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AN APPLICATION OF A FAMILY HOMOTOPY EXTENSION THEOREM TO ANR SPACES

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The first of the writers, on p. 206 of Introduction to the Theory of Block Assemblages and Related Topics in Topology, NSF Research Report, University of Kansas, 1956, defined a clean-cut pair to be any pair (X, A) in which X is a metrizable space, A is a closed subset of X, A is a strong deformation neighborhood retract of X, and $X - A$ is an ANR. It is shown in the present paper that for each clean-cut pair (X, A), X is an ANR if and only if A is an ANR. A consequence is that for each locally step-finite clean-cut block assemblage (cf. the report cited above), the underlying space is an ANR. One of the central tools is a family homotopy extension theorem.

Consider a topological space X and a set $A \subset X$.

Suppose $A \subset N \subset X$. A *strong deformation retraction in X of N onto A* is a retraction r of N onto A such that there is a homotopy $H: N \times I \to X$ between the identity map on N and r which leaves A pointwise fixed at each stage. Also, A is a *strong deformation retract in X of N* if and only if there is a strong deformation retraction in X of N onto A. (These definitions are handled more generally in [4, pp. 109–111].) A is a *strong deformation neighborhood retract of X* if and only if for each neighborhood U of A in X there is a neighborhood V of A in U such that A is a strong deformation neighborhood retract in U of V. (This definition is taken from [4, p. 127].) It is observed in [4, pp. 127–128] that A is a strong deformation neighborhood retract of X if and only if A is a strong deformation retract in X of some neighborhood of A.

By an ANR we shall mean an ANR relative to the class of all metrizable spaces.

In [4, p. 206] the pair (X, A) is defined to be *clean-cut* if and only if X is metrizable, A is a closed subset of X, A is a strong deformation neighborhood retract of X, and $X - A$ is an ANR.

In § 2 it will be shown that if (X, A) is a clean-cut pair, then X is an ANR if and only if A is an ANR. The "only if" part is trivial. The proof of the "if" part will be based on the usual LC characterization of an ANR and the following proposition from [4, p. 181] (the hypothesis there that $\{X_j\}_{j \in J}$ covers X is inessential since X and K may be added to the respective families).

PROPOSITION 1.1. Suppose that X is a topological space and that

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$\{X_j\}_{j\in J}$ is a family of subsets of X. Suppose that K is a simplicial complex ($|K|$ having the usual CW-topology) and that $\{K_j\}_{j\in J}$ is a family of subcomplexes of K. Suppose that

$$f\colon (|K|, |K_j|)_{j\in J} \to (X, X_j)_{j\in J}$$

is a continuous map, L is a subcomplex of K, and

$$H\colon (|L| \times I, |L \cap K_j| \times I)_{j\in J} \to (X, X)_{j\in J}$$

is a homotopy from $f \mid |L|$ to some map

$$g\colon (|L|, |L \cap K_j|)_{j\in J} \to (X, X_j)_{j\in J}\ .$$

Then H has an extension

$$H'\colon (|K| \times I, |K_j| \times I)_{j\in J} \to (X, X_j)_{j\in J}$$

which is a homotopy from f to some extension of g.

The reader may read § 2 on the basis of 1.1 and standard results from *ANR* theory. In [4, p. 181], 1.1 is done with CW-complexes in place of simplicial complexes. If the set J in 1.1 is empty, we get one of several homotopy extension theorems. We may call 1.1 a family homotopy extension theorem. For a general treatment of homotopy extension theorems and family homotopy extension theorems, see [4, pp. 210–217].

2. Results for pairs (X, A). Each simplicial complex will have the CW-topology. Consider any class $\mathscr{K}$ of simplicial complexes. As in [4, pp. 231–232], $\mathscr{K}$ is *admissible* if and only if $\mathscr{K}$ is closed under subcomplexes and isomorphic images. Suppose that X is a topological space, $\mathscr{K}$ is an admissible class of simplicial complexes, and m is a nonnegative integer. Then, as in [4, p. 232], X is *LC from m upward relative to* $\mathscr{K}$ if and only if for each covering $\mathscr{U}$ of X by open subsets of X there is a covering $\mathscr{V}$ of X by open subsets of X such that (*) below holds.

