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FIXED SETS OF INVOLUTIONS

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In their work *Differentiable Periodic Maps*, Conner and Floyd posed the following question: Given a closed smooth n-manifold M^n, for what values of k does there exist a closed $(n+k)$-manifold V^{n+k} with smooth involution T whose fixed point set is diffeomorphic to M^n? In this paper we show that for many values of k there is a closed manifold with involution (T, V^{n+k}) whose fixed point set is cobordant to M^n.

We begin by defining I_n^k to be the set of classes in the n-dimensional unoriented cobordism group $\mathfrak{N}_n$ which are represented by an n-manifold which is the fixed point set of a closed $(n+k)$-manifold with smooth involution. Some properties of I_n^k are easy to see—for instance, that I_n^k is a subgroup of $\mathfrak{N}_n$, that $I_n^0 = \mathfrak{N}_n$, and that $I_*^k = \sum_{n=0}^{\infty} I_n^k$ is an ideal in $\mathfrak{N}_*$. It follows from [4] that if the manifold with involution (T, V^{n+1}) has fixed point set F^n, then F^n bords; hence $I_n^1 = (0)$. It is well-known that if the manifold with involution (T, V^{n+k}) has fixed point set F^n, then the mod 2 Euler characteristics $w_n(F^n)$ and $w_{n+k}(V^{n+k})$ are equal; hence for k odd I_n^k is contained in χ_n, the subgroup of classes in $\mathfrak{N}_n$ with zero Euler characteristic.

The main result of this paper is the following:

THEOREM. *For $2 \leqq k \leqq n$ and k even, $I_n^k = \mathfrak{N}_n$; for $2 < k \leqq n$ and k odd, $I_n^k = \chi_n$.*

To prove this result we first verify that I_n^k is as claimed for $k = 2, 3$ and that I_n^k contains an indecomposable cobordism class for each n not of the form $2^r - 1$ and each k such that $4 \leqq k \leqq n$. Once these facts are established, the theorem itself follows easily by induction.

It is tempting to conjecture that $I_n^k = (0)$ for $k > n$. In fact, the techniques employed in Section 2 of this paper originally appeared in a dissertation written under the suprevision of R. E. Stong at the University of Virginia which verified this conjecture for $n \leqq 5$. In this regard, the author wishes to express his gratitude and indebtedness to Professor Stong for the generous advice which underlies most of this work.

2. The structure of I_n^2. Because a smooth involution on a closed manifold can not fix an odd number of points, $I_0^k = (0)$ for $k > 0$. In this section we shall prove that for $n > 0$, $I_n^2 = \mathfrak{N}_n$, by using the

Boardman homomorphism J' introduced in [1]. In what follows we adopt the notation and terminology of [4].

Let $\mathcal{M}_m = \Sigma_{j=0}^{m} \mathfrak{N}_j(\mathrm{BO}(m-j))$. We define a map J': $\mathcal{M}_* \to \mathfrak{N}_*[[\theta]]$, where $\mathfrak{N}_*[[\theta]]$ denotes the ring of formal power series in one variable, as follows: If x is an element of $\mathcal{M}_m$, set $J'_n(x) = \Delta^m JI^{n+1}(x)$. As an element of $\mathfrak{N}_n(\mathbf{Z}_2)$, $J'_n(x)$ may be written as a sum $\Sigma_{i=0}^{n} \beta_i [A, S^{n-i}]$ for a unique choice of classes $\beta_i \in \mathfrak{N}_i$. We define $J'(x) = \Sigma_{i=0}^{\infty} \beta_i \theta^i$. Arguments similar to those found in [3] prove that J' is a homomorphism of rings.

LEMMA 2.1. *Let* $\bigcup_{j=0}^{m} \nu^{m-j} \to F^j$ *be a disjoint union of* $(m-j)$-*plane bundles. Let* β *be an element of* $\mathfrak{N}_m$. *There exists a manifold with involution* (T, V^m) *such that* β *is the class of* V^m *and* $\bigcup_{j=0}^{m} \nu^{m-j} \to F^j$ *is the normal bundle to the fixed point set of* (T, V^m) *if and only if* $J'(\Sigma_{j=0}^{m} [\nu^{m-j} \to F^j]) = \beta\theta^m +$ *higher power terms.*

