GAME THEORETIC PROOF THAT CHEBYSHEV INEQUALITIES ARE SHARP

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1. Summary. This paper is concerned with showing that Chebyshev inequalities obtained by the standard method are sharp. The proof is based on relating the bound to the solution of a game. An optimum strategy yields a portion of the extremal distribution, and the remainder is obtained as a solution of the relevant moment problem.

2. Introduction. Let X be a random vector taking values in $\mathscr{X} \subset R^k$, and suppose that $Ef(X) \equiv E(f_1(X), \cdots, f_r(X)) = (\varphi_1, \cdots, \varphi_r) \equiv \varphi$ is given, where f_j is a real valued function on $\mathscr{X}$. For convenience, we suppose $f_1 \equiv 1$. An upper bound for $P\{X \in \mathscr{T}\}$, $\mathscr{T} \subset \mathscr{X}$, may be obtained as follows. If $a = (a_1, \cdots, a_r) \in R^r$ and $\chi_{\mathscr{T}}$ is the indicator of $\mathscr{T}$ then $af' \geqq \chi_{\mathscr{T}}$ on $\mathscr{X}$ implies $P\{X \in \mathscr{T}\} \leqq a\varphi'$, and if $\mathscr{A}_0 = \{a\colon af' \geqq \chi_{\mathscr{T}}$ on $\mathscr{X}\}$, a "best" bound is given by

$$P\{X \in \mathscr{T}\} \leqq \inf_{a \in \mathscr{A}_0} a\varphi' . \tag{2.1}$$

In general, a bound is called sharp if it cannot be improved. For some cases, when $\mathscr{T}$ is assumed to be closed, the bound can actually be attained by a distribution satisfying the moment hypotheses.

The main result of this paper is

THEOREM 2.1. *Inequality* (2.1) *is sharp in the following cases.*

(I) $X = (X_1, \cdots, X_k)$ with EX_iX_j or EX_i and EX_iX_j given, $i, j = 1, \cdots, k$.

(II) X has range $(-\infty, \infty), [0, \infty)$, or $[0, 1]$, and EX^j is given, $j = 1, \cdots, m$.

(III) X is a random angle in $[0, 2\pi)$ and the trigonometric moments $Ee^{i\alpha X}, \alpha = \pm 1, \cdots, \pm m$ are given.

Sharpness has been shown in (I) by Marshall and Olkin [6] when $\mathscr{T}$ is convex, and by Isii [3, 4] in the unbounded cases of (II). Sharpness has also been proved in a number of specialized situations.

In § 3 the proof for (I) will be given in detail. The necessary alterations for each of the remaining cases will be given in § 4, 5, 6, 7. The solution of certain moment problems depend on conditions on Hankel matrices, i.e., matrices of the form $H = (h_{i+j})$, and some results concerning these matrices are given in § 8.

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The notation $A > 0 \, (\geqq 0)$ is used to mean that the matrix A is symmetric and positive definite (p.s.d).

3. The multivariate case. The relation between inequality (2.1) and a game can be greatly simplified if we use matrix theoretic arguments. This is true in part because functions of the form af', $a \in \mathscr{A}_0$, can be written very naturally as quadratic or bilinear forms.

Let $X = (X_1, \cdots, X_k)$ be a random vector on R^k with $EX = \mu$ and moment matrix $EX'X = \Sigma$. If $u \equiv u(x) = (1, x)$ for $x \in R^k$, then $Eu'(X) u(X) = \begin{pmatrix} 1 & \mu \\ \mu' & \Sigma \end{pmatrix} = \Pi$. We assume $\Pi > 0$, for otherwise the dimensionality of X can be reduced.

