcal{L}_{\mathcal{A}}(S)=\mathcal{L}_G(S)$.

(b) implies (c): This follows from Theorem 2.9.

(c) implies (a): Assume $S=(G,I)$ is not power joined. Then there exists $\alpha\in G$ of infinite order. Let $K_1=\langle((1,\epsilon))\rangle$ and $K_2=\langle((1,\alpha))\rangle$. Then $K_i\cap S\neq\Box$ for $i=1,2$ so that K_1 and K_2 correspond to group congruences on S. But since α has infinite order, $K_1\cap K_2=\{((0,\epsilon))\}$ is an $\mathfrak{R}$-kernel and so the group congruences of S are not closed under intersection. Hence $\mathcal{L}_G(S)$ is not a sublattice of $\mathcal{L}(S)$.

Thus if S is a power joined $\mathfrak{R}$-semigroup we have $\mathcal{L}_C(S)$ is the disjoint union of the two sublattices $\mathcal{L}_G(S)$ and $\mathcal{L}_{\mathfrak{R}}(S)$. Although power joinedness is a sufficient condition on S to make $\mathcal{L}_{\mathfrak{R}}(S)$ a sublattice of $\mathcal{L}(S)$, it is not necessary because every irreducible $\mathfrak{R}$-semigroup (i.e. an $\mathfrak{R}$-semigroup with no proper $\mathfrak{R}$-homomorphic images) has only i as an $\mathfrak{R}$-congruence.

REMARK 2.2. The correspondence from $\mathfrak{R}$-kernels to triples $(H, h, 0)$ is one-to-one. Also, as is obvious from the proof of (a) implies (b) in Theorem 2.10, a triple $(H, h, 0)$ gives an $\mathfrak{R}$-kernel if and only if h maps H into $Z^{-,0}$. (Compare this with Lemma 2.3.)

PROPOSITION 2.11. *Let ρ_i correspond to $(H_i, h_i, 0)$ for $i = 1,2$ be two $\mathfrak{R}$-congruences on S then $\rho_1 \vee \rho_2$ is an $\mathfrak{R}$-congruence if and only if for all $\alpha \in H_1$ and $\beta \in H_2$ we have*

$$h_1(\alpha) + h_2(\beta) + I(\alpha, \beta) \leqq 1. \tag{2.7}$$

Proof. Let $K_i = (H_i, h_i, 0)$ be the subgroup of $\mathscr{G}(S)$ associated with ρ_i $(i = 1,2)$. Then $\rho_1 \vee \rho_2$ corresponds to $K_1K_2 = \{k_1k_2 \colon k_i \in K_i \ (i = 1,2)\}$. Then H_1H_2 is clearly the subgroup of G determined by K_1K_2 (see the proof of the converse half of Theorem 2.4). Now if $\rho_1 \vee \rho_2$ is an $\mathfrak{R}$-congruence then for each $\gamma \in H_1H_2$ there is a unique integer $h(\gamma) \leqq 0$ such that $((h(\gamma), \gamma)) \in K_1K_2$. Let $\alpha \in H_1$ and $\beta \in H_2$ with $\alpha\beta = \gamma$ then $((h_1(\alpha), \alpha)) \in K_1$ and $((h_2(\beta), \beta)) \in K_2$. Hence

$$((h_1(\alpha), \alpha))((h_2(\beta), \beta)) = ((h_1(\alpha) + h_2(\beta) + f(\alpha, \beta), \alpha\beta)) \in K_1K_2$$

but since $\alpha\beta = \gamma$ we have $h_1(\alpha) + h_2(\beta) + f(\alpha, \beta) = h(\gamma) \leqq 0$ or $h_1(\alpha) + h_2(\beta) + I(\alpha, \beta) \leqq 1$.

Conversely, if for all $\alpha \in H_1$ and $\beta \in H_2$ we have $h_1(\alpha) + h_2(\beta) + I(\alpha, \beta) \leqq 1$ then if $((x, \gamma)) \in K_1K_2$ there must be $((h_1(\alpha), \alpha)) \in K_1$ and $((h_2(\beta), \beta)) \in K_2$ such that

$$\begin{aligned} ((x, \gamma)) &= ((h_1(\alpha), \alpha))((h_2(\beta), \beta)) \\ &= ((h_1(\alpha) + h_2(\beta) + I(\alpha, \beta) - 1, \alpha\beta)). \end{aligned}$$

Hence $x = h_1(\alpha) + h_2(\beta) + I(\alpha, \beta) - 1 \leqq 0$ so that K_1K_2 is an $\mathfrak{R}$-kernel and $\rho_1 \vee \rho_2$ is an $\mathfrak{R}$-congruence.

The following corollary to Proposition 2.11 gives us a large number of examples of $\mathfrak{R}$-semigroups for which $\mathscr{L}_{\mathfrak{R}}(S)$ is a sublattice of $\mathscr{L}(S)$. Those for which the structure group G is not a periodic group show that power joinedness is not necessary to make $\mathscr{L}_{\mathfrak{R}}(S)$ a sublattice of $\mathscr{L}(S)$.

COROLLARY 2.12. *If $S = Z^+ \times G$ for some abelian group G then $\mathscr{L}_{\mathfrak{R}}(S)$ is a sublattice of $\mathscr{L}(S)$.*

Proof. $Z^+ \times G$ is isomorphic onto the $\mathfrak{R}$-semigroup (G, I) where $I(\alpha, \beta) = 1$ for all $\alpha, \beta \in G$. Hence given $h_i \colon H_i \to Z^{-,0}$ $(i = 1,2)$ we have (2.7) is obviously satisfied.

REMARK 2.3. Note that condition (2.7) implies that $h_1(\alpha) = h_2(\alpha)$ for all $\alpha \in H_1 \cap H_2$ since h is unique for an $\mathfrak{N}$-congruence. This can be shown directly also.

3. General congruences. Here we make a study of the general congruences on an $\mathfrak{N}$-semigroup $S = (G, I)$. We first study ideals of S and relate them to certain functions from G into $Z^{+,0}$. This yields a characterization of $\mathcal{L}_R(S)$. Next, we associate a particular ideal J_ρ of S with each congruence ρ on S and use its emptiness or nonemptiness to separate the congruences on S into two types. The first being those which do not relate elements of S with the same second coordinate denoting the collection of those congruences by $\mathcal{L}_\Box(S)$. Then the other type is shown to be the intersection of a group-congruence with a nil-congruence. We go on to give characterizations of members of $\mathcal{L}_\Box(S)$ and $\mathcal{L}_N(S)$.

Throughout this section we will assume that $S = (G, I)$. Let J be an ideal of S. For each $\alpha \in G$ let $G_\alpha = \{m \in Z^{+,0}\colon (m, \alpha) \in J\}$.

(3.1) If $x \in Z^{+,0}$ and $m \in G_\alpha$ then $m + x \in G_\alpha$.

Proof. $m \in G_\alpha$ implies $(m, \alpha) \in J$. Thus for all $(x, \epsilon) \in S (x, \epsilon)(m, \alpha) = (m + x + 1, \alpha) \in J$ as J is an ideal.

(3.2) If $J \neq \Box$ then $G_\alpha \neq \Box$ for all $\alpha \in G$.

