A MODIFICATION OF THE NEUMANN-POINCARÉ METHOD FOR MULTIPLY CONNECTED REGIONS

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1. **Introduction.** Some interest has been attached to the problem of effective computation of the solution to boundary value and conformal mapping problems. Birkhoff [3; 4] has given an excellent iteration precedure for the solution of the conformal mapping problem for simply connected regions. However, the convergence of his method is easily seen to be the same as that for the classical Neumann-Poincaré method in potential theory, which, while converging for all simply connected regions [1; 5], fails to converge for the computation of the harmonic measures of multiply connected regions [1; 2]. Since it is primarily these harmonic measures which are needed in the conformal mapping problem, we derive in the present paper a modification of the Neumann-Poincaré method which will converge in this case and apply it to the conformal mapping of doubly connected regions. While the formulas involve certain ϑ-series, these should not present a major problem for numerical computation since the series are very rapidly convergent and only a few terms need be taken.

2. **General formulation.** Let Ω be a multiply connected region which is bounded by a smooth curve C. Let $U(z)$ be a continuous real-valued function defined for $z \in C$. We wish to consider the problem of finding a function $u + iv$ which is analytic in Ω and for which

$$u(\zeta) \longrightarrow U(z), \qquad \text{as } \zeta \longrightarrow z \in C. \tag{1}$$

We do not require that v be single valued in Ω, and we shall assume that at a fixed point $\zeta_0 \in C$ we have

$$U(\zeta_0) = 0. \tag{2}$$

Let W be a Riemann surface which contains Ω and for which $\tilde{\Omega} = W - \Omega$ is also a connected region. For example, if Ω is the plane region exterior to the contours $C_1, \cdots, C_{n-1}$ and interior to C_n, we may take W to be the double of the region R which is bounded by circles $\gamma_1, \cdots, \gamma_n$, where γ_i lies inside

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C_i for $i = 1, \cdots, n-1$, and γ_n lies outside C_n.

If $g(z;\zeta,\zeta_0)$ is the Green's function for W, that is, a harmonic function of the variable z which has a positive logarithmic pole at ζ and a negative logarithmic pole at ζ_0, we have the symmetry relation

$$(3) \qquad g(z;\zeta,\zeta_0) - g(z_0;\zeta,\zeta_0) = g(\zeta; z, z_0) - g(\zeta_0; z, z_0).$$

From this it follows that the derivatives of g with respect to x and y (where $z = x + iy$) are harmonic functions of ζ.

Thus if $\mu(z)$ is a continuous real-valued function defined for $z \in C$, then the function

$$(4) \qquad u_i(\zeta) = \frac{1}{2\pi} \int_C \mu(z) \frac{\partial g}{\partial n_z}\, ds_z \qquad (\zeta \in \Omega),$$

(where n is the inner normal to Ω) is a harmonic function of ζ. If further we define

$$(5) \qquad u_e(\zeta) = \frac{1}{2\pi} \int_C \mu(z) \frac{\partial g}{\partial n_z}\, ds_z \qquad (\zeta \in \tilde{\Omega}),$$

then we know by the well-known boundary behavior of double layer distributions that

$$(6) \qquad \frac{\partial u_e(\zeta)}{\partial n} = \frac{\partial u_i(\zeta)}{\partial n} \qquad (\zeta \in C),$$

while

$$(7) \qquad u_i(\zeta) = -\frac{1}{2}\mu(\zeta) + \frac{1}{2}\mu(\zeta_0) + \frac{1}{2\pi}\int_C \mu(z) \frac{\partial g}{\partial n_z}\, ds_z$$

and

$$(8) \qquad u_e(\zeta) = \frac{1}{2}\mu(\zeta) - \frac{1}{2}\mu(\zeta_0) + \frac{1}{2\pi}\int_C \mu(z) \frac{\partial g}{\partial n_z}\, ds_z .$$

If we suppose that $\mu(\zeta) = 0$, then the solution of our problem is determined if we can solve the integral equation

$$(9) \qquad 2U(\zeta) = -\mu(\zeta) + \frac{1}{\pi}\int_C \mu(z) \frac{\partial g}{\partial n_z}\, ds_z ,$$

for μ subject to the condition that $\mu(\zeta_0) = 0$. If this is done, then u is given by