Functions of the form $af', a \in \mathscr{A}_0$ can be written as uAu', A: $k+1 \times k+1$, $A \in \mathscr{A} = \{A; A \geqq 0, uAu' \geqq 1 \text{ for } x \in \mathscr{T}\}$. Hence

$$P\{X \in \mathscr{T}\} \leqq \inf_{a \in \mathscr{A}_0} a\varphi' = \inf_{A \in \mathscr{A}} \operatorname{tr} A\Pi \,. \tag{3.1}$$

Let $x_1, \cdots, x_m$ be points (row vectors) in R^k, $u_i = u(x_i)$, $p_1, \cdots, p_m$, $\Sigma p_i = 1$ be probabilities, $T = (u_1', \cdots, u_m')$, $D_p = \operatorname{diag}(p_1, \cdots, p_m)$, and $H = TD_pT'$. By $H \sim \mathscr{T}$ we mean that all $x_i \in \mathscr{T}$. The condition $uAu' \geqq 1$ for $x \in \mathscr{T}$ can then be written as $\operatorname{tr} AH \geqq 1$ for $H \sim \mathscr{T}$, so that $\mathscr{A} = \{A: A \geqq 0, \operatorname{tr} AH \geqq 1 \text{ for } H \sim \mathscr{T}\}$.

With this notation, we can rewrite the bound (3.1) in a form which is suggestive of a game.

$$\begin{aligned} \inf_{A \in \mathscr{A}} \operatorname{tr} A\Pi &= \inf_{\{A:\ \inf_{H \sim \mathscr{T}} \operatorname{tr} AH \geqq 1,\ A \geqq 0\}} \operatorname{tr} A\Pi \\ &= \inf_{S \geqq 0} \left(\frac{\operatorname{tr} S\Pi}{\inf_{H \sim \mathscr{T}} \operatorname{tr} SH} \right) = \left(\sup_{S \geqq 0} \inf_{H \sim \mathscr{T}} \frac{\operatorname{tr} SH}{\operatorname{tr} S\Pi} \right)^{-1} \\ &= \Big(\sup_{\{S:\ S \geqq 0,\ \operatorname{tr} S H \leqq 1\}} \inf_{\Pi \sim \mathscr{T}} \operatorname{tr} SH \Big)^{-1} \equiv \nu^{-1} \,. \end{aligned} \tag{3.2}$$

In view of (3.2) it is natural to consider the game $G = (\mathscr{S}, \mathscr{H}, g)$, where $\mathscr{S} = \{S: S \geqq 0, \operatorname{tr} S\Pi \leqq 1\}$ and $\mathscr{H} = \{H: H \sim \mathscr{T}\}$ are the strategy spaces for players I and II, respectively, and $g(S, H) = \operatorname{tr} SH$ is the payoff to player I.

Clearly $\mathscr{S}$ and $\mathscr{H}$ are closed and convex. Further, $\mathscr{S}$ is bounded since

$$||S||^2 \equiv (\operatorname{tr} SS') \leqq (\operatorname{tr} S) c_M(S) \leqq (\operatorname{tr} S)^2 \leqq (\operatorname{tr} S\Pi)^2 / c_m^2(\Pi) \leqq 1/c_m^2(\Pi) \,,$$

where $c_m(A)$, $c_M(A)$ are the minimum and maximum characteristic roots of A. For the present we assume that $\mathscr{H}$ is bounded, then by [2, Section 2.5], G has a value and there exist optimal strategies $S_0 \in \mathscr{S}$, $H_0 \in \mathscr{H}$, such that

$$\operatorname{tr} SH_0 \leqq \operatorname{tr} S_0 H_0 = \nu \leqq \operatorname{tr} S_0 H\,, \qquad \text{for all } S \in \mathscr{S}, H \in \mathscr{H}. \tag{3.3}$$

The optimal strategy S_0 has the property that $\inf_{A \in \mathscr{A}} \operatorname{tr} A\Pi = \operatorname{tr} A_0 \Pi$, where $A_0 = S_0/\nu$.

To prove sharpness of (3.1), we must show that there exists a distribution for X such that $P\{X \in \mathscr{T}\} = 1/\nu$, and $Eu'u = \Pi$. H_0 is the moment matrix of a distribution F_1 on points in $\mathscr{T}$. If we can prove the existence of a probability distribution F for X of the form $F = F_1/\nu + F_2$, and with moment matrix Π, then this distribution attains equality in (3.1). To see this, note that F assigns at least probability ν to $\mathscr{T}$, and by (3.1) it can assign at most probability ν to $\mathscr{T}$.

To show the above, we need only show that a distribution F_2 exists with total variation $1 - 1/\nu$ and moment matrix $\Psi = \Pi - H_0/\nu$. The following Lemma yields this result.