Proof. Since $J = \bigcup_{\alpha \in G}(G_\alpha \times \{\alpha\})$ if $J \neq \Box$ then there exists $\alpha_0 \in G$ such that $G_{\alpha_0} \neq \Box$. Let $(m_0, \alpha_0) \in J$. Choose $\alpha \in G$. G being a group, there is a $\beta \in G$ such that $\alpha = \alpha_0\beta$ (i.e. $\beta = \alpha_0^{-1}\alpha$). Then

$$(m_0, \alpha_0)(0, \beta) = (m_0 + I(\alpha_0, \beta), \alpha_0\beta) = (m_0 + I(\alpha_0, \beta), \alpha) \in J$$

so that $m_0 + I(\alpha_0, \beta) \in G_\alpha$ and $G_\alpha \neq \Box$. Thus if $J \neq \Box$ for each $\alpha \in G$ G_α is a segment $[\psi(\alpha), \infty)$ of $Z^{+,0}$, that is, for each $\alpha \in G$ there exists $\psi(\alpha) \in Z^{+,0}$ such that $G_\alpha = \{m \in Z^{+,0}\colon m \geqq \psi(\alpha)\}$. If $J = \Box$ then for each $\alpha \in G$ $G_\alpha = \Box$ and so we will define $\psi(\alpha) = \infty$ for all $\alpha \in G$ where ∞ is adjoined to $Z^{+,0}$ as a largest element under $\leqq$ and as a zero under addition (that is, $x + \infty = \infty + x = \infty + \infty = \infty$ for all $x \in Z^{+,0}$). The resulting ordered semigroup being denoted by $Z_\infty^{+,0}$. In any case, an ideal J determines a function $\psi\colon G \to Z_\infty^{+,0}$ such that

(3.3) $\psi(\alpha) + I(\alpha, \beta) \geqq \psi(\alpha\beta)$ for all $\alpha, \beta \in G$.

Proof. $(\psi(\alpha), \alpha) \in J$ implies that

$$(\psi(\alpha), \alpha)(0, \beta) = (\psi(\alpha) + I(\alpha, \beta), \alpha\beta) \in J$$

for all $\alpha, \beta \in G$. Hence $\psi(\alpha) + I(\alpha, \beta) \in G_{\alpha\beta}$, so (3.3) holds and we are done. A function $\psi: G \to Z_\infty^{+,0}$ satisfying (3.3) is called an ideal function on G.

Conversely, given an ideal function ψ we obtain an ideal J of S by

$$J = \{(m, \alpha) \in S : m \geqq \psi(\alpha)\}. \tag{3.4}$$

REMARK 3.1. Constant functions $\psi: G \to Z_\infty^{+,0}$ are ideal functions, and translates of ideal functions are ideal functions. (That is, if ψ is an ideal function on G then $\psi_k(\alpha) = \psi(\alpha) + k$, where $k \in Z^{+,0}$, is also an ideal function on G.)

REMARK 3.2. Since $\psi(\alpha) + I(\alpha, \alpha^{-1}\beta) \geqq \psi(\beta)$ for all $\alpha, \beta \in G$, we see that $\psi(\alpha)$ finite for one $\alpha \in G$ implies that $\psi(\alpha)$ is finite for all $\alpha \in G$. Moreover by (3.3) we see that $\psi(\epsilon) + I(\epsilon, \beta) = \psi(\epsilon) + 1 \geqq \psi(\beta)$ for all $\beta \in G$ where ϵ is the identity element of G. Hence if ψ maps into $Z^{+,0}$ then ψ is bounded.

Let ψ_G denote the collection of all ideal functions on G. ψ_G becomes a lattice under the natural ordering (i.e. $\psi_1 \leqq \psi_2$ if for all α in G $\psi_1(\alpha) \leqq \psi_2(\alpha)$) and the join and meet are given by $(\psi_1 \vee \psi_2)(\alpha) = \max\{\psi_1(\alpha), \psi_2(\alpha)\}$ and $(\psi_1 \wedge \psi_2)(\alpha) = \min\{\psi_1(\alpha), \psi_2(\alpha)\}$. If ψ_i corresponds to the ideal J_i of S for $i = 1, 2$ then $\psi_1 \vee \psi_2$ corresponds to $J_1 \cap J_2$, and $\psi_1 \wedge \psi_2$ corresponds to $J_1 \cup J_2$. Hence $\mathscr{L}_R(S)$ is dually isomorphic onto the lattice $(\psi_G, \leqq) = (\psi_G, \vee, \wedge)$.

Again we assume that we have our $\mathfrak{N}$-semigroup S determined as (G, I). Let ρ be a congruence on S. Define

$$J_\rho = \{(m, \alpha) \in S : \text{there exists } (n, \alpha) \in S \text{ with } n \neq m \text{ and } (m, \alpha)\rho(n, \alpha)\}. \tag{3.5}$$

Claim. J_ρ is an ideal of S (possibly empty).

Proof. If $J_\rho \neq \square$ let $(m, \alpha) \in J_\rho$ then there exists $n \neq m$ such that $(m, \alpha)\rho(n, \alpha)$. Let $(l, \gamma) \in S$ then $(l, \gamma)(m, \alpha)\rho(l, \gamma)(n, \alpha)$ and so $(l + m + I(\gamma, \alpha), \gamma\alpha)\rho(l + n + I(\gamma, \alpha), \gamma\alpha)$ and $l + m + I(\gamma, \alpha) \neq l + n + I(\gamma, \alpha)$ since $m \neq n$. Hence $(l, \gamma)(m, \alpha) \in J_\rho$. Thus J_ρ is an ideal.

LEMMA 3.1. $J_\rho = \{(m, \alpha) \in S : (m, \alpha)\rho(m + l, \alpha)$ *for some* $l \in Z^+\}$.

Proof. Let $J = \{(m, \alpha) \in S : (m, \alpha)\rho(m + l, \alpha)$ for some $l \in Z^+\}$. By the definition of J_ρ we have $J \subseteq J_\rho$. Let $(m, \alpha) \in J_\rho$. Then $(m, \alpha)\rho(l, \alpha)$ for some $l \in Z^{+,0}$ with $l \neq m$. If $l > m$ then $(m, \alpha) \in J$ by the definition of J so assume $l < m$. Then we have

$$(m, \alpha)(m - l - 1, \epsilon)\rho(l, \alpha)(m - l - 1, \epsilon)$$

so that $(2m - l, \alpha)\rho(m, \alpha)$ and $2m - l > m$. Thus $(m, \alpha) \in J$. And we have shown $J_\rho \subseteq J$ and hence $J = J_\rho$.

LEMMA 3.2. *If $(m, \alpha)\rho(n, \beta)$ then $(m + k, \alpha)\rho(n + k, \beta)$ for all $k \in Z^{+,0}$.*

Proof. If $k > 0$ then

$$(m + k, \alpha) = (m, \alpha)(k - 1, \epsilon)\rho(n, \beta)(k - 1, \epsilon) = (n + k, \beta).$$

LEMMA 3.3. *If $J_\rho \neq \square$ and $(m, \alpha)\rho(n, \beta)$ with $(m, \alpha) \in J_\rho$ then $(n, \beta) \in J_\rho$.*

Proof. Assume $(m, \alpha) \in J_\rho$ and $(m, \alpha)\rho(n, \beta)$. By Lemma 3.1 there is $l \in Z^+$ such that $(m, \alpha)\rho(m + l, \alpha)$ and by Lemma 3.2 $(m + l, \alpha)\rho(n + l, \beta)$. Hence, since ρ is a transitive relation we have $(n, \beta)\rho(n + l, \beta)$ and so by Lemma 3.1 $(n, \beta) \in J_\rho$.