(*) If $K \in \mathscr{K}$, if L is a subcomplex of K, if $K^m \subset L$ (K^m is the m-skeleton of K), if $g\colon |L| \to X$ is a $\mathscr{V}$-subordinate partial realization of K in X (thus, for each $\sigma \in K$, $g(\bar\sigma \cap |L|) \subset$ some member of $\mathscr{V}$), then g extends to a $\mathscr{U}$-subordinate full realization $f\colon |K| \to X$ of K in X.

Also, X is *LC relative to* $\mathscr{K}$ if and only if X is LC from 0 upward relative to $\mathscr{K}$. Also, X is LC if and only if X is LC relative to the class of all simplicial complexes.

The following lemma is probably well-known and follows immediately from standard theorems (e.g., cf. [3, (A), p. 86]). In fact, one could replace I by any compact space.

LEMMA 2.1. *Suppose that N and X are spaces and A is a closed subset of N. Suppose that $H: N \times I \to X$ is continuous. Suppose that $\mathscr{U}$ is a covering of $H(A \times I)$ by open subsets of X and that for each $a \in A$, $H(\{a\} \times I) \subset U$ for some $U \in \mathscr{U}$. Then there exists a covering $\mathscr{V}$ of A by open subsets of N such that for each $V \in \mathscr{V}$, $H(V \times I) \subset$ some member of $\mathscr{U}$.*

Suppose that X is a space and $\mathscr{V}$ is a set of subsets of X. If $A \subset X$, the *star of A with respect to $\mathscr{V}$* is the union of those elements of $\mathscr{V}$ which meet A and will be denoted $St(A; \mathscr{V})$. If $\mathscr{U}$ and $\mathscr{V}$ are sets of subsets of X, $\mathscr{V}$ will be said to *star-refine* (or **-refine*) $\mathscr{U}$ if and only if for each $V \in \mathscr{V}$, $St(V; \mathscr{V})$ is a subset of a member of $\mathscr{U}$. In this case, $\mathscr{V}$ will be called a *star-refinement* (or **-refinement*) *of* $\mathscr{U}$.

THEOREM 2.2. *Suppose that X is a normal and paracompact space and A is a nonvoid closed subset of X which is a strong deformation neighborhood retract of X. Let $\mathscr{K}$ be an admissible class of simplicial complexes, and let m be a nonnegative integer. Suppose that A and $X - A$ are LC from m upward relative to $\mathscr{K}$. Then X is LC from m upward relative to $\mathscr{K}$.*

Proof. Consider any covering $\mathscr{U}$ of X by open subsets of X. Let $\mathscr{U}'$ be a *-refinement of $\mathscr{U}$ by open subsets of X which covers X. Let N be an open neighborhood of A in X such that A is a strong deformation retract in X of N. Thus there is a homotopy $H: N \times I \to X$ such that $H(u, t) = u$ for each $u \in A$ and each $t \in I$ and such that $H(u, 0) = u$ and $H(u, 1) = r(u)$ for each $u \in N$, where $r: N \to A$ is some retraction onto A. Let $\mathscr{V}_1$ be a covering of A by open subsets of N which refines $\mathscr{U}'$ such that if $K \in \mathscr{K}$, if L is a subcomplex of K, if $K^m \subset L$, if $g: |L| \to A$ is a partial realization of K in A subordinate to $\mathscr{V}_1$, then g can be extended to a full realization of K in A subordinate to $\mathscr{U}'$. Using 2.1, let $\mathscr{V}_2$ be a covering of A by open subsets of N such that for each $V \in \mathscr{V}_2$, $H(V \times I) \subset$ some member of $\mathscr{V}_1$. Observe that $\mathscr{V}_2$ refines $\mathscr{V}_1$. Let $\mathscr{V}_3$ be a *-refinement of $\mathscr{V}_2$ by open subsets of N which covers A. Let $\mathscr{V}_4$ be a refinement of $\mathscr{V}_3$ by open subsets of N which covers A. Let $N_3 = \bigcup \mathscr{V}_3$ and $N_4 = \bigcup \mathscr{V}_4$. We may and do require that $\bar{N}_4 \subset N_3$. Let $\mathscr{W}_1$ be a covering of $X - \bar{N}_4$ by open subsets of $X - \bar{N}_4$ which refines $\mathscr{U}'$. Let $\mathscr{W} = \mathscr{V}_3 \cup \mathscr{W}_1$. Let $\mathscr{V}_{X-A}$ be an open covering of $X - A$ such that if $K \in \mathscr{K}$, if L is a subcomplex of K, if $K^m \subset L$, if $g: |L| \to X - A$ is a partial realization of K in $X - A$ subordinate to $\mathscr{V}_{X-A}$, then g can be extended to a full realization of K in $X - A$

subordinate to $\mathscr{W}$. $\mathscr{V} = \mathscr{V}_A \cup \mathscr{V}_{X-A}$; $\mathscr{V}$ is a covering of X by open subsets of X.