LEMMA 3.1. *Let* $\Pi > 0$, $\mathscr{S} = \{S\colon S \geqq 0, \operatorname{tr} S\Pi \leqq 1\}$.
(i) *If* $\operatorname{tr} SH \leqq \nu$ *for all* $S \in \mathscr{S}$, *then* $\Psi = \Pi - H/\nu \geqq 0$.
(ii) *If* $\operatorname{tr} SH = \nu$ *for some* $S_0 \in \mathscr{S}$, *then* Ψ *is not strictly* > 0.
(iii) *If* $\operatorname{tr} SH < \nu$ *for all* $S \in \mathscr{S}$, *then* $\Psi > 0$.

Proof. There exists a representation $\Pi = WW'$, $H = WD_\theta W'$, $|W| \neq 0$, $D_\theta = \operatorname{diag}(\theta_0, \cdots, \theta_k)$, and hence $\Psi \geqq 0$ if and only if $\theta_i \leqq \nu$, $i = 0, \cdots, k$. If $W'SW = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $S \in \mathscr{S}$, and from $\operatorname{tr} SH = tr\, W'SWD_\theta \leqq \nu$, we obtain $\theta_0 \leqq \nu$. Part (i) follows using permutations. If $\operatorname{tr} SH < \nu$, then in the above argument, each $\theta_i < \nu$. If $\operatorname{tr} S_0 H = \operatorname{tr}(W'S_0W)D_\theta = \nu$ and $\operatorname{tr} W'S_0W \leqq 1$, then at least one of the θ_i is equa to ν.

The condition that $\mathscr{H}$ be bounded now can be removed, since $\|H_0\|^2 \leqq (\operatorname{tr} H_0)^2 \leqq [\nu \operatorname{tr} \Pi]^2$, by Lemma 3.1.

REMARK 3.1. We note that $\operatorname{tr} S_0\Pi = 1$, for if not, αS_0 for $\alpha > 1$ would violate (3.3).

S_0 and H_0 are related by $\nu S_0 \Pi = S_0 H_0$. This follows from the fact that $\operatorname{tr} S_0 \Psi = \operatorname{tr} S_o(\Pi - H_0/\nu) = 0$ and $\Psi \geqq 0$ implies that $S_0^{1/2}\Psi S_0^{1/2} = 0$, or equivalently that $S_0^{1/2}\Psi^{1/2} = 0$, which yields the result.

REMARK 3.2. In the above development we assumed that $EX = \mu$ was given. If this is not the case, then choose $\mathscr{S} = \{S = \begin{pmatrix} \alpha & 0 \\ 0 & S_1 \end{pmatrix}\colon S > 0$, $\operatorname{tr} S\Pi \leqq 1\}$, $S_1\colon k \times k$, and the entire development remains unchanged with S_1 replacing S, since $S \geqq 0$ if and only if $\alpha > 0$, $S_1 \geqq 0$ and $\operatorname{tr} S\Pi = \alpha + \operatorname{tr} S_1\Sigma$.

We now summarize the essential points of the proof which are appropriately modified in each of the remaining cases.

(1) Introduce vectors $u(x)$ and $v(x)$ ($u = v$ in the above) such that

(i) $Ev'(X)u(X) = \Pi$ is a matrix of given moments,

(ii) $af', a \in \mathscr{A}_0$ can be written as uAv' with $A \in \mathscr{A}$.

To define $\mathscr{A}$ we first must characterize $\mathscr{A}_0$.

(2) Define $\mathscr{H}$, a set of moment matrices of the same kind as Π, but corresponding to distributions on $\mathscr{T}$.

(3) Define $\mathscr{S}$ and show that $\mathscr{S}$ is bounded.

(4) Use the game to assert that H_0 exists, and show that the moment problem with moments defined by $\Psi = \Pi - H_0/\nu$ has a solution with $\psi_{11} = 1 - 1/\nu$.