We will denote the ideal function associated with J_ρ by ψ_ρ.

LEMMA 3.4. *If $J_\rho \neq \square$ then $\mu = \rho | J_\rho$ is a group-congruence on J_ρ.*

Proof. Let $(m, \alpha) \in J_\rho$ and let $l \in Z^+$ such that $(m, \alpha)\rho(m + l, \alpha)$. Note that we can assume that $l \geqq \psi(\epsilon)$ because $(m, \alpha)\rho(m + l, \alpha)$ implies that

$$(m, \alpha)\rho(m + l, \alpha)\rho(m + 2l, \alpha)\rho(m + 3l, \alpha)\cdots$$

and $\psi(\epsilon) < \infty$ because $J_\rho \neq \square$. Let $(m, \alpha)\mu$ denote the μ-class (= the ρ-class) of (m, α) for each element (m, α) in J_ρ. Then we have $(m, \alpha)_\mu = (m + l, \alpha)_\mu = (m, \alpha)_\mu(l - 1, \epsilon)_\mu$ and by our choice of $l \geqq \psi(\epsilon)$ we have $(l - 1, \epsilon) \in J_\rho$. Therefore

$$(m, \alpha)_\mu = (m, \alpha)_\mu(l - 1, \epsilon)_\mu = (m, \alpha)_\mu(l - 1, \epsilon)^2_\mu = \cdots$$

so that

$$(m, \alpha)_\mu \in \bigcap_{n=1}^{\infty} (l - 1, \epsilon)^n_\mu(J_\rho/\mu).$$

Hence by Fact 1.1 we have $(m, \alpha)_\mu$ is in the kernel of J_ρ/μ which is a group. But (m, α) was an arbitrary element of J_ρ. Thus J_ρ/μ is a group. This proves the lemma.

By Theorem 1.2 μ extends uniquely to a group-congruence ν on S.

LEMMA 3.5. *$\nu \supseteq \rho$ if $J_\rho \neq \square$.*

Proof. If $a, b \in S$ with $a\rho b$ then choose $e \in J_\rho$ such that e_μ is the identity class of J_ρ/μ then $ae\rho be$ hence $ae\mu be$ and $a\nu b$.

Now J_ρ/ρ is an ideal of S/ρ (since J_ρ is an ideal of S). Let $\mathcal{N}$ be the nil-congruence on S determined by the composition of the following two natural maps:

$$S \to S/\rho \to (S/\rho)/(J_\rho/\rho).$$

That is, $\mathcal{N}$ equals ρ outside of J_ρ and $\mathcal{N}$ collapses J_ρ. In symbols,

$$\mathcal{N} \mid (S \backslash J_\rho) = \rho \mid (S \backslash J_\rho) \quad \text{and} \quad \mathcal{N} \mid J_\rho = \omega_{J_\rho}. \tag{3.6}$$

Hence, since $\nu \mid J = \mu = \rho \mid J_\rho$ we have

THEOREM 3.6. *If $J_\rho \neq \square$ then $\rho = \nu \cap \mathcal{N}$.*

Our problem of determining all congruences on S is reduced to determining the nil-congruences on S and the congruences with $J_\rho = \square$. We know the group-congruences on S from §2. We will denote the collection of those congruences ρ on S with $J_\rho = \square$ by $\mathcal{L}_\square(S)$.

We will now show that the congruences in $\mathcal{L}_\square(S)$ are quite simply determined by the pairs of columns in S which contain ρ-related elements of S and by the first elements of these columns which are ρ-related.

Let $\rho \in \mathcal{L}_\square(S)$. Define a relation σ_ρ on G by

$$\begin{array}{l} \text{For } \alpha, \beta \in G\ \alpha\sigma_\rho\beta \text{ if and only if there exists } m, n \in Z^{+,0} \\ \text{such that } (m, \alpha)\rho(n, \beta). \end{array} \tag{3.7}$$

Claim. σ_ρ is a congruence on G.

Proof. Reflexivity and symmetry of σ_ρ follow from the same properties for the congruence ρ. To see that σ_ρ is transitive let $\alpha\sigma_\rho\beta$ and $\beta\sigma_\rho\gamma$ for $\alpha, \beta, \gamma \in G$. Then there are $m, n, k, l \in Z^{+,0}$ such that $(m, \alpha)\rho(n, \beta)$ and $(k, \beta)\rho(l, \gamma)$. Assume $k \geqq n$ then

$$(m + k - n, \alpha)\rho(n + k - n, \beta) = (k, \beta)\rho(l, \gamma)$$

so that by transitivity of ρ we have $\alpha\sigma_\rho\gamma$. The case where $k < n$ is handled in the same way. Thus σ_ρ is transitive. To check that σ_ρ is compatible let $\alpha, \beta \in G$ with $\alpha\sigma_\rho\beta$ and let $\gamma \in G$. We need to show that $\alpha\gamma\sigma_\rho\beta\gamma$. Well, $\alpha\sigma_\rho\beta$ implies that there are $m, n \in Z^{+,0}$ such that $(m, \alpha)\rho(n, \beta)$ then $(m + I(\alpha, \gamma), \alpha\gamma) = (m, \alpha)(0, \gamma)\rho(n, \beta)(0, \gamma) = (n + I(\beta, \gamma), \beta\gamma)$ so that $\alpha\gamma\sigma_\rho\beta\gamma$. Thus σ_ρ is a congruence on G.

Next we define a function $\phi_\rho\colon \sigma_\rho \to Z^{+,0}$ (first used by Dickinson **[4]** to determine $T4$-congruences) by considering σ_ρ as a subset of $G \times G$. If $(\alpha, \beta) \in \sigma_\rho$ then define

$$\phi_\rho(\alpha, \beta) = \min\{m : (m, \alpha)\rho(n, \beta) \text{ for some } n \in Z^{+,0}\}. \tag{3.8}$$

It then follows that $(\phi_\rho(\alpha, \beta), \alpha)\rho(\phi_\rho(\beta, \alpha), \beta)$ whenever $\alpha\sigma_\rho\beta$ and hence by Lemma (3.2) we have

$$(\phi_\rho(\alpha, \beta) + k, \alpha)\rho(\phi_\rho(\beta, \alpha) + k, \beta) \text{ for all } k \in Z^{+,0}.$$

Furthermore, since $J_\rho = \square$ we have

$$\rho = \{((\phi_\rho(\alpha, \beta) + k, \alpha), (\phi_\rho(\beta, \alpha) + k, \beta)) \in S \times S : \alpha\sigma_\rho\beta \text{ and } k \in Z^{+,0}\}, \tag{3.9}$$

or

$$(m, \alpha)\rho(n, \beta) \text{ if and only if } \alpha\sigma_\rho\beta \text{ and there exists } k \in Z^{+,0} \text{ such that } m = \phi_\rho(\alpha, \beta) + k \text{ and } n = \phi_\rho(\beta, \alpha) + k. \tag{3.9$'$}$$

Thus ρ is determined by a congruence σ_ρ on G and a map $\phi_\rho\colon \sigma_\rho \to Z^{+,0}$. The natural question is when do a congruence σ on G and a map $\phi\colon \sigma \to Z^{+,0}$ determine a congruence relation by (3.9) or (3.9′)? That is, what are the defining properties of ϕ_ρ and its relation to ρ_ρ? We now present a collection of lemmas which determine these properties and relations, and then in Theorem 3.12 we choose from among this collection of necessary conditions on σ and ϕ those which are sufficient to determine a congruence on S by (3.9′).