Now consider $K \in \mathscr{K}$. Consider any partial realization $\alpha: |L| \to X$ of K in X subordinate to $\mathscr{V}$ such that $K^m \subset L$. Define

$$K_A = \{\sigma \in K: \alpha(\bar{\sigma} \cap |L|) \subset \text{some member of } \mathscr{V}_A\} ,$$
$$K_{X-A} = \{\sigma \in K: \alpha(\bar{\sigma} \cap |L|) \subset \text{some member of } \mathscr{V}_{X-A}\} .$$

K_A and K_{X-A} are subcomplexes of K, and $K_A \cup K_{X-A} = K$.

Now $\alpha \mid |K_{X-A} \cap L| : |K_{X-A} \cap L| \to X - A$ is a partial realization of K_{X-A} in $X - A$ subordinate to $\mathscr{V}_{X-A}$, and $(K_{X-A})^m \subset K_{X-A} \cap L$. Hence $\alpha \mid |K_{X-A} \cap L|$ extends to a full realization $\beta: |K_{X-A}| \to X - A$ of K_{X-A} in $X - A$ subordinate to $\mathscr{W}$.

Define $\hat{\beta}: |K_{X-A} \cup L| \to X$ by

$$\hat{\beta} \mid \bar{\sigma} = \begin{cases} \alpha \mid \bar{\sigma} \text{ if } \sigma \in L , \\ \beta \mid \bar{\sigma} \text{ if } \sigma \in K_{X-A} . \end{cases}$$

$\hat{\beta}$ is obviously continuous.

Set $M = K_A \cap (K_{X-A} \cup L)$ and $\tilde{\beta} = \hat{\beta} \mid |M|$. Consider $\sigma \in K_A$. Now $\tilde{\beta}(\bar{\sigma} \cap |L|) \subset V$ for some $V \in \mathscr{V}_A$. Consider a face $\tau \in K_{X-A}$ of σ. Now $\hat{\beta}(\bar{\tau}) \subset W$ for some $W \in \mathscr{W}$. Since also $\hat{\beta}(\bar{\tau} \cap |L|) = \alpha(\bar{\tau} \cap |L|) \subset \alpha(\bar{\sigma} \cap |L|) \subset$ some member of $\mathscr{V}_A$, $W \cap N_A$ is nonvoid. Hence $W \in \mathscr{V}_3$. It follows (since also $\mathscr{V}_A$ refines $\mathscr{V}_3$ and $\mathscr{V}_3$ *-refines $\mathscr{V}_2$) that

$$\tilde{\beta}(\bar{\sigma} \cap |K_{X-A} \cup L|) \subset St(V; \mathscr{V}_3) \subset \text{some member of } \mathscr{V}_2 .$$

Thus $\tilde{\beta}: |M| \to N$ is a partial realization of K_A in N subordinate to $\mathscr{V}_2$.

For each $\sigma \in K_A$, $H(\tilde{\beta}(\bar{\sigma} \cap |M|) \times I) \subset H(V_2 \times I) \subset V_1$ for some $V_2 \in \mathscr{V}_2$, $V_1 \in \mathscr{V}_1$. For each $u \in |M|$ and each $t \in I$ put