4. Univariate distributions on $(-\infty, \infty)$. Let $u(x) = v(x) = (1, x, \cdots, x^n)$. Then polynomials $af'(x)$ of degree $\leqq 2n$ which are nonnegatve in $(-\infty, \infty)$ can be expressed as $uAu', A \geqq 0$, [7, p. 82]. Hence $\mathscr{A} = \{A: A \geqq 0, uAu' \geqq 1 \text{ for } x \in \mathscr{T}\}$, and (3.1) holds. Note that $\Pi = (\pi_{i+j-2}) = (EX^{i+j-2}), i, j = 1, \cdots, n+1$. Let $-\infty < t_i < \infty$, $u_i = u(t_i), i = 1, \cdots, m$, $T = (u_1', \cdots, u_m')$, $D_p = \operatorname{diag}(p_1, \cdots, p_m) \geqq 0$, $\operatorname{tr} D_p = 1$, $H = TD_pT' = (h_{i+j-2}), i, j = 1, \cdots, n+1$. Define $\mathscr{H} = \{H: t_i \in \mathscr{T}, i = 1, \cdots, m\}$, $\mathscr{S} = \{S: S \geqq 0, \operatorname{tr} S\Pi \leqq 1\}$. We assume that the moment problem corresponding to the given moments $\{\pi_0, \cdots, \pi_{2n}\}$ is not determined so that $\Pi > 0$, [8, Th. 3.3], and the previous argument that $\mathscr{S}$ is bounded holds. Assuming that $\mathscr{H}$ is bounded, there exists an S_0 and $H_0 = (h^0_{i+j-2})$ satisfying (3.3), and with Lemma 3.1 we conclude as before that the boundedness condition on $\mathscr{H}$ can be removed.

Since $\pi_0 = h_0^0 = 1, \psi_0 = 1 - 1/\nu$. Define $\Delta_r = |\psi_{i+j-2}|_{i,j=1}^{r+1}$; then since $\Psi \geqq 0$, by Theorem 8.1 it follows that $\Delta_1 > 0, \cdots, \Delta_{r-1} > 0, \Delta_r = 0, \cdots, \Delta_n = 0$, for some r. The reduced (Hamburger) moment problem has a solution if and only if $\Psi \geqq 0$, in which case there exists a (unique) representation $\psi_j = \sum_{i=1}^r p_i \xi_i^j$, $j = 0, 1, \cdots, 2n-1$, and $\psi_{2n} = \sum_{i=1}^r p_i \xi_i^{2n} + c$, $c \geqq 0$, and $c = 0$ if $r = n$, [8, p. 85].

In the event $c > 0$, by using an ε-good strategy for player II to guarantee Ψ strictly > 0, we obtain a distribution with moments $\{\pi_0, \cdots, \pi_{2n}\}$, which assigns probability $1/(\nu + \varepsilon)$ to $\mathscr{T}$.

REMARK 4.1. The representation obtained from [7, p. 82] is of the form $(\Sigma u_i c_i)^2 + (\Sigma u_i d_i)^2$, which is expressible as uAu', where $A = c'c + d'd$. However, the same class of polynomials is obtained if we include all $A \geqq 0$.

REMARK 4.2. If $\mathscr{T}$ is bounded, there exists an extremal distribution with a spectrum consisting of at most $2(n+1)$ points. This follows from the fact that the least number of points contributing to H_0 is at most $(n+1)$, [2, § 2.5], and to Ψ is at most $(n+1)$ points by the previous argument.