REMARK 3.3. The reader will notice that some of the following lemmas are quite similar to statements proven by Dickinson **[4]**. In Dickinson's proofs he used the link between "ϕ-functions" and "d-functions" (d-functions were introduced by Tamura in characterizing $\mathfrak{R}$-congruences). Note that here I have deleted the need for the d-function and its properties by essentially restating the d-function properties in terms of ϕ. That is, d is really determined by ϕ.

LEMMA 3.7. *$\phi_\rho(\alpha, \alpha) = 0$ for all $\alpha \in G$.*

Proof. $(0, \alpha)\rho(0, \alpha)$ since ρ is reflexive.

LEMMA 3.8. *If $(m, \alpha)\rho(n, \beta)$ then $m - n = \phi_\rho(\alpha, \beta) - \phi_\rho(\beta, \alpha)$. This follows from the fact that $J_\rho = \square$.*

LEMMA 3.9. *If $\alpha\sigma_\rho\beta$ and $\beta\sigma_\rho\gamma$ for $\alpha, \beta, \gamma \in G$ then we have $\phi_\rho(\alpha, \beta) - \phi_\rho(\beta, \alpha) + \phi_\rho(\beta, \gamma) - \phi_\rho(\gamma, \beta) = \phi_\rho(\alpha, \gamma) - \phi_\rho(\gamma, \alpha)$.*

Proof. We will give a proof assuming that $\phi_\rho(\beta, \gamma) \geqq \phi_\rho(\beta, \alpha)$. The proof for the opposite inequality is identical. Let $\phi_\rho(\beta, \gamma) = \phi_\rho(\beta, \alpha) + k$ with $k \in Z^{+,0}$. Then by Lemma 3.2

$$(\phi_\rho(\alpha, \beta) + k, \alpha)\rho(\phi_\rho(\beta, \alpha) + k, \beta) = (\phi_\rho(\beta, \gamma), \beta)\rho(\phi_\rho(\gamma, \beta), \gamma).$$

Hence by Lemma 3.8

$$\begin{aligned}\phi_\rho(\alpha, \gamma) - \phi_\rho(\gamma, \alpha) &= \phi_\rho(\alpha, \beta) + k - \phi_\rho(\gamma, \beta)\\ &= \phi_\rho(\alpha, \beta) + k - \phi_\rho(\beta, \gamma) + \phi_\rho(\beta, \gamma) - \phi_\rho(\gamma, \beta)\\ &= \phi_\rho(\alpha, \beta) - \phi_\rho(\beta, \alpha) + \phi_\rho(\beta, \gamma) - \phi_\rho(\gamma, \beta)\end{aligned}$$

and we are done.

LEMMA 3.10. *For any $\alpha, \beta, \gamma \in G$ such that $\alpha\sigma_\rho\beta$ and $\beta\sigma_\rho\gamma$ let $k, l \in Z^{+,0}$ with $\min\{k, l\} = 0$ and $\phi_\rho(\beta, \alpha) + k = \phi_\rho(\beta, \gamma) + l$. Then there exists $m \in Z^{+,0}$ such that $\phi_\rho(\alpha, \gamma) + m = \phi_\rho(\alpha, \beta) + k$ and $\phi_\rho(\gamma, \alpha) + m = \phi_\rho(\gamma, \beta) + l$.*

Proof. Assume that $l = 0$, then

(a) $$\phi_\rho(\beta, \alpha) + k = \phi_\rho(\beta, \gamma).$$

Suppose $k = 0$, also, then

$$(\phi_\rho(\alpha, \beta), \alpha)\rho(\phi_\rho(\beta, \alpha), \beta) = (\phi_\rho(\beta, \gamma), \beta)\rho(\phi_\rho(\gamma, \beta), \gamma)$$

so by transitivity of ρ we have $(\phi_\rho(\alpha, \beta), \alpha)\rho(\phi_\rho(\gamma, \beta), \gamma)$. Thus $\phi_\rho(\alpha, \gamma) \leqq \phi_\rho(\alpha, \beta)$ and there exists $m \in Z^{+,0}$ such that $\phi_\rho(\alpha, \gamma) + m = \phi_\rho(\alpha, \beta) = \phi_\rho(\alpha, \beta) + k$. If $k > 0$ then $(\phi_\rho(\alpha, \beta), \alpha)\rho(\phi_\rho(\beta, \alpha), \beta)$ implies that

$$\begin{aligned}(\phi_\rho(\alpha, \beta) + k, \alpha) &= (\phi_\rho(\alpha, \beta), \alpha)(k - 1, \epsilon)\rho(\phi_\rho(\beta, \alpha), \beta)(k - 1, \epsilon)\\ &= (\phi_\rho(\beta, \alpha) + k, \beta) = (\phi_\rho(\beta, \gamma), \beta)\rho(\phi_\rho(\gamma, \beta), \gamma).\end{aligned}$$

Hence $\phi_\rho(\alpha,\gamma) \leqq \phi_\rho(\alpha,\beta)+k$ and so there exists $m \in Z^{+,0}$ such that

(b) $$\phi_\rho(\alpha,\gamma)+m = \phi_\rho(\alpha,\beta)+k.$$

From (a) we have

$$\phi_\rho(\beta,\alpha)-\phi_\rho(\alpha,\beta)+\phi_\rho(\alpha,\beta)+k = \phi_\rho(\beta,\gamma)-\phi_\rho(\gamma,\beta)+\phi_\rho(\gamma,\beta)$$

which implies from (b) that

$$\phi_\rho(\beta,\alpha)-\phi_\rho(\alpha,\beta)+\phi_\rho(\alpha,\gamma)+m+\phi_\rho(\gamma,\beta)-\phi_\rho(\beta,\gamma)=\phi_\rho(\gamma,\beta).$$

By Lemma 3.9 we have

$$\phi_\rho(\beta,\alpha)-\phi_\rho(\alpha,\beta)+\phi_\rho(\alpha,\gamma)+m+\phi_\rho(\gamma,\beta)-\phi_\rho(\beta,\gamma)=\phi_\rho(\gamma,\alpha)+m.$$

Hence $\phi_\rho(\gamma,\alpha)+m = \phi_\rho(\gamma,\beta)=\phi_\rho(\gamma,\beta)+l.$

LEMMA 3.11. *If $\alpha\sigma_\rho\beta$ then for all $\gamma \in G$*

$$\phi_\rho(\alpha,\beta)+I(\alpha,\gamma)-\phi_\rho(\alpha\gamma,\beta\gamma)=\phi_\rho(\beta,\alpha)+I(\beta,\gamma)-\phi_\rho(\beta\gamma,\alpha\gamma)\geqq 0.$$

Proof. This follows from the compatibility of ρ, Lemma 3.8 and the definition of ϕ_ρ.

We now state the theorem which gives us the converse to the above characterization of $\rho \in \mathscr{L}_\square(S)$.