$$G_t(u) = G(u, t) = H(\tilde{\beta}(u), t) ,$$

and observe that $G_1(u) = r(\tilde{\beta}(u)) \in A$ for each $u \in |M|$. Thus $G_1: |M| \to A$ is a partial realization of K_A in A subordinate to $\mathscr{V}_1$. Hence $G_1: |M| \to A$ extends to a full realization $J_1: |K_A| \to A$ subordinate to $\mathscr{U}'$. Consider $\sigma \in K_A$. We have $J_1(\bar{\sigma}) \subset U'$ for some $U' \in \mathscr{U}'$. Also $G((|M| \cap \bar{\sigma}) \times I) \subset V$ for some $V \in \mathscr{V}_1$. Hence $G_1(|M| \cap \bar{\sigma}) \subset V$. Hence $J_1(\bar{\sigma}) \subset St(U'; \mathscr{U}') \subset U_\sigma$ for some $U_\sigma \in \mathscr{U}$ and likewise $G((|M| \cap \bar{\sigma}) \times I) \subset U_\sigma$. Thus

$$G: (|M| \times I, (|M| \cap \bar{\sigma}) \times I)_{\sigma \in K_A} \to (X, U_\sigma)_{\sigma \in K_A}$$

is a homotopy from

$$G_0: (|M|, |M| \cap \bar{\sigma}))_{\sigma \in K_A} \to (X, U_\sigma)_{\sigma \in K_A}$$

to

$$J_1 \mid |M|: (|M| \cap \bar{\sigma})_{\sigma \in K_A} \to (X, U_\sigma)_{\sigma \in K_A} .$$

By 1.1, G extends to

$$G': (|K_A| \times I, \bar{\sigma} \times I)_{\sigma \in K_A} \to (X, U_\sigma)_{\sigma \in K_A}$$

which is a homotopy from an extension G_0' of G_0 to J_1.

Define $\phi: |K| \to X$ by

$$\phi \mid \bar{\sigma} = \begin{cases} \beta \mid \bar{\sigma} \text{ if } \sigma \in K_{X-A} , \\ G_0' \mid \bar{\sigma} \text{ if } \sigma \in K_A . \end{cases}$$

It is easily seen that ϕ is a full realization of K in X subordinate to $\mathscr{U}$ which extends $\alpha: |L| \to X$.

The proof is complete.

For another interesting application of 1.1, cf. the proof of [4, 4.3, p. 249], the application occurring in [4, p. 251]. The theorem proved there is used to characterize a metric space being $\mathscr{U}$-dominated by simplicial complexes (for open coverings $\mathscr{U}$) in terms of LC properties. See also [4, 3.6, p. 277].

THEOREM 2.3. *Suppose that (X, A) is a clean-cut pair. Then X is an ANR if and only if A is an ANR.*

Proof. If X is an ANR, then so is the closed neighborhood retract A of X by standard ANR theory (also, cf. [4, 1.3, p. 206]). Suppose now that A is an ANR. By [4, 3.2, p. 275] or [1, p. 364], a metrizable space is an ANR if and only if it is LC. Hence $X - A$ and A are LC. By 2.2, X is LC. Thus X is an ANR.

3. Results for clean-cut block assemblages. The definitions pertinent to this section are too long to be given here and may be found in [4, p. 70] (for *block assemblage*), [4, p. 94] (for *locally step-finite*), and [4, p. 207] (for *clean-cut* applied to block assemblages). We remark here only that *clean-cut block assemblage* is essentially a generalization of *CW-complex*, suitably embedded Euclidean cells being replaced by suitably embedded *ANRs*.

THEOREM 3.1. *Suppose that $(X, \mathscr{B})$ is a locally step-finite clean-cut block assemblage. Then X is an ANR.*

Proof. The notation of [4, p. 70] will be used. By [4, 8.6, p. 98], X is metrizable. It suffices to show that S_μ is an ANR for each $\mu \leqq \nu$. Assume the contray. Thus we have some $\mu \leqq \nu$ with S_μ not an ANR and with S_λ an ANR for each $\lambda < \mu$. If $\mu = \gamma + 1$, then

S_γ is an *ANR* and $S_\mu = (B_\mu - S_\gamma) \cup S_\gamma$ is an *ANR* by [4, p. 207] and 2.3, contradiction. Hence μ has no immediate predecessor. Each point of S_μ has S_λ for a neighborhood in S_μ for some $\lambda < \mu$ (cf. [4, p. 94]). Hence S_μ is locally *ANR*. Hence S_μ is an *ANR* by [2, 19.2 or 19.3, p. 341].

COROLLARY 3.2. *Suppose that* $(X, \mathscr{B})$ *is a clean-cut block assemblage with only finitely many blocks. Then* X *is an ANR.*

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