(7), and v is given by

(10) $$v = \frac{1}{2\pi} \int_C \mu(z) \frac{\partial h}{\partial n_z} ds_z,$$

where $\partial h/\partial n_z$ $(z; \zeta, \zeta_0)$ is the harmonic conjugate with respect to ζ of $\partial g/\partial n_z$ $(z; \zeta, \zeta_0)$; that is, such that

$$\frac{\partial g}{\partial n} + i \frac{\partial h}{\partial n}$$

is an analytic function of ζ.

3. **Solution of the integral equation.** Because of the smoothness of C, we know that $\partial g/\partial n_z$ is a continuous function on C, and the Fredholm theory states that either

(11) $$2U(\zeta) = -\mu(\zeta) + \frac{\lambda}{\pi} \int_C \mu(z) \frac{\partial g}{\partial n_z} ds_z$$

has a unique continuous solution for all continuous functions $U(z)$, or else

(12) $$\mu(\zeta) = \frac{\lambda}{\pi} \int_C \mu(z) \frac{\partial g}{\partial n_z} ds_z$$

has a nontrivial solution. Moreover, if we set

(13) $$T[\phi] = \frac{1}{\pi} \int_C \phi \frac{\partial g}{\partial n_z} ds_z,$$

we have

(14) $$\mu(z) = -2 \sum_{\nu=0}^{\infty} \lambda^\nu T^\nu U,$$

uniformly in z for all λ with $|\lambda| < |\lambda_0|$, where λ_0 is the absolutely smallest value of λ for which (12) has a nontrivial solution. Thus if we can show that $|\lambda_0| \geq M > 1$, we know that the series

(15) $$\mu(z) = -2 \sum_{\nu=0}^{\infty} T^\nu U$$

converges with order at least $1/M$ and gives a solution to the integral equation (9). For computational purposes, the series (15) is more conveniently expressed by the following iteration:

$$\mu_1(\zeta) = -\,2U(\zeta)$$
(16)
$$\mu_n(\zeta) = -\,2U(\zeta) + T\,\mu_{n-1}.$$

In order to ensure the validity of this iterative method, we need only to bound the eigenvalues of (12) away from unity. Suppose that we have an eigenfunction

$$\phi = \lambda_1\, T\phi\,.$$

Then

$$u_i(\zeta) = \frac{1}{2\pi} \int_C \phi(z) \frac{\partial g}{\partial n_z}\, ds_z$$

and

$$u_e(\zeta) = \frac{1}{2\pi} \int_C \phi(z) \frac{\partial g}{\partial n_z}\, ds_z$$

are harmonic functions in Ω and $\tilde{\Omega}$, respectively. For $\zeta \in C$ we have, by (7) and (8),

$$u_i(\zeta) = -\frac{1}{2}\,\phi(\zeta) + \frac{1}{2\pi} \int_C \phi(z) \frac{\partial g}{\partial n}\, ds = -\frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)\phi(\zeta).$$

Also,

$$u_e = +\frac{1}{2}\,\phi(\zeta) + \frac{1}{2\pi} \int_C \phi(z) \frac{\partial g}{\partial n}\, ds = \frac{1}{2}\left(1 + \frac{1}{\lambda_1}\right)\phi(\zeta),$$

whence

(17) $$u_i(\zeta) = \frac{1 - \lambda_1}{1 + \lambda_1}\, u_e(\zeta).$$

By (6) we have

(18) $$\frac{\partial u_i}{\partial n} = \frac{\partial u_e}{\partial n},$$

and so

$$D(u_i) = \int_C u_i \frac{\partial \bar{u}_i}{\partial n}\, ds = \frac{1 - \lambda_1}{1 + \lambda_1} \int_C u_e \frac{\partial \bar{u}_e}{\partial n}\, ds = \frac{\lambda_1 - 1}{\lambda_1 + 1}\, D(u_e),$$

where

$$D(u_i) = \iint_{\Omega} \left\{ \left| \frac{\partial u_i}{\partial x} \right|^2 + \left| \frac{\partial u_i}{\partial y} \right|^2 \right\} dx\, dy$$

and

$$D(e_u) = \iint_{\tilde{\Omega}} \left\{ \left| \frac{\partial u_e}{\partial x} \right|^2 + \left| \frac{\partial u_e}{\partial y} \right|^2 \right\} dx\, dy.$$