THEOREM 3.12. *Let σ be a congruence on G and let $\phi\colon \sigma \to Z^{+,0}$ be a map such that*

($\square$.1) $$\phi(\alpha,\alpha)=0 \quad \text{for all} \quad \alpha \in G.$$

($\square$.2) *For all $\alpha,\beta,\gamma \in G$ with $\alpha\sigma\beta$ and $\beta\sigma\gamma$ let $k,l \in Z^{+,0}$ with $\min\{k,l\}=0$ and $\phi(\beta,\alpha)+k=\phi(\beta,\gamma)+l$. Then there exists $m \in Z^{+,0}$ such that $\phi(\alpha,\gamma)+m=\phi(\alpha,\beta)+k$ and $\phi(\gamma,\alpha)+m=\phi(\gamma,\beta)+l$.*

($\square$.3) *If $\alpha,\beta \in G$ with $\alpha\sigma\beta$ then $\phi(\alpha,\beta)+I(\alpha,\gamma)-\phi(\alpha\gamma,\beta\gamma)=\phi(\beta,\alpha)+I(\beta,\gamma)-\phi(\beta\gamma,\alpha\gamma)\geqq 0$ for all $\gamma \in G$.*

If we define a relation ρ on S by (3.9′) then $\rho \in \mathscr{L}_\square(S)$. We denote $\rho=(\sigma,\phi)$.

Proof. Clearly, if ρ defined from σ and ϕ by (3.9′) is a congruence on S then $J_\rho = \square$ and so $\rho \in \mathscr{L}_\square(S)$. To see that ρ is in $\mathscr{L}(S)$, note that ($\square$.1) implies that ρ is reflexive and the definition of ρ clearly makes ρ a symmetric relation. Now let $(m,\alpha)\rho(n,\beta)$ and $(n,\beta)\rho(l,\gamma)$ then $m = \phi(\alpha,\beta)+k$, $n = \phi(\beta,\alpha)+k = \phi(\beta,\gamma)+r$, and $l = \phi(\gamma,\beta)+r$ for some $k, r \in Z^{+,0}$. Hence $\phi(\beta,\alpha)+k-\min\{k,r\} = \phi(\beta,\gamma)+r-\min\{k,r\}$ and $\min\{k-\min\{k,r\}, r-\min\{k,r\}\} = 0$ so by ($\square$.2) there exists $s \in Z^{+,0}$ such that $\phi(\alpha,\gamma)+s = \phi(\alpha,\beta)+k-\min\{k,r\}$ and $\phi(\gamma,\alpha)+s = \phi(\gamma,\beta)+r-\min\{k,r\}$. Therefore $m = \phi(\alpha,\gamma)+s+\min\{k,r\}$ and $l = \phi(\gamma,\alpha)+s+\min\{k,r\}$. Since σ is a congruence on G, $\alpha\sigma\gamma$. Thus $(m,\alpha)\rho(l,\gamma)$ and ρ is transitive. Next let $(m,\alpha)\rho(n,\beta)$ and $(l,\gamma) \in S$. Then $m = \phi(\alpha,\beta)+k$ and $n = \phi(\beta,\alpha)+k$ with $k \in Z^{+,0}$. We need to show that $(m,\alpha)(l,\gamma)\rho(n,\beta)(l,\gamma)$, that is,

$$(m+l+I(\alpha,\gamma), \alpha\gamma)\rho(n+l+I(\beta,\gamma), \beta\gamma).$$

Well, we have $m+l+I(\alpha,\gamma) = \phi(\alpha,\beta)+k+l+I(\alpha,\gamma)$ and $n+l+I(\beta,\gamma) = \phi(\beta,\alpha)+k+l+I(\beta,\gamma)$. By ($\square$.3) there exists $r \in Z^{+,0}$ such that $\phi(\alpha,\beta)+I(\alpha,\gamma) = \phi(\alpha\gamma,\beta\gamma)+r$ and $\phi(\beta,\alpha)+I(\beta,\gamma) = \phi(\beta\gamma,\alpha\gamma)+r$. Thus $m+l+I(\alpha,\gamma) = \phi(\alpha\gamma,\beta\gamma)+r+k+l$ and $n+l+I(\beta,\gamma) = \phi(\beta\gamma,\alpha\gamma)+r+k+l$. Also, $\alpha\gamma\sigma\beta\gamma$ since σ is compatible. Hence

$$(m+l+I(\alpha,\gamma), \alpha\gamma)\rho(n+l+I(\beta,\gamma), \beta\gamma).$$

Thus ρ is a congruence on S.

If σ and ϕ satisfy conditions ($\square$.1) thru ($\square$.3) then define a map $\bar{\phi}: \sigma \to Z^{+,0}$ by

$$\bar{\phi}(\alpha,\beta) = \phi(\alpha,\beta) - \min\{\phi(\alpha,\beta), \phi(\beta,\alpha)\}. \tag{3.10}$$

Then we have,

THEOREM 3.13. *If σ and ϕ satsify ($\square$.1) thru ($\square$.3) then σ and $\bar{\phi}$ satisfy ($\square$.1) thru ($\square$.3). The congruence $(\sigma, \bar{\phi})$ is in $\mathscr{L}_C(S)$ and is the smallest cancellative congruence on S containing (σ, ϕ).*

Proof. ($\square$.1) for $\bar{\phi}$: $\bar{\phi}(\alpha,\alpha) = \phi(\alpha,\alpha) - \min\{\phi(\alpha,\alpha), \phi(\alpha,\alpha)\} = 0$. ($\square$.2) for $\bar{\phi}$: Let $k, l \in Z^{+,0}$ with $\min\{k,l\} = 0$ and $\bar{\phi}(\beta,\alpha)+k = \bar{\phi}(\beta,\gamma)+l$. Then by (3.10) we have

$$\phi(\beta,\alpha) - \min\{\phi(\beta,\alpha), \phi(\alpha,\beta)\} + k$$
$$= \phi(\beta,\gamma) - \min\{\phi(\beta,\gamma), \phi(\gamma,\beta)\} + l.$$

Set $r = k - \min\{\phi(\beta, \alpha), \phi(\alpha, \beta)\}$, $s = l - \min\{(\beta, \gamma), \phi(\gamma, \beta)\}$ and $t = \min\{r, s\}$. Then $\phi(\beta, \alpha) + r - t = \phi(\beta, \gamma) + s - t$ with $r - t, s - t \in Z^{+,0}$ and $\min\{r - t, s - t\} = 0$; therefore, by ($\square$.2) for there exists $m \in Z^{+,0}$ such that $\phi(\alpha, \gamma) + m = \phi(\alpha, \beta) + r - t$ and $\phi(\gamma, \alpha) + m = \phi(\gamma, \beta) + s - t$. Hence

$$\begin{aligned}\bar{\phi}(\alpha, \gamma) + m &= \phi(\alpha, \gamma) - \min\{\phi(\alpha, \gamma), \phi(\gamma, \alpha)\} + m \\ &= \phi(\alpha, \beta) + r - t - \min\{\phi(\alpha, \gamma), \phi(\gamma, \alpha)\}\end{aligned}$$

and

$$\begin{aligned}\bar{\phi}(\gamma, \alpha) + m &= \phi(\gamma, \alpha) - \min\{\phi(\gamma, \alpha), \phi(\alpha, \gamma)\} + m \\ &= \phi(\gamma, \beta) + s - t - \min\{\phi(\gamma, \alpha), \phi(\alpha, \gamma)\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\bar{\phi}(\alpha, \gamma) + m + \min\{\phi(\alpha, \gamma), \phi(\gamma, \alpha)\} + t = \phi(\alpha, \beta) + r \\ &\quad = \phi(\alpha, \beta) - \min\{\phi(\beta, \alpha), \phi(\alpha, \beta)\} + k \\ &\quad = \bar{\phi}(\alpha, \beta) + k,\end{aligned}$$

and

$$\begin{aligned}&\bar{\phi}(\gamma, \alpha) + m + \min\{\phi(\gamma, \alpha), \phi(\alpha, \gamma)\} + t = \phi(\gamma, \beta) + s \\ &\quad = \phi(\gamma, \beta) - \min\{\phi(\beta, \gamma), \phi(\gamma, \beta)\} + l \\ &\quad = \bar{\phi}(\gamma, \beta) + l.\end{aligned}$$

Thus ($\square$.2) holds for $\bar{\phi}$.