5. Univariate Case on $[0, \infty)$**.** Consider first the case $m = 2n - 1$, and let $u(x) = (1, x, \cdots, x^{n-1})$, $v(x) = (1, x, \cdots, x^n)$. Then polynomials $af'(x)$ of degree $\leqq 2n - 1$ can be expressed as $u[(B, 0) + (0, C)]v' \equiv uAv'$, where $B \geqq 0, C \geqq 0$ are $n \times n$ matrices (See [7, p. 82] and Remark 4.1). Hence $\mathscr{A} = \{A\colon B \geqq 0, C \geqq 0,\ uAv' \geqq 1 \text{ for } x \in \mathscr{T}\}$, and (3.1) holds. Now $\Pi = (\pi_{i+j-2}) = (EX^{i+j-2})$, $i = 1, \cdots, n+1$; $j = 1, \cdots, n$. Let $0 < t_i < \infty$, $u_i = u(t_i)$, $v_i = v(t_i)$, $i = 1, \cdots, m$, $T_1 = (u_1', \cdots, u_m')$, $T_2 = (v_1', \cdots, v_m')$, $D_p = \operatorname{diag}(p_1, \cdots, p_m) \geqq 0$, $\operatorname{tr} D_p = 1$, $H = T_1 D_p T_2' = (h_{i+j-2})$, $i = 1, \cdots, n+1$; $j = 1, \cdots, n$. Define $\mathscr{H} = \{H : t_i \in \mathscr{T},\ i = 1, \cdots, m\}$, $\mathscr{S} = \{S = (S_1, S_2) : S_1 \geqq 0, S_2 \geqq 0, \operatorname{tr}[(S_1, 0) + (0, S_2)]\Pi \leqq 1\}$, $S_1, S_2 : n \times n$, $0 : n \times 1$. Assuming that the moment problem corresponding to Π is not determined, i.e., $\Pi_{(1)} = (\pi_{i+j-2})$, $i, j = 1, \cdots, n$, $\Pi^{(1)} = (\pi_{i+j-1})$, $i, j = 1, \cdots, n$, are positive definite, [8, p. 6], the argument of § 3 that $\mathscr{S}$ is bounded holds, with $\|S\| \equiv \|(S_1, 0) + (0, S_2)\|$.

Assuming that $\mathscr{H}$ is bounded, there exists an $S_0 = (S_{10}, 0) + (0, S_{20})$ and $H_0 = (h_{i+j-2}^0)$, $i = 1, \cdots, n+1$; $j = 1, \cdots, n$, satisfying (3.3). Define $H_{0(1)}$ and $H_0^{(1)}$ in the some manner as $\Pi_{(1)}$ and $\Pi^{(1)}$. An application of Lemma 3.1 yields $\Psi_{(1)} = \Pi_{(1)} - H_{0(1)}/\nu \geqq 0$ and $\Psi^{(1)} = \Pi^{(1)} - H_0^{(1)}/\nu \geqq 0$. The boundedness condition of $\mathscr{H}$ can now be removed since $\|H_0\|^2 \leqq \|H_{01}\|^2 + \|H_0^{(1)}\|^2 \leqq \nu\, tr(\Pi_{(1)} + \Pi^{(1)})$. Also $\psi_0 = \pi_0 - h_0/\nu = 1 - 1/\nu$.

In order for the reduced (Stieltjes) moment problem to have a solution, it is necessary that both $\Psi_{(1)}$ and $\Psi^{(1)}$ be $\geqq 0$.[1]

Recall from § 4 that $\Delta_r = |\psi_{i+j-2}|$, $i, j = 1, \cdots, r+1$. Now define $\Delta_r^{(1)} = |\psi_{i+j-1}|$, $i, j = 1, \cdots, r+1$. From Theorem 8.1 it follows that either

(i) $\Delta_0 > 0, \cdots, \Delta_r > 0$, $\Delta_{r+1} = \cdots = \Delta_n = 0$ and $\Delta_0^{(1)} > 0, \cdots, \Delta_r^{(1)} > 0$, $\Delta_{r+1}^{(1)} = \cdots = \Delta_n^{(1)} = 0$, or

(ii) $\Delta_0 > 0, \cdots, \Delta_r > 0$, $\Delta_{r+1} = \cdots = \Delta_n = 0$ and $\Delta_0^{(1)} > 0, \cdots, \Delta_{r-1}^{(1)} > 0$, $\Delta_r^{(1)} = \cdots = \Delta_n^{(1)} = 0$, for some r. But these are the conditions that there exist a distribution whose spectrum consists of $r + 1$ points distinct from 0 in case (i) and including 0 in (ii).

If $m = 2n$, let $u(x) = v(x) = (1, x, \cdots, x^n)$. Then polynomials $af'(x)$ of degree $\leqq 2n$ can be expressed as $v\left[B + \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}\right]v'$, where $B\colon n+1 \times n+1$, $C : n \times n$, $B \geqq 0$, $C \geqq 0$, [7, p. 82]. The remainder of the proof is essentially the same as for the case $m = 2n - 1$ above.