Thus

$$\lambda_1 = \frac{r+1}{r-1},$$

where

$$r = \frac{D(u_e)}{D(u_i)}.$$

Since $D(u_e)$ and $D(u_i)$ are always nonnegative, we conclude that λ_1 is real and

(19) $$|\lambda_1| \geq 1.$$

In order for equality to occur in (19), we must have either u_e constant or u_i constant. In either case the constant must be zero since $u_i(\zeta_0) = 0$ and $u_e(\zeta_0) = 0$, while Ω and $\tilde{\Omega}$ are connected. Thus equality is excluded in (19) and we have, taking λ_1 to be λ_0, the absolutely smallest eigenvalue,

(20) $$|\lambda_0| > 1;$$

this ensures the convergence and correctness of the iteration (16).

In order to obtain an estimate for $|\lambda_0|$ and consequently for the rate of convergence of (20), we suppose that Ω and $\tilde{\Omega}$ are topologically equivalent. Then following Ahlfors [1] we take a quasi-conformal mapping $f(z)$ of Ω onto $\tilde{\Omega}$ with dilation quotient $\leq K$. Then the functions v_i and v_e, which are the harmonic conjugates of u_i and u_e with the conditions that $v_i(\zeta_0) = v_e(\zeta_0) = 0$, are harmonic functions in Ω and $\tilde{\Omega}$ with the same values on C by (6). Moreover, $D(v_i) = D(u_i)$; $D(v_e) = D(u_e)$.

But now $v_e[f(z)]$ is a continuous function in Ω which has the same periods and boundary values as v_i. Hence by the Dirichlet principle we have

$$D\{v_e[f(z)]\} \geq D\{v_i\}.$$

But

$$D\{v_e[f(z)]\} \le KD\{v_e\}.$$

Hence

$$K \ge \frac{D\{v_i\}}{D\{v_e\}} = \frac{\lambda_0 - 1}{\lambda_0 + 1}.$$

In a similar manner, using f^{-1}, we obtain

$$K \ge \frac{D\{v_e\}}{D\{v_i\}} = \frac{\lambda_0 + 1}{\lambda_0 - 1}.$$

Hence either

$$\lambda_0 \le \frac{-(K+1)}{(K+1)}, \quad \text{or } \lambda_0 \ge \frac{K+1}{K-1};$$

that is,

$$|\lambda_0| \ge \frac{K+1}{K-1}. \tag{21}$$

4. The doubly connected case. We now suppose that Ω is a doubly connected region in the plane contained in the circle $|z| \le 1$ and containing the circle $|z| \le q^2$ within its inner contour. We take our Riemann surface W to be the torus formed by identifying the points z and $q^2 z$.

Now we shall show that the Green's function for W is

$$g(z;\zeta,\zeta_0) = \log\left|\frac{\vartheta_*(qz/\zeta_0)}{\vartheta_*(qz/\zeta)}\right| - \frac{1}{2}\,\frac{\log|z|}{\log q}\cdot\log\left|\frac{\zeta_0}{\zeta}\right|, \tag{22}$$

where

$$\vartheta_*(t) = \vartheta_4\left(\frac{1}{2i}\log t, q\right) = 1 + \sum_{n=1}^{\infty} (-)^n q^{n^2}(t^n + t^{-n}). \tag{23}$$