Proof. Without loss of generality we may suppress the fact we are considering a disjoint union of bundles. Assume (T, V^m) fixes $\nu^{m-j} \to F^j$. Then $J'_m([\nu^{m-j} \to F^j]) = \Delta^m JI^{m+1}([\nu^{m-j} \to F^j]) = [V^m][A, S^0]$ by [4]; so $J'([\nu^{m-j} \to F^j]) = [V^m]\theta^m +$ higher power terms. Assume $J'([\nu^{m-j} \to F^j]) = \beta\theta^m +$ higher power terms. By definition,

$$0 = J'_{m-1}([\nu^{m-j} \to F^j]) = \Delta^m JI^m([\nu^{m-j} \to F^j]) = [A, S(\nu^{m-j})].$$

Suppose $(A, S(\nu^{m-j}))$ bounds (S, M^m). Let

$$V^m = (D(\nu^{m-j}) \cup M^m)/(S(\nu^{m-j}) \equiv \partial M^m)$$

and $T = A \cup S$. The normal bundle to the fixed point set of (T, V^m) is $\nu^{m-j} \to F^j$ and hence $\beta = [V^m]$.

We use Lemma 2.1 to explicitly compute J' on a basis for $\mathcal{M}_*$: Let T be the involution on $\mathbf{R}P(n+1)$ defined by mapping $[x_0, \cdots, x_{n+1}]$ to $[-x_0, x_1, \cdots, x_{n+1}]$. The normal bundle to the fixed point set of $(T, \mathbf{R}P(n+1))$ is $\mathbf{R}^{n+1} \to \mathbf{R}P(0) \cup \lambda \to \mathbf{R}P(n)$, where λ is the canonical line bundle. Let λ_n denote the cobordism class of $\lambda \to \mathbf{R}P(n)$. Then by 2.1, $J'(\lambda_n) = 1 + \Sigma_{i=0}^{\infty} [V(n+1, i)]\theta^{n+i+1}$, where the $V(n+1, i)$ are the manifolds studied in [5]. In particular, $[V(n+1, 0)] = [\mathbf{R}P(n+1)]$ and

$$[V(n+1, i)] = [\mathbf{R}P(n+i+1)] + [\mathbf{R}P(\lambda \oplus \mathbf{R}^{i+1})] + \sum_{k=0}^{i-1} [\mathbf{R}P(i-k)]\,[V(n+1, k)],$$

where $\mathbf{R}P(\lambda \oplus \mathbf{R}^{i+1})$ is used here to denote the total space of the projective space bundle associated to $\lambda \oplus \mathbf{R}^{i+1} \to \mathbf{R}P(n)$.

LEMMA 2.2. *Let $\alpha \in \mathfrak{N}_n$. α belongs to I_n^2 if and only if there exists a 2-plane bundle $\nu^2 \to F^n$ such that $\alpha = [F^n]$ and the first nonzero term appearing in the power series expansion of $J'([\nu^2 \to F^n])$ is at least $(n+1)$-dimensional.*

Proof. Lemma 2.1 implies that α belongs to I_n^2 if and only if there exists a 2-plane bundle $\nu^2 \to F^n$ such that $\alpha = [F^n]$ and the first nonzero term appearing in the power series expansion of $J'([\nu^2 \to F^n])$ is at least $(n+2)$-dimensional. By [**5**; 2.1], $J(\mathfrak{N}_n(\mathrm{BO}(2))) \subset \mathfrak{N}_{n+1}(\mathbf{Z}_2)$. Thus, requiring that the first nonzero term be at least $(n+1)$-dimensional is sufficient.

LEMMA 2.3. *$I_n^2 = \mathfrak{N}_n$ for $n \geqq 1$.*

Proof. We use Lemma 2.2 to show that for each positive dimension n, not of the form $2^r - 1$, I_n^2 contains an indecomposable cobordism class; the result then follows from [**7**]. Because conjugation on $\mathbf{C}P(2)$ fixes $\mathbf{R}P(2)$, I_2^2 contains the class of $\mathbf{R}P(2)$. Because $J'(\lambda_{4n+2}\lambda_0 + \lambda_{2n+1}^2) = ([V(4n+3,1)] + [V(2n+2,0)]^2)\theta^{4n+4} +$ higher power terms, I_{4n+2}^2 contains the class of $\mathbf{R}P(4n+2)$. Because $J'(\lambda_{4n}\lambda_0 + \lambda_{2n}^2) = [V(4n+1,1)]\theta^{4n+2} +$ higher power terms, I_{4n}^2 contains the class of $\mathbf{R}P(4n) \cup \mathbf{R}P(2n) \times \mathbf{R}P(2n)$. Suppose $n = 2^p(2q+1) - 1$ for $p, q > 0$. For each j, $0 \leqq j \leqq n$, let the cobordism class γ_j be defined by

$$\gamma_j = \begin{cases} 1 & j = 0 \\ 0 & 1 \leqq j \leqq 2^p + 1 \\ [V(2^p+1, j-2^p-1)] & 2^p + 2 \leqq j \leqq 2^{p+1}q - 1 \\ [V(2^p+1, j-2^p-1)] + [V(2^{p+1}q, j-2^{p+1}q)] & 2^{p+1}q \leqq j \leqq n. \end{cases}$$

