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EXISTENCE OF BOREL TRANSVERSALS IN GROUPS

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If M is a closed subgroup of a locally compact group G, we consider the problem of finding a measurable transversal for the cosets $G/M = \{gM: q \in G\}$—a measurable subset $T \subset G$ which meets each coset just once. To each transversal T corresponds a unique cross-section map $\tau: G/M \to T \subset G$ such that $\pi \circ \tau = id$, where $\pi: G \to G/M$ is the canonical mapping. For many purposes it is important to produce reasonably well behaved cross sections for the cosets G/M, and the generality of results obtained is often limited by one's ability to prove that such cross-sections exist. It is well known that, even if G is a connected Lie group, smooth (continuous) cross-sections need not exist; however Mackey ([3], pp. 101–139) showed, using the theory of standard Borel spaces, that a Borel measurable cross section exists if G is a separable (second countable) locally compact group. In this paper topological methods, independent of the theory of standard Borel spaces, are applied to show that Borel measurable cross-sections exist if G is any locally compact group and M any closed subgroup which is metrizable (first countable). The constructions become very simple if G is separable, and give a direct proof that Borel cross-sections exist in this familiar situation.

It is hoped that results of this sort will be helpful in efforts to remove separability restrictions in the study of induced group representations. There are several other areas where one is limited, in part, to studying separable groups by reliance on standard Borel space methods in producing cross-section maps. Cross-sections are widely used in classifying group extensions and their representations—see Mackey [4], [5] (and Rieffel [7], where some attempts are made to extend results of [4] to nonseparable groups). In another direction authors such as Leptin [2] have tried to represent group algebras $L^1(G)$ as vector-valued group algebras $L^1(M, X)$ for a subgroup M. These constructions are successful provided we can insure that there is a measurable cross-section of G/M.

2. We define measurability following Halmos [1], but must distinguish between several commonly used variants of this definition.

DEFINITION. Let X be a topological space. By $\boldsymbol{B}(X)$ will be meant the σ-algebra generated by the closed sets, by $\boldsymbol{C}(X)$ the σ-algebra generated by the compact sets. By $\boldsymbol{B}_0(X)$ and $\boldsymbol{C}_0(X)$ will be meant the σ-algebras generated by the closed G_δ sets and compact G_δ

sets, respectively. If X is σ-compact then $\boldsymbol{B}(X) = \boldsymbol{C}(X)$ and $\boldsymbol{B}_0(X) = \boldsymbol{C}_0(X)$; if X is metrizable then $\boldsymbol{B}(X) = \boldsymbol{B}_0(X)$ and $\boldsymbol{C}(X) = \boldsymbol{C}_0(X)$.

Let G be a locally compact group, M a closed metrizable subgroup. Then M has a complete left-invariant metric ρ which gives it the left uniformity induced from G (see [6], pp. 34–37). A metric ρ_{gM} may be introduced in the coset gM, by setting

$$\rho_{gM}(gm_1, gm_2) = \rho(m_1, m_2) = \rho(e, m_1^{-1}m_2) .$$

This is well-defined, because of left-invariance of ρ. Furthermore, it gives to gM the left uniformity from G. Write G/M for the space of left cosets $\{gM: g \in G\}$ with its usual topology and let $\pi: G \to G/M$ be the canonical map.

DEFINITION. We will say that $A \subset G$ is of *height* $<\varepsilon$ if the diameter of $A \cap gM$ is less than ε for the metric ρ_{gM}, for all $g \in G$.

LEMMA 1. *For any $\varepsilon > 0$, the sets of height $<\varepsilon$ form a basis for the topology of G.*

Proof. Let U be a neighborhood of the identity in G such that $U^{-1}U \cap M$ has diameter $<\varepsilon$. Then the diameter of $g_0U \cap gM$ for the metric ρ_{gM} is also $<\varepsilon$ for all g_0 and g in G since $\rho_{gM}(gm_1, gm_2) = \rho(e, m_1^{-1}m_2)$, and if both gm_1 and gm_2 are in g_0U then $m_1^{-1}m_2$ is in $U^{-1}U$.

We recall that any locally compact group G has a σ-compact open subgroup F.

THEOREM 1. *Let G be a locally compact group, M a closed metrizable subgroup. Then*

(a) *There is a transversal T for the left cosets of M which is in $\boldsymbol{B}(G)$. Furthermore, the corresponding cross-section, $\tau: G/M \to G$, is measurable with respect to $\boldsymbol{C}(G/M)$ and $\boldsymbol{C}(G)$.*

(b) *If there exists an open subgroup of G which contains M and whose image in G/M is σ-compact, then the cross-section can also be made measurable with respect to $\boldsymbol{B}(G/M)$ and $\boldsymbol{B}(G)$.*

REMARK. The conditions on M in (b) are quite weak—see Remark 3 at the end of this note.

Proof. Let F be an open σ-compact subgroup of G. Although F need not include M (there might not be any subgroups F satisfying (b) for arbitrary G), $\pi(F)$ is σ-compact in G/M since π is continuous. Let $L = F \cap M$ and let $\theta: F \to F/L$ be the canonical map to the space of left cosets $\{fL: f \in F\}$ with its usual topology. Then θ is continuous, so $\theta(F) = F/L$ is σ-compact.

We shall begin by constructing a transversal S for the cosets of L in F. S will be of the form $\bigcap_{n=0}^{\infty} S_n$, where S_n is of height $<1/2^n$ (with respect to some preassigned left-invariant metric on L), $S_0 \supset S_1 \supset \cdots$, and each $S_n \cap fL$ will be a nonempty closed set in fL. From this it is clear that S will be a transversal.