From (23), we see that ϑ_* is a single-valued function of t and that

$$\vartheta_*(q^2 t) = \sum_{n=-\infty}^{\infty} (-)^n q^{n^2+2n} t^n = -(qt)^{-1} \sum_{n=-\infty}^{\infty} (-)^{n+1} q^{(n+1)} t^{n+1}$$

$$= -(qt)^{-1}\vartheta_*(t).$$

Hence

$$g(q^2z;\zeta,\zeta_0)=\log\frac{|\vartheta_*(qz/\zeta_0)|\;|\zeta_0|}{|\vartheta_*(qz/\zeta)|\;|\zeta|}-\frac{1}{2}\frac{\log|z|}{\log q}\log\left|\frac{\zeta_0}{\zeta}\right|-\log\left|\frac{\zeta_0}{\zeta}\right|$$

$$=g(z;\zeta,\zeta_0),$$

so that g is a single-valued function of z on the torus. By (23) we see that g has poles at ζ and ζ_0, since

$$\vartheta_*(q)=0.$$

Moreover, ϑ_4 has only one zero for $(1/2i)\log t$ in the rectangle whose corners are a, $a+\pi$, $a+\pi-i\log q$, $a-i\log q$ [6, p. 465]. Thus $\vartheta_*(t)$ has only one zero for $q^2\leq|t|\leq 1$, and consequently g is the Green's function of W.

5. **Computation.** The iteration (16) becomes now

$$\mu_1(\zeta)=-2U(\zeta),\tag{24}$$

$$\mu_n(\zeta)=-2U(\zeta)-\frac{2}{\pi}\int_C\mu_{n-1}(z)\,\Im\frac{\partial g}{\partial z}z'(s)\,ds,$$

since

$$\frac{\partial g}{\partial n}=\frac{\partial g}{\partial z}\frac{\partial z}{\partial n}+\frac{\partial g}{\partial\bar z}\frac{\partial\bar z}{\partial n}=2\Re\frac{\partial g}{\partial z}\frac{\partial z}{\partial n},$$

while

$$\frac{\partial z}{\partial n}=i\frac{\partial z}{\partial s},$$

and so

$$\frac{\partial g}{\partial n}=2\Re\left\{i\frac{\partial g}{\partial z}\frac{\partial z}{\partial s}\right\}$$

$$=-2\Im\frac{\partial g}{\partial z}z'.$$

Clearly (24) is independent of the choice of the parameter s, and hence s need not be arc-length, but may be any conveniently chosen parameter σ.

Now

$$g(z;\zeta,\zeta_0)=\log\left|\frac{\vartheta_*(qz/\zeta_0)}{\vartheta_*(qz/\zeta)}\right|=\frac{1}{2}\frac{\log|z|}{\log q}\log\left|\frac{\zeta_0}{\zeta}\right|,\tag{25}$$

whence

$$2\,\frac{\partial g}{\partial z} = f(z,\zeta) - f(z,\zeta_0), \tag{26}$$

where

$$f(z,\zeta) = \frac{-\vartheta_*'(qz/\zeta)\;q/\zeta}{\vartheta_*(qz/\zeta)} + \frac{1}{2z}\cdot\frac{\log|\zeta|}{\log q}. \tag{27}$$

Thus (24) becomes

$$\mu_1(\zeta) = -2U(\zeta) \tag{28}$$

$$\mu_n(\zeta) = -2U(\zeta) - \frac{1}{\pi}\int_C \mu_{n-1}(z)\,\mathcal{I}[f(z,\zeta) - f(z,\zeta_0)]\,z'ds.$$

This is equivalent to

$$-\mu_n^{(0)} = -2U(\zeta) - \frac{1}{\pi}\int_C \mu_{n-1}(z)\,\mathcal{I}\,f(z,\zeta)\,z'ds \tag{28'}$$

$$\mu_n(\zeta) = \mu_n^{(0)}(\zeta) - \mu_n^{(0)}(\zeta).$$

Near $z = \zeta$ the boundary has the following expansion in terms of arc length s:

$$z = \zeta + \zeta'\left[s + \frac{1}{2}\,i\kappa s^2 + O(s^3)\right], \tag{29}$$

where primes denote differentiation with respect to arc-length and κ is the curvature at ζ. Then, since

$$\frac{\vartheta_*''(q)}{\vartheta_*'(q)} = -\frac{2}{q},$$

we have

$$\vartheta_*\left(\frac{qz}{\zeta}\right) = \vartheta_*'(q)\left[q\,\frac{\zeta'}{\zeta}\left(s + \frac{1}{2}\,i\kappa s^2\right) - \frac{1}{q}\,\frac{q^2\zeta'^2}{\zeta^2}\,s^2 + O(s^3)\right].$$