Let $\gamma = \Sigma_{j=0}^n \gamma_j \lambda_{n-j} \lambda_0$. Then $J'(\lambda_2^p \lambda_{2^{p+1}q-1} + \gamma) = \beta\theta^{n+1} +$ higher power terms, for some class $\beta \in \mathfrak{N}_{n+1}$. By Lemma 2.2, the base of $\lambda_2^p \lambda_{2^{p+1}q-1} + \gamma$ belongs to I_n^2; by [**5**; 4.2] and [**6**; 3.4], this class is indecomposable.

3. The structure of I_n^k, $2 < k \leqq n$.

Let $\xi^k \to M^{n-k+1}$ be an arbitrary k-plane bundle and let $\mathbf{R}P(\xi^k)$ denote the total space of the associated projective space bundle.

LEMMA 3.1. *I_n^k contains the cobordism class of $\mathbf{R}P(\xi^k) \cup M^{n-k+1} \times \mathbf{R}P(k-1)$.*

Proof. Consider the Whitney sum $\xi^k \oplus \mathbf{R}^k \to M^{n-k+1}$ and the total space $\mathbf{R}P(\xi^k \oplus \mathbf{R}^k)$ of the associated projective space bundle. Multiplication by -1 in the fibers of ξ^k induces an involution on $\mathbf{R}P(\xi^k \oplus \mathbf{R}^k)$ whose fixed point set is $\mathbf{R}P(\xi^k) \cup M^{n-k+1} \times \mathbf{R}P(k-1)$.

LEMMA 3.2. $I_n^3 = \chi_n$.

Proof. Recall from §1 that I_*^3 is contained in $\chi_* = \Sigma_{n=0}^{\infty} \chi_n$, the ideal of classes in $\mathfrak{N}_*$ with zero Euler characteristic. It is not hard to see that χ_n contains an indecomposable cobordism class for each dimension $n \geqq 4$, $n \neq 2^r - 1$, and that χ_* is generated by these elements. In [6; 8.1] Stong exhibited for each $n \geqq 4$, $n \neq 2^r - 1$, a 3-plane bundle $\xi^3 \to M^{n-2}$ such that the cobordism class of $\mathbf{R}P(\xi^3)$ is indecomposable. Thus by Lemma 3.1 I_n^3 contains the indecomposable class $\mathbf{R}P(\xi^3) \cup M^{n-2} \times \mathbf{R}P(2)$, and therefore $I_*^3 = \chi_*$.

To prove that I_n^k is as stated in §1 we need finally to show that I_n^k contains an indecomposable cobordism class for each dimension n not of the form $2^r - 1$ and each k such that $4 \leqq k \leqq n$.

LEMMA 3.3. *I_n^k contains an indecomposable cobordism class for each $n \neq 2^r - 1$ and each k such that $4 \leqq k \leqq \alpha(n)$, where $\alpha(n)$ denotes the number of ones in the dyadic expansion of n.*

Proof. Recall the Stong manifolds from [6]: Let $(n_1, \cdots, n_k)$ be a partition of $n + k - 1$ and let $p: \mathbf{R}(P(n_1, \cdots, n_k) \to \mathbf{R}P(n_1) \times \cdots \times \mathbf{R}P(n_k)$ be the projective space bundle associated to $\lambda_1 \oplus \cdots \oplus \lambda_k \to \mathbf{R}P(n_1) \times \cdots \times \mathbf{R}P(n_k)$, where λ_i is the pullback of the canonical line bundle over the ith factor. By Lemma 3.1 I_n^k contains the cobordism class of $\mathbf{R}P(n_1, \cdots, n_k) \cup \mathbf{R}P(n_1) \times \cdots \times \mathbf{R}P(n_k) \times \mathbf{R}P(k-1)$; and by [6; 3.4] this class is indecomposable if and only if $\binom{n-1}{n_1} + \cdots + \binom{n-1}{n_k} = 1 \bmod 2$. It suffices then to exhibit for each choice of n and k a partition $(n_1, \cdots, n_k)$ of $n - k + 1$ such that $\binom{n-1}{n_1} + \cdots + \binom{n-1}{n_k} = 1 \bmod 2$. If $n = 2^{r_1} + \cdots + 2^{r_t}$, $r_1 > \cdots > r_t > 0$, and $4 \leqq k \leqq t$, then