For each $f \in F$ let $C(f)$ and $D(f)$ be compact neighborhoods of f in F, with $C(f)$ of height <1 and $D(f) \subset \operatorname{int} C(f)$. let $\{f_j: j = 1, 2, \cdots\}$ be a sequence such that $\bigcup_{j=1}^{\infty} \theta(D(f_j)) = F/L$. Such a sequence exists, by σ-compactness of F. Let $C_j = C(f_j)$, $D_j = D(f_j)$.

Inductively, suppose we are given compact sets $C_{j_0,\dots,j_k}$ and $D_{j_0,\dots,j_k}$ in F, for $k = 0, \cdots, n$, where the subscripts j_i range over the positive integers, and the sets satisfy

(1) $C_{j_0,\dots,j_k}$ has height $<1/2^k$ if $0 \leqq k \leqq n$,
(2) $C_{j_0,\dots,j_k} \subset C_{j_0,\dots,j_{k-1}}$ if $1 \leqq k \leqq n$,
(3) $D_{j_0,\dots,j_k} \subset \operatorname{int} C_{j_0,\dots,j_k}$ if $0 \leqq k \leqq n$,
(4) $\bigcup_{j_k=1}^{\infty} \theta(D_{j_0,\dots,j_k}) \supset \theta(D_{j_0,\dots,j_{k-1}})$ if $1 \leqq k \leqq n$.

Then $C_{j_0,\dots,j_{n+1}}$ and $D_{j_0,\dots,j_{n+1}}$ may be defined so that (1), $\cdots$, (4) still hold with n replaced by $n+1$. The construction is as follows. For each $f \in D_{j_0,\dots,j_n}$ choose a compact neighborhood $C_{j_0,\dots,j_n}(f)$ lying in the interior of $C_{j_0,\dots,j_n}$, and a closed subneighborhood $D_{j_0,\dots,j_n}(f)$ in the interior of $C_{j_0,\dots,j_n}(f)$. Then by compactness of $D_{j_0,\dots,j_n}$ a finite subfamily $f_1, \cdots, f_k$ may be chosen so that

$$\bigcup_{j=1}^{k} \theta(D_{j_0,\dots,j_n}(f_j)) \supset \theta(D_{j_0,\dots,j_n}) .$$

For $1 \leqq j \leqq k$, let $C_{j_0,\dots,j_n,j} = C_{j_0,\dots,j_n}(f_j)$ and $D_{j_0,\dots,j_n,j} = D_{j_0,\dots,j_n}(f_j)$. If $j > k$, simply set $D_{j_0,\dots,j_n,j} = C_{j_0,\dots,j_n,j} = \phi$. It is then straightforward that (1), $\cdots$, (4) are still satisfied.

For convenience, we will also define $D_{j_0,\dots,j_n} = \phi$ whenever $j_n = 0$. Now set

$$E_j = \theta(D_j)\backslash\theta\Big(\bigcup_{i<j} D_i\Big) = \theta(D_j)\backslash\bigcup_{i<j} \theta(D_i)$$

(so $E_1 = \theta(D_1)\backslash\phi = \theta(D_1)$), and inductively

$$\begin{aligned} E_{j_0,\dots,j_n} &= \Big[\theta(D_{j_0,\dots,j_n})\backslash\theta\Big(\bigcup_{i<j_n} D_{j_0,\dots,i}\Big)\Big] \cap E_{j_0,\dots,j_{n-1}} \\ &= \Big[\theta(D_{j_0,\dots,j_n})\Big\backslash \bigcup_{i<j_n} \theta(D_{j_0,\dots,i})\Big] \cap E_{j_0,\dots,j_{n-1}} . \end{aligned}$$

Note that only finitely many $E_{j_0,\dots,j_n}$ are nonempty for fixed $(j_0, \cdots, j_{n-1})$. Furthermore $E_{j_0,\dots,j_{n-1}}$ is a disjoint union of the sets

$\{E_{j_0,\cdots,j_n}: j_n = 1, 2 \cdots\}$ and each set $E_{j_0,\cdots,j_n}$ is a difference of compacta in F/L.

Now look at $C_{j_0,\cdots,j_n} \cap \theta^{-1}(E_{j_0,\cdots,j_n})$. Since

$$E_{j_0,\cdots,j_n} \subset \theta(D_{j_0,\cdots,j_n}) \subset \theta(C_{j_0,\cdots,j_n}) ,$$

it follows that $\theta(C_{j_0,\cdots,j_n} \cap \theta^{-1}(E_{j_0,\cdots,j_n}))$ is precisely $E_{j_0,\cdots,j_n}$. Since $E_{j_0,\cdots,j_n}$ is a difference of compacta in F/L, $\theta^{-1}(E_{j_0,\cdots,j_n})$ is a difference of closed L-saturated sets in F and, since $C_{j_0,\cdots,j_n}$ is compact, $C_{j_0,\cdots,j_n} \cap \theta^{-1}(E_{j_0,\cdots,j_n})$ may be written as a difference $A_{j_0,\cdots,j_n}\backslash B_{j_0,\cdots,j_n} = A_{j_0,\cdots,j_n}\backslash B_{j_0,\cdots,j_n}L$ where $A_{j_0,\cdots,j_n}$ and $B_{j_0,\cdots,j_n}$ are compacta in F with $A_{j_0,\cdots,j_n