($\square$.3) for $\bar{\phi}$: By ($\square$.3) for ϕ we have

$$\phi(\alpha, \beta) + I(\alpha, \gamma) - \phi(\alpha\gamma, \beta\gamma) = \phi(\beta, \alpha) + I(\beta, \gamma) - \phi(\beta\gamma, \alpha\gamma) \geqq 0$$

for all $\alpha, \beta, \gamma \in G$ with $\alpha\sigma\beta$. Therefore, we have

$$\begin{aligned}\text{(a)} \quad &\phi(\alpha, \beta) - \min\{\phi(\alpha, \beta), \phi(\beta, \alpha)\} + I(\alpha, \gamma) - \phi(\alpha\gamma, \beta\gamma) \\ &\quad + \min\{\phi(\alpha\gamma, \beta\gamma), \phi(\beta\gamma, \alpha\gamma)\} \\ &= \phi(\beta, \alpha) - \min\{\phi(\alpha, \beta), \phi(\beta, \alpha)\} + I(\beta, \gamma) - \phi(\beta\gamma, \alpha\gamma) \\ &\quad + \min\{\phi(\alpha\gamma, \beta\gamma), \phi(\beta\gamma, \alpha\gamma)\}.\end{aligned}$$

Which by (3.10) becomes

$$\text{(b)} \quad \bar{\phi}(\alpha, \beta) + I(\alpha, \gamma) - \bar{\phi}(\alpha\gamma, \beta\gamma) = \bar{\phi}(\beta, \alpha) + I(\beta, \gamma) - \bar{\phi}(\beta\gamma, \alpha\gamma).$$

We need only show that this number on either side of (b) is nonnegative, equivalently, the number on either side of (a) is nonnegative. Note that $-\phi(\alpha\gamma,\beta\gamma)+\min\{\phi(\alpha\gamma,\beta\gamma),\phi(\beta\gamma,\alpha\gamma)\}=0$ or $-\phi(\beta\gamma,\alpha\gamma)+\min\{\phi(\alpha\gamma,\beta\gamma),\phi(\beta\gamma,\alpha\gamma)\}=0$; hence, in the first case the left-hand side of (a) is clearly nonnegative and in the second case the right-hand side of (a) is clearly nonnegative. Thus (□.3) holds for $\bar{\phi}$.

Proof that $\bar{\rho}=(\sigma,\phi)$ is in $\mathscr{L}_C(S)$: Let $(m,\alpha)(l,\gamma)\bar{\rho}(n,\beta)(l,\gamma)$ for (m,α), (n,β) and (l,γ) in S. That is,

(c) $$(m+l+I(\alpha,\gamma),\alpha\gamma)\bar{\rho}(n+l+I(\beta,\gamma),\beta\gamma).$$

Hence $\alpha\gamma\sigma\beta\gamma$ and so $\alpha\sigma\beta$. Thus $(\bar{\phi}(\alpha,\beta),\alpha)\bar{\rho}(\bar{\phi}(\beta,\alpha),\beta)$ and so

$$(\bar{\phi}(\alpha,\beta),\alpha)(0,\gamma)\bar{\rho}(\bar{\phi}(\beta,\alpha),\beta)(0,\gamma)$$

or

$$(\bar{\phi}(\alpha,\beta)+I(\alpha,\gamma),\alpha\gamma)\bar{\rho}(\bar{\phi}(\beta,\gamma)+I(\beta,\gamma),\beta\gamma)$$

and

$$(\bar{\phi}(\alpha,\beta)+I(\alpha,\gamma)+I(\alpha\gamma,\gamma^{-1}),\alpha)\bar{\rho}(\bar{\phi}(\beta,\alpha)+I(\beta,\gamma)+I(\beta\gamma,\gamma^{-1}),\beta).$$

But by (3.9′) this implies that

(d) $$I(\alpha,\gamma)+(\alpha\gamma,\gamma^{-1})=I(\beta,\gamma)+I(\beta\gamma,\gamma^{-1}).$$

Now from (c) if we multiply both sides by $(0,\gamma^{-1})$ we get

$$(m+l+I(\alpha,\gamma)+I(\alpha\gamma,\gamma^{-1}),\alpha)\bar{\rho}(n+l+I(\beta,\gamma)+I(\beta\gamma,\gamma^{-1}),\beta).$$

Therefore, $m+l+I(\alpha,\gamma)+I(\alpha\gamma,\gamma^{-1})=\bar{\phi}(\alpha,\beta)+r$ and $n+l+I(\beta,\gamma)+I(\beta\gamma,\gamma^{-1})=\phi(\beta,\alpha)+r$ for some $r\in Z^{+,0}$. Rewriting the last line we have $m=\bar{\phi}(\alpha,\beta)+r-l-I(\alpha,\gamma)-I(\alpha\gamma,\gamma^{-1})$ and $n=\phi(\beta,\alpha)+r-l-I(\beta,\gamma)-I(\beta\gamma,\gamma^{-1})$. Let

$$s=r-l-I(\alpha,\gamma)-I(\alpha\gamma,\gamma^{-1})=r-l-I(\beta,\gamma)-I(\beta\gamma,\gamma^{-1})$$

by (d). This number s is nonnegative because $\bar{\phi}(\alpha,\beta)$ or $\bar{\phi}(\beta,\alpha)$ is 0 and $m,n\in Z^{+,0}$. Hence by (3.9′) we have $(m,\alpha)\bar{\rho}(n,\beta)$.

Proof that $\bar{\rho}$ is the smallest cancellative congruence on S containing $\rho=(\sigma,\phi)$: Let $(m,\alpha)\tau(n,\beta)$ where τ is any cancellative congruence on S containing ρ. We have $(m,\alpha)=(m-1,\alpha)(0,\epsilon)$ and $(n,\beta)=(n-1,\beta)(0,\epsilon)$ if $m-1$ and $n-1$ are nonnegative. Hence

$(m,\alpha)\tau(n,\beta)$ implies $(m-1,\alpha)\tau(n-1,\beta)$ if $m-1$ and $n-1$ are nonnegative. Similarly, $(m-2,\alpha)\tau(n-2,\beta)$ if $m-2$ and $n-2$ are nonnegative, etc. Thus $(\bar{\phi}(\alpha,\beta),\alpha)\tau(\bar{\phi}(\beta,\alpha),\beta)$ and so $\tau \supseteq \rho$. This completes the proof.

COROLLARY 3.14. *Let $(\sigma,\phi)\in\mathcal{L}_\square(S)$. Then $(\sigma,\phi)\in\mathcal{L}_C(S)$ if and only if for all $\alpha,\beta\in G$ with $\alpha\sigma\beta$ we have $\phi(\alpha,\beta)=0$ or $\phi(\beta,\alpha)=0$.*

The proof of this corollary is contained in the proof that $\bar{\rho}$ is a cancellative congruence in Theorem 3.13.