$$(2^{r_1} + \cdots + 2^{r_{t-k+2}}, 2^{r_{t-k+3}} - 1, \cdots, 2^{r_{t-1}} - 1, 2^{r_t-1} - 1, 2^{r_t-1} - 1)$$

is as required. If $n = 2^{r_1} + \cdots + 2^{r_t}$, where $r_1 > \cdots > r_t = 0$ and there exists an i, $2 \leqq i \leqq t$, such that $r_{i-1} > r_i + 1$, and $4 \leqq k \leqq t$, then

$$(2^{r_1} - 2, 2^{r_2} - 1, \cdots, 2^{r_{k-2}} - 1, 2^{r_{k-1}} + \cdots + 2^{r_{t-1}}, 1)$$

is as required.

To prove that I_n^k contains an indecomposable class for each $n \neq 2^r - 1$ and each k such that $\alpha(n) < k \leqq n$ we must use a different technique, provided by the following:

LEMMA 3.4. *If M^n is a closed manifold such that $w_i(M^n) = 0$ for $i > \alpha(n) + 1$, then I_n^k contains the class of M^n for $\alpha(n) < k \leqq n$.*

Proof. The twist involution on $M^n \times M^n$ is defined by sending (x, y) to (y, x) and has fixed point set M^n; furthermore, the normal bundle to M^n in $M^n \times M^n$ is the tangent bundle $\tau M^n \to M^n$. By Lemma 2.1 $J'([\tau M^n \to M^n]) = [M^n \times M^n]\theta^{2n} +$ higher power terms. By [4], since $w_i(M^n) = 0$ for $i > \alpha(n) + 1$ there exists an $(\alpha(n) + 1)$-plane bundle $\xi \to N^n$ such that $\xi \oplus \mathbf{R}^{n-\alpha(n)-1} \to N^n$ is cobordant to $\tau M^n \to M^n$. Therefore, $J'([\xi \to N^n]) = J'([\tau M^n \to M^n]) = [M^n \times M^n]\theta^{2n} +$ higher power terms. By Lemma 2.1, for each k such that $\alpha(n) < k \leqq n$ there exists a manifold with involution (T, V^{n+k}) such that the normal bundle to the fixed point set of T is $\xi \oplus \mathbf{R}^{k-\alpha(n)-1} \to N^n$. Therefore the cobordism class of M^n, which is the same as that of N^n, belongs to I_n^k for $\alpha(n) < k \leqq n$.

It remains then to show that for each dimension $n \neq 2^r - 1$ there is an indecomposable manifold M^n such that $w_i(M^n) = 0$ for $i > \alpha(n) + 1$. For this purpose we define generalized Stong manifolds as follows: Let $N = (N_1, \cdots, N_k)$ be a k-tuple where for each i, $1 \leqq i \leqq k$, N_i is a t_i-tuple $(n_{i1}, \cdots, n_{it_i})$ of nonnegative integers. Define $\mathbf{R}P(N_1, \cdots, N_k)$ to be the total space of the projective space bundle associated to $\lambda_1 \oplus \cdots \oplus \lambda_k \to \mathbf{R}P(N_1) \times \cdots \times \mathbf{R}P(N_k)$, where λ_i is the pullback of the canonical line bundle over the Strong manifold $\mathbf{R}P(N_i)$. Letting $|N_i|$ denote $n_{i1} + \cdots + n_{it_i} + t_i - 1$ and $|N| = |N_1| + \cdots + |N_k| + k - 1$, we see that $\mathbf{R}P(N_1, \cdots, N_k)$ is an $|N|$-dimensional manifold.