Also

$$\vartheta_*'\left(\frac{qz}{\zeta}\right) = \vartheta_*'(q)\left[1 - \frac{2\zeta'}{\zeta}\,s + O(s^2)\right],$$

while

$$\frac{qz'}{\zeta} = \frac{q\zeta'}{\zeta}\,[1 + i\kappa s + O(s^2)].$$

Hence

$$\frac{\vartheta_*'(qz/\zeta)\; qz'/\zeta}{\vartheta_*(qz/\zeta)} = \frac{1}{s}\left[1 + \frac{1}{2}\, i\kappa s - \frac{\zeta'}{\zeta}\, s + O(s^2)\right], \tag{30}$$

whence

$$\mathcal{I} f(z, z)\, z' = \frac{1}{2}\,\kappa(z) - \mathcal{I}\,\frac{z'}{z}\left(1 - \frac{1}{2}\,\frac{\log|z|}{\log q}\right). \tag{31}$$

In terms of an arbitrary parameter σ, this becomes

$$\mathcal{I} f(z, z)\, z' = \frac{1}{2}\,\kappa(z)\,\frac{ds}{d\sigma} - \mathcal{I}\,\frac{z'}{z}\left(1 - \frac{1}{2}\,\frac{\log|z|}{\log q}\right), \tag{31$'$}$$

where now the primes denote differentiation with respect to σ.

For computational purposes, it would seem best to calculate

$$\vartheta_*(t) = 1 + \sum_{n=1}^{\infty} (-)^n\, q^{n^2}\,(t^n + t^{-n}) \tag{32}$$

and

$$t\vartheta_*'(t) = \sum_{n=1}^{\infty} (-)^n\, n q^{n^2}\,(t^n - t^{-n}) \tag{33}$$

for the different values of $t = qz/\zeta$ which occur. Then

$$f(z, \zeta)\, z' = \left(\frac{t\vartheta_*'(t)}{\vartheta_*(t)} + \frac{1}{2}\,\frac{\log|\zeta|}{\log q}\right)\frac{z'}{z}. \tag{34}$$

The conjugate function v is given by the integral

$$v = \frac{1}{2\pi}\int_C \mu(z)\,\mathcal{I}\,\tilde{f}(z, \zeta)\, z'\, d\sigma, \tag{35}$$

where

$$\tilde{f}(z, \zeta)\, z' = \left(\frac{-it\,\vartheta_*'(t)}{\vartheta_*(t)} + \frac{1}{2}\,\frac{\arg\zeta}{\log q}\right)\frac{z'}{z} \tag{36}$$

and

$$\mathcal{J}\,\tilde{f}(z, z)\, z' \sim \frac{1}{\sigma}\,, \tag{37}$$

with primes denoting differentiation with respect to σ. The integral (35) has to be taken to mean its Cauchy principle value if $\zeta \in C$. However, since

$$\int_C \mathcal{J}\, f(z, \zeta)\, z'\, d\sigma = 0,$$

we may write (35) as

$$v = \frac{1}{2\pi} \int_C [\mu(z) - \mu(\zeta)]\, \mathcal{J}\, f(z, \zeta)\, z' \cdot d\sigma, \tag{38}$$

with

$$[\mu(z) - \mu(\zeta)]\, \mathcal{J}\, f(z, \zeta)\, z' \Big|_{z=\zeta} = \frac{\partial \mu}{\partial \zeta}\,.$$

6. **An application to conformal mapping.** In order to map the doubly connected region Ω onto the annulus $r \le |w| \le 1$, it is only necessary to find, say using the method of the preceding section, that analytic function $u + iv$ in Ω for which $u = 0$ on the outer boundary of Ω and $u = 1$ on the inner contour. For if ω is the period of v, then

$$w = e^{(2\pi/\omega)\,(u + iv)}$$

maps Ω onto the annulus, and v/ω gives the angular correspondence on the boundaries.

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