6. Univariate distribution on $[0, 1]$**.** We first deal with the case when an odd number of moments is given. Let $u(x) = (1, x, \cdots, x^{n-1})$, $v(x) = (1, x, \cdots, x^n)$. Now $\Pi = (\pi_{i+j-2}) = (EX^{i+j-2})$, $i = 1, \cdots, n+1$; $j = 1, \cdots, n$. Then polynomials $af'(x)$ of degree $\leqq 2n - 1$ which are

[1] This result was communicated to the authors by S. Karlin. The proof is similar to that for the reduced Hausdorff moment problem given in [5].

nonnegative in $[0, 1]$ can be expressed as $u[(B, 0) + (0, C - B)]v' \equiv uAv'$, where B and C are $n \times n$ matrices, $B \geqq 0$, $C \geqq 0$, (See [7, p. 82] and Remark 4.1). Hence $\mathscr{A} = \{A : B \geqq 0,\ C \geqq 0,\ uAv' \geqq 1 \text{ for } x \in \mathscr{T}\}$, and (3.1) holds. We assume that the moment problem corresponding to the given moments $\{\pi_0, \cdots, \pi_{2n-1}\}$ is not determined. This means that $\Pi^{(1)} = (\pi_{i+j-1})$, $i, j = 1, \cdots, n$, and $\Pi_{(2)} = (\pi_{i+j-2} - \pi_{i+j-1})$, $i, j = 1, \cdots, n$, are both positive definite, [5, p. 55] or [8, p. 77]. (In the latter reference the conditions are presented for the interval $[-1, 1]$.)

Let $0 \leqq t_i \leqq 1$, $u_i = u(t_i)$, $v_i = v(t_i)$, $i = 1, \cdots, m$, $T_1 = (u_1', \cdots, u_m')$, $T_2 = (v_1', \cdots, v_m')$, $D_p = \operatorname{diag}(p_1, \cdots, p_m) \geqq 0$, $\operatorname{tr} D_p = 1$, $H = T_1 D_p T_2' = (h_{i+j-2})$, $i = 1, \cdots, n+1$; $j = 1, \cdots, n$. Define $\mathscr{H} = \{H : t_i \in \mathscr{T}, i = 1, \cdots, m\}$, $\mathscr{S} = \{(S_1, S_2) \colon S_1 \geqq 0,\ S_2 \geqq 0,\ \operatorname{tr}[(S_1\ 0) + (0, S_2 - S_1)]\Pi \leqq 1\}$. We first show that $\mathscr{S}$ is bounded:

$$||S||^2 \equiv ||(S_1, 0) + (0, S_2 - S_1)|| \leqq 2 \operatorname{tr} S_1^2 + \operatorname{tr} S_2^2 \leqq 2(\operatorname{tr} S_1)^2 + (\operatorname{tr} S_2)^2 .$$

But $\operatorname{tr} S\Pi = \operatorname{tr} S_1 \Pi_{(2)} + \operatorname{tr} S_2 \Pi^{(1)} \leqq 1$, and $\Pi_{(2)} > 0$, $\Pi^{(1)} > 0$, so that $\operatorname{tr} S_1 \leqq 1/c_m(\Pi_{(2)})$, $\operatorname{tr} S_2 \leqq 1/c_m(\Pi^{(1)})$, and $\mathscr{S}$ is bounded.

Assuming that $\mathscr{H}$ is bounded, there exists an $S_0 = (S_{10}, 0) + (0, S_{20} - S_{10})$ and $H_0 = (h^0_{i+j-2})$, $i = 1, \cdots, n+1$; $j = 1, \cdots, n$, satisfying (3.3). Define $H_{0(2)}$ and $H_0^{(1)}$ as for $\Pi_{(2)}$ and $\Pi^{(1)}$; then an application of Lemma 3.1 yields

$$\Psi_{(2)} = \Pi_{(2)} - H_{0(2)}/\nu \geqq 0 \ , \ \Psi^{(1)} = \Pi^{(1)} - H_0^{(1)}/\nu \geqq 0 \ .$$

The boundedness condition on $\mathscr{H}$ can now be removed since $||H_0||^2 \leqq 2||H_{0(2)}||^2 + 2||H_0^{(1)}||^2 \leqq \nu \operatorname{tr}(\Pi_{(2)} + \Pi^{(1)})$. Also $\psi_0 = \pi_0 - ho/\nu = 1 - 1/\nu$.