LEMMA 3.5. *$\mathbf{R}P(N_1, \cdots, N_k)$ represents an indecomposable cobordism class if and only if*

$$\binom{|N|-1}{|N_1|} + \cdots + \binom{|N|-1}{|N_k|} \quad \textit{is odd.}$$

Proof. There is a degree one map $\mathbf{R}P(N_1) \times \cdots \times \mathbf{R}P(N_k) \to \mathbf{R}P(|N_1|) \times \cdots \times \mathbf{R}P(|N_k|)$ such that the pullback of $\lambda_1 \oplus \cdots \oplus \lambda_k \to \mathbf{R}P(|N_1|) \times \cdots \times \mathbf{R}P(|N_k|)$ is $\lambda_1 \oplus \cdots \oplus \lambda_k \to \mathbf{R}P(N_1) \times \cdots \times \mathbf{R}P(N_k)$. By [6; 2.4], $\mathbf{R}P(N_1, \cdots, N_k)$ is indecomposable if and only if $\mathbf{R}P(|N_1|, \cdots, |N_k|)$ is; but, by [6; 3.4] $\mathbf{R}P(|N_1|, \cdots, |N_k|)$ is indecomposable if and only if

$$\binom{|N|-1}{|N_1|}+\cdots+\binom{|N|-1}{|N_k|} \quad \text{is odd.}$$

The cohomology and Stiefel–Whitney classes of $\mathbf{R}P(N_1,\cdots,N_k)$ are explicitly computable from [4]. In fact, let $H^*(\mathbf{R}P(n_{ij});\mathbf{Z}_2)=\mathbf{Z}_2[\alpha_{ij}]/(\alpha_{ij}^{n_{ij}+1}=0)$ and c_i and e represent the characteristic class of the canonical line bundle over $\mathbf{R}P(N_i)$ and $\mathbf{R}P(N_1,\cdots,N_k)$ respectively. Suppressing all bundle maps, we may write

$$w(\mathbf{R}P(N_1,\cdots,N_k))=\prod_{i=1}^{k}\prod_{j=1}^{t_i}(1+\alpha_{ij})^{n_{ij}+1}(1+c_i+\alpha_{ij})(1+e+c_i).$$

LEMMA 3.6. *For each dimension $n\neq 2^r-1$ there is an indecomposable manifold M^n such that $w_i(M^n)=0$ for $i>\alpha(n)+1$.*

Proof. If $n=2^{r_1}+\cdots+2^{r_t}$, $r_1>\cdots>r_t>0$, let $M^n=\mathbf{R}P((2^{r_1}-1,\cdots,2^{r_{t-1}}-1,0),\ (2^{r_t-1}-1),\ (2^{r_t-1}-1))$. If $n=2^{r_1}+\cdots+2^{r_t}+2^j+2^{j-1}+\cdots+1$, $r_1>\cdots>r_t>j+1$, let $M^n=\mathbf{R}P((2^{r_1}-1,\cdots,2^{r_{t-1}}-1,2^{r_t-1}-1,0),\ (2^{r_t-1}-1),(2^j-1),\cdots,(2^0-1))$. That these manifolds are indecomposable is a direct consequence of Lemma 3.5. That $w_i(M^n)=0$ for $i>\alpha(n)+1$ is immediate from the given expansion of $w(\mathbf{R}P(N_1,\cdots,N_k))$ taken with the fact that multiplication in $H^*(\mathbf{R}P(N_1,\cdots,N_k);\mathbf{Z}_2)$ is subject to the relations $\prod_{j=1}^{t_i}(c_i+\alpha_{ij})=0$ for each i, $1\leqq i\leqq k$, and $\prod_{i=1}^{k}(e+c_i)=0$.

Let now assemble the above lemmas to prove:

THEOREM. *For $2\leqq k\leqq n$ and k even, $I_n^k=\mathfrak{N}_n$; for $2<k\leqq n$ and k odd, $I_n^k=\chi_n$.*

Proof. Let $4\leqq k\leqq n$ and assume inductively that for $2\leqq j<k\leqq n$ and j even, $I_n^j=\mathfrak{N}_n$, while for $2<j<k\leqq n$ and j odd, $I_n^j=\chi_n$. We must show that I_n^k is as claimed. Let $\alpha\in\mathfrak{N}_n$, with $w_n(\alpha)=0$ if k is odd. If α is decomposable, say $\alpha=\beta\gamma$ where $\beta\in\mathfrak{N}_p$ and $\gamma\in\mathfrak{N}_q$ with $w_q(\gamma)=0$ if k is odd, then by induction $\beta\in I_p^2$ and $\gamma\in I_q^{k-2}$. Clearly $I_p^2I_q^{k-2}\subset I_n^k$, so $\alpha\in I_n^k$. If α is indecomposable, then by Lemmas 3.3–3.6 α belongs to I_n^k mod decomposables; but, since I_n^k contains all decomposables, $\alpha\in I_n^k$.

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