We will now determine the triple (H,h,A), from Theorem 2.4, associated with $\bar{\rho}=(\sigma,\bar{\phi})$.

THEOREM 3.15. *Let $\rho=(\sigma,\bar{\phi})\in\mathcal{L}_\square(S)$. Let $H=\ker\sigma=\{\alpha\in G\colon \alpha\sigma\epsilon\}$ and let $h\colon H\to Z$ be given by $h(\alpha)=\bar{\phi}(\alpha,\epsilon)-\bar{\phi}(\epsilon,\alpha)=\phi(\alpha,\epsilon)-\phi(\epsilon,\alpha)$ for all $\alpha\in H$. Then h satisfies (2.2) with $A=0$ and $(H,h,0)=(\sigma,\bar{\phi})$.*

Proof. Let $\alpha,\beta\in H$. By ($\square$.3) $\phi(\alpha,\epsilon)+I(\alpha,\beta)-\phi(\alpha\beta,\beta)=\phi(\epsilon,\alpha)+1-\phi(\beta,\alpha\beta)$ so that $1-I(\alpha,\beta)=\phi(\alpha,\epsilon)-\phi(\epsilon,\alpha)+\phi(\beta,\alpha\beta)-\phi(\alpha\beta,\beta)$. Let ϕ_ρ be defined by (3.8) then $\phi_\rho=\phi$ if $\rho=(\sigma,\phi)$. Hence Lemma 3.9 gives $\phi(\beta,\alpha\beta)-\phi(\alpha\beta,\beta)=\phi(\beta,\epsilon)-\phi(\epsilon,\beta)+\phi(\epsilon,\alpha\beta)-\phi(\alpha\beta,\epsilon)$ because $\beta\sigma\epsilon$ and $\epsilon\sigma\alpha\beta$ for $\alpha,\beta\in H$. Thus

$$\begin{aligned}1-I(\alpha,\beta)&=\phi(\alpha,\epsilon)-\phi(\epsilon,\alpha)+\phi(\beta,\epsilon)-\phi(\epsilon,\beta)+\phi(\epsilon,\alpha\beta)-\phi(\alpha\beta,\epsilon)\\&=h(\alpha)+h(\beta)-h(\alpha\beta)\end{aligned}$$

and (2.2) holds for h.

To see that $(H,h,0)=(\sigma,\bar{\phi})$ let $\tau=(H,h,0)$ and $\bar{\rho}=(\sigma,\bar{\phi})$. Then $(m,\alpha)\tau(n,\beta)$ if and only if $\alpha\beta^{-1}\in H$ and $m-n=I(\beta,\beta^{-1})-I(\alpha,\beta^{-1})+h(\alpha\beta^{-1})$ by (2.4). That is, $(m,\alpha)\tau(n,\beta)$ if and only if $\alpha\sigma\beta$ and $m-n=I(\beta,\beta^{-1})-I(\alpha,\beta^{-1})+\phi(\alpha\beta^{-1},\epsilon)-\phi(\epsilon,\alpha\beta^{-1})$. Now ($\square$.3) implies that $\phi(\alpha\beta^{-1},\epsilon)-\phi(\epsilon,\alpha\beta^{-1})=\phi(\alpha,\beta)+I(\alpha,\beta^{-1})-\phi(\beta,\alpha)-I(\beta,\beta^{-1})$. So $(m,\alpha)\tau(n,\beta)$ if and only if $\alpha\sigma\beta$ and $m-n=\phi(\alpha,\beta)-\phi(\beta,\alpha)=\bar{\phi}(\alpha,\beta)-\bar{\phi}(\beta,\alpha)$. Since $\bar{\phi}(\alpha,\beta)$ or $\bar{\phi}(\beta,\alpha)$ is 0 this implies that $m-\phi(\alpha,\beta)=n-\bar{\phi}(\beta,\alpha)$ are nonnegative integers. Hence $(m,\alpha)\tau(n,\beta)$ if and only if $(m,\alpha)\bar{\rho}(n,\beta)$.

We now turn to the characterization of the nil-congruences on S.

Let $\rho\in\mathcal{L}_N(S)$. Let J_ρ be the ρ-class of S which is the zero of S/ρ. Let $\pi\colon S\to S/\rho$ be the natural map then $J_\rho=\pi^{-1}(0)$; hence, J_ρ is an ideal of S and $\rho\,|\,J_\rho=\omega_{J_\rho}$. From this it is easy to see that J_ρ is that ideal of S defined by (3.5). Let $\psi_\rho\colon G\to Z^{+,0}$ be the ideal function on G associated with J_ρ. Note that the relation σ_ρ on G defined by (3.7) is

ω_G. We define $\phi_\rho \colon \omega_G = G \times G \to Z^{+,0}$ by (3.8). It then follows that ρ is determined from ψ_ρ and ϕ_ρ as

$$(m,\alpha)\rho(n,\beta) \text{ if and only if } \begin{cases} m \geqq \psi_\rho(\alpha) \text{ and } n \geqq \psi_\rho(\beta) \\ \qquad\qquad \text{or} \\ m < \psi_\rho(\alpha),\ n < \psi_\rho(\beta) \text{ and there} \\ \text{exists } k \in Z^{+,0} \text{ such that} \\ m = \phi_\rho(\alpha,\beta) + k \text{ and } n = \phi_\rho(\beta,\alpha) + k. \end{cases}$$

The necessary relationships between ψ_ρ and ϕ_ρ in order that (3.11) yield a $\rho \in \mathscr{L}_N(S)$ are stated in the next few lemmas.

LEMMA 3.16. *$\phi_\rho(\alpha,\alpha) = 0$ for all $\alpha \in G$.* (This is just a repeat of Lemma 3.7).

LEMMA 3.17. *For all $\alpha, \beta \in G$ $\phi_\rho(\alpha,\beta) \leqq \psi_\rho(\alpha)$.*

Proof. $(\psi_\rho(\alpha),\alpha)\rho(\psi_\rho(\beta),\beta)$ for all $\alpha,\beta \in G$ since $\rho \mid J_\rho = \omega_{J_\rho}$. Thus the result follows from (3.8).

LEMMA 3.18. *If $(m,\alpha) \notin J_\rho$ and $(m,\alpha)\rho(n,\beta)$ then $m - n = \psi_\rho(\alpha) - \psi_\rho(\beta) = \phi_\rho(\alpha,\beta) - \phi_\rho(\beta,\alpha)$, and the last equality holds for all $\alpha,\beta \in G$.*

Proof. By Lemma 3.3, $(m,\alpha) \notin J_\rho$ implies that $(n,\beta) \notin J_\rho$. Hence $\psi_\rho(\alpha) > m$ and $\psi_\rho(\beta) > n$. Let $r = \max\{\psi_\rho(\alpha) - m, \psi_\rho(\beta) - n\}$. Then $r \in Z^+$. Now, $(m,\alpha)\rho(n,\beta)$ implies by Lemma 3.2 that $(m+r,\alpha)\rho(n+r,\beta)$. Hence, $m + r = \psi_\rho(\alpha)$ if and only if $n + r = \psi_\rho(\beta)$ by Lemma 3.2 again. That is, $\max\{\psi_\rho(\alpha) - m, \psi_\rho(\beta) - n\} = \psi_\rho(\alpha) - m = \psi_\rho(\beta) - n$ and $m - n = \psi_\rho(\alpha) - \psi_\rho(\beta)$. In particular, if $\phi_\rho(\alpha,\beta) < \psi_\rho(\alpha)$ we get $\phi_\rho(\alpha,\beta) - \phi_\rho(\beta,\alpha) = \psi_\rho(\alpha) - \psi_\rho(\beta)$; and if $\phi_\rho(\alpha,\beta) = \psi(\alpha)$ then $\phi_\rho(\beta,\alpha) = \psi(\beta)$, by Lemma 3.2, so that again $\phi_\rho(\alpha,\beta) - \phi_\rho(\beta,\alpha) = \psi_\rho(\alpha) - \psi_\rho(\beta)$.