In order for the reduced (Hausdorff) moment problem to have a solution, it is necessary that both $\Psi_{(2)}$ and $\Psi^{(1)}$ be $\geqq 0$, [5, p. 55].

If an even number of moments is given, we let $u(x) = v(x) = (1, x, \cdots, x^n)$. Now $\Pi = (\pi_{i+j-2})$, $i, j = 1, \cdots, n+1$. Polynomials $af'(x)$ of degree $\leqq 2n$ which are nonnegative in $[0, 1]$ can be expressed as $u\left[\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -C \end{pmatrix}\right]u' \equiv uAu'$, where B and C are $n \times n$ matrices, $B \geqq 0$, $C \geqq 0$, (See [7, p. 82] and Remark 4.1). Hence $\mathscr{A} = \{A \colon B \geqq 0,\ C \geqq 0,\ uAu' \geqq 1 \text{ for } x \in \mathscr{T}\}$, and (3.1) holds. We assume that the moment problem corresponding to the given moments $\{\pi_0, \cdots, \pi_{2n}\}$ is not determined. This means that Π and $\Pi_{(3)} = (\pi_{i+j-1} - \pi_{i+j})$, $i, j = 1, \cdots, n$, are positive definite, [5, p. 55] or [8, p. 77].

The remainder of the argument is analogous to the odd moment case.

REMARK 6.1. As in Remark 4.1, if $\mathscr{T}$ is bounded, there exists an extremal distribution with a spectrum consisting of at most $2(n+1)$ points. This follows from [2, § 2.5] and [5, § 17].

REMARK 6.2. A condition for the solution of the Hausdorff moment problem with an infinite number of moments is the condition that

$$\Delta^k\mu_j = \mu_j - \binom{k}{1}\mu_{j+1} + \binom{k}{2}\mu_{j+2} + \cdots + (-1)^k\mu_{j+k} \geqq 0 \,, \quad k, j = 0, 1, \cdots$$

However, this condition with $k, j = 0, 1, \cdots, n$ is not sufficient for a solution of the reduced moment problem. It is interesting to note that this condition enters naturally using an alternative formulation. Polynomials $af'(x)$ which are nonnegative in $[0, 1]$ may be represented as $\Sigma a_{ij}(1-x)^i x^j$, where $a_{ij} \geqq 0$. If we let $u(x) = (1, (1-x), \cdots, (1-x)^n)$, $v(x) = (1, x, \cdots, x^n)$, then the representation is uAv', $a_{ij} \geqq 0$. Now $\Pi = (E(1-X)^{i-1}X^{j-1}) = (\Delta^{i-1}\mu_{j-1})$, $i, j = 1, \cdots, n+1$. Using a similar development as before, $\mathscr{S} = \{S : s_{ij} \geqq 0,\ \operatorname{tr} S\Pi \leqq 1\}$, and from Lemma 3.1, $\Psi = \Pi - H_0/\nu = (\Delta^{i-1}\mu_{j-1}) - (\Delta^{i-1}\mu_{j-1}/\nu) \geqq 0$. Let $\psi_j = \mu_j - h_j/\nu$, $\Psi = (\Delta^{i-1}\psi_{j-1})$; we wish to show that $\Delta^{i-1}\psi_{j-1} \geqq 0$. By choosing S to have all zeros except $s_{ij} = 1/\Delta^{i-1}\pi_{j-1}$, $\operatorname{tr} S\Pi = 1$. The result follows after using (3.3).

7. **Random angle in** $[0, 2\pi)$. If $u(x) = v(x) = (1, e^{inx}, \cdots, e^{inx})$, then polynomials $af'(x)$ which are nonnegative in $[0, 2\pi)$ can be expressed as uAu', $A \geqq 0$, (See [7, p. 82] and Remark 4.1). Hence $\mathscr{A} = \{A\colon A \geqq 0,\ uAu' \geqq 1 \text{ for } x \in \mathscr{T}\}$, and (3.1) holds. Now $\Pi = (\pi_{j-k}) = (Ee^{i(j-k)X})$, $j, k = 1, \cdots, n+1$.

The proof is virtually that of § 4, noting only that the reduced trigonometric (Herglotz) moment problem has a solution if the Toeplitz matrix $\Pi > 0$. (See footnote, § 5.)