LEMMA 3.19. *For all $\alpha,\beta,\gamma \in G$ we have $\phi_\rho(\alpha,\beta) - \phi_\rho(\beta,\alpha) + \phi_\rho(\beta,\gamma) - \phi_\rho(\gamma,\beta) = \phi_\rho(\alpha,\gamma) - \phi_\rho(\gamma,\alpha)$.*

Proof. Use the proof of Lemma 3.9 and replace the use of Lemma 3.8 by Lemma 3.18.

LEMMA 3.20 = *Lemma* 3.10 (with $\sigma_\rho = \omega_G$) *holds in this case,* *also. Just replace the use of Lemma* 3.9 *in the proof by Lemma* 3.19 *above.*

LEMMA 3.21. *for all* $\alpha, \beta, \gamma \in G$ $\phi_\rho(\alpha, \beta) + I(\alpha, \gamma) \geqq \phi_\rho(\alpha\gamma, \beta\gamma)$ *and if* $\phi_\rho(\alpha, \beta) + I(\alpha, \gamma) < \psi(\alpha, \gamma)$ *then* $\phi_\rho(\alpha, \beta) + I(\alpha, \gamma) - \phi_\rho(\alpha\gamma, \beta\gamma) = \phi_\rho(\beta, \alpha) + I(\beta, \gamma) - \phi_\rho(\beta\gamma, \alpha\gamma)$.

Proof. This follows from the compatibility of ρ, Lemma 3.18 and the definition of ϕ_ρ.

We are now ready to state and prove the theorem which characterizes all nil-congruences on S.

THEOREM 3.22. *If* $\psi\colon G \to Z^{+,0}$ *is an ideal function on* G *and if* $\phi\colon G \times G \to Z^{+,0}$ *is a function such that*

(N.1) $$\phi(\alpha, \alpha) = 0 \text{ for all } \alpha \in G.$$

(N.2) $$\phi(\alpha, \beta) \leqq \psi(\alpha) \text{ for all } \alpha, \beta \in G.$$

(N.3) $$\text{For all } \alpha, \beta \in G\ \psi(\alpha) - \psi(\beta) = \phi(\alpha, \beta) - \phi(\beta, \alpha).$$

(N.4) *For all* $\alpha, \beta, \gamma \in G$ *if* $k, l \in Z^{+,0}$ *with* $\min\{k, l\} = 0$ *and* $\phi(\beta, \alpha) + k = \phi(\beta, \gamma) + l$ *then there exists* $m \in Z^{+,0}$ *such that* $\phi(\alpha, \gamma) + m = \phi(\alpha, \beta) + k$ *and* $\phi(\gamma, \alpha) + m = \phi(\gamma, \beta) + l$.

(N.5) *For all* $\alpha, \beta, \gamma \in G$ $\phi(\alpha, \beta) + I(\alpha, \gamma) \geqq \phi(\alpha\gamma, \beta\gamma)$ *and if* $\phi(\alpha, \beta) + I(\alpha, \gamma) < \psi(\alpha\gamma)$ *then* $\phi(\alpha, \beta) + I(\alpha, \gamma) - \phi(\alpha\gamma, \beta\gamma) = \phi(\beta, \alpha) + I(\beta, \gamma) - \phi(\beta\gamma, \alpha\gamma)$.

Then the relation ρ *defined by* (3.11) *is in* $\mathscr{L}_N(S)$, *and every congruence in* $\mathscr{L}_N(S)$ *is obtained in this way. We denote* ρ *by* (ψ, ϕ).

Proof. (3.11) and (N.1) make it immediate that ρ is reflexive and symmetric. To see that ρ is transitive let $(m, \alpha)\rho(n, \beta)$ and $(n, \beta)\rho(l, \gamma)$. If $m \geqq \psi(\alpha)$, $n \geqq \psi(\beta)$, and $l \geqq \psi(\gamma)$ then by (3.11) we have $(m, \alpha)\rho(l, \gamma)$. Thus we will assume that $m < \psi(\alpha)$, $n < \psi(\beta)$ and $l < \psi(\gamma)$. Let $r, s \in Z^{+,0}$ be such that $m = \phi(\alpha, \beta) + r$, $n = \phi(\beta, \gamma) + r = \phi(\beta, \gamma) + x$, and $l = \phi(\gamma, \beta) + s$. Using (N.4) here as we used (□.2) in the proof of Theorem 3.12, we have the transitivity of ρ. We now turn to the compatibility of ρ.

Let $(m, \alpha)\rho(n, \beta)$ and $(l, \gamma) \in S$. If $m \geqq \psi(\alpha)$ and $n \geqq \psi(\beta)$ then $m + l + I(\alpha, \gamma) \geqq \psi(\alpha) + l + I(\alpha, \gamma) \geqq l + \psi(\alpha\gamma) \geqq \psi(\alpha\gamma)$ and $n + l + I(\beta, \gamma) \geqq \psi(\beta) + l + I(\beta, \gamma) \geqq l + \psi(\beta, \gamma) \geqq \psi(\beta\gamma)$. Hence

$(m, \alpha)(l, \gamma)\rho(n, \beta)(l, \gamma)$. If $m < \psi(\alpha)$ and $n < \psi(\beta)$ then there exists $r \in Z^{+,0}$ such that $m = \phi(\alpha, \beta) + r$ and $n = \phi(\beta, \alpha) + r$. If $m + l + I(\alpha, \gamma) < \psi(\alpha\gamma)$ then $\phi(\alpha, \beta) + I(\alpha, \gamma) < \psi(\alpha\gamma)$ so by (N.5) there is $s \in Z^{+,0}$ such that $\phi(\alpha, \beta) + I(\alpha, \gamma) = \phi(\alpha\gamma, \beta\gamma) + s$ and $\phi(\beta, \alpha) + (\beta, \gamma) = \phi(\beta\gamma, \alpha\gamma) + s$. Thus

$$m + l + I(\alpha, \gamma) = \phi(\alpha, \beta) + r + l + I(\alpha, \gamma) = \phi(\alpha\gamma, \beta\gamma) + r + l + s,$$

and

$$n + l + I(\beta, \gamma) = \phi(\beta\gamma, \alpha\gamma) + r + l + s.$$

Now $r + l + s \in Z^{+,0}$ and by (N.3) we have $n + l + I(\beta, \gamma) < \psi(\beta\gamma)$. Hence $m + l + I(\alpha, \gamma) < \psi(\alpha\gamma)$ if and only if $n + l + I(\beta, \gamma) < \psi(\beta\gamma)$. Thus in all cases we have $(m, \alpha)(l, \gamma)\rho(n, \beta)(l, \gamma)$ and so ρ is a congruence on S. Clearly the ρ-class of $(\psi(\alpha), \alpha)$, $\alpha \in G$, is a zero of S/ρ. Hence S/ρ is a nil semigroup and $\rho \in \mathscr{L}_N(S)$.

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