7.1. **An example.** The authors are unaware of any Chebyshev inequalities when trigonometric moments are available, and we present a simple illustration.

THEOREM 7.1. *If X is a random angle in $[0, 2\pi)$ and $E\sin X = \alpha$, $E\cos X = \beta$, then*

$$P\{2\theta < X < 2\varphi\} \geqq 1 - \frac{1 - \alpha\sin(\theta+\varphi) - \beta\cos(\theta+\varphi)}{1 - \cos(\varphi - \theta)}\,, \tag{7.1}$$

$$P\{2\theta \leqq X \leqq 2\varphi\} \leqq \frac{1 + \alpha\sin(\theta+\varphi) + \beta\cos(\theta+\varphi)}{1 + \cos(\varphi - \theta)}\,, \tag{7.2}$$

$$0 \leqq \theta \leqq \varphi \leqq \pi\,.$$

Proof. Choose $f(x) = c_1 + c_2 \sin x + c_3 \cos x$. The conditions $f(\theta+\varphi) = 0$, $f(2\theta) = f(2\varphi) = 1$ lead to (7.1), and the conditions $f(\theta+\varphi+\pi) = 0$, $f(2\theta) = f(2\varphi) = 1$ lead to (7.2).

8. Properties of Hankel matrices.

In this section we obtain several properties of Hankel matrices which were required in §§ 4 and 5. These properties are known as a consequence of the solution of moment problems, but it may be of interest to present matrix theoretic proofs. We need the following preliminaries.

A matrix $U = (u_{i+j-2})$, $i, j = 1, \cdots, n$ is called a Hankel matrix. By the rth compound, $A^{(r)}$, of a matrix A: $n \times n$ we mean the matrix whose elements are the rth order minors of A arranged in lexicographic order; thus $A^{(r)} : \binom{n}{r} \times \binom{n}{r}$. The following properties of compound matrices are well-known, e.g., [1].

(8.1) Let A be symmetric. The characteristic roots of $A^{(r)}$ are the $\binom{n}{r}$ products of r characteristic roots of A. Thus, $A^{(r)} \geqq 0$ if and only if $A \geqq 0$.

$$|A^{(r)}| = |A|^{\binom{n-1}{r-1}} . \tag{8.2}$$

THEOREM 8.1. *If the Hankel matrix* $U = (u_{i+j-2})$, $i, j = 1, \cdots, r + 1$, *is* $\geqq 0$, *and if* $\Delta_r = |u_{i+j-2}|_{i,j=1}^{r} = 0$, *then* $\Delta_{r+1} = 0$.

Proof. Suppose $u_0 = 0$, then by nonnegativity of each 2×2 principal minor, it follows that $u_0 = u_1 = \cdots = u_{2n-1} = 0$, $u_{2n} \geqq 0$. But $U^{(r)} \geqq 0$ has first element 0, and hence its first row is 0, so that $\Delta_r = 0$.

THEOREM 8.2. *Let* $U = (u_{i+j-2})$, $i, j = 1, \cdots, r+1$, $V = (u_{i+j-1})$, $i, j = 1, \cdots, r + 1$, $U \geqq 0$, $V \geqq 0$. *Then* $\Delta_r = 0 \Rightarrow \Delta_r^{(1)} = 0 \Rightarrow \Delta_{r+1} = 0$, *where* $\Delta_m = |u_{i+j-2}|$, $i, j = 1, \cdots, m$; $\Delta_m^{(1)} = |u_{i+j-1}|$, $i, j = 1, \cdots, m$.

Proof. In the rth compound $U^{(r)}$, $\Delta_r = u_{11}^{(r)} = 0$ implies that $u_{12}^{(r)} = \Delta_r^{(1)} = 0$. In the rth compound $V^{(r)}$, $\Delta_r^{(1)} = v_{11}^{(r)} = 0$, and hence all $v_{ij}^{(r)} = 0$, except possibly the last diagonal element, which is a function of u_{2r+1}. In $U^{(r+1)}$, the last column does not depend on u_{2r+1}, and its elements are the $v_{ij}^{(r)}$ which are zero. Hence $|U^{(r+1)}| = 0$, so that $\Delta_{r+1} = 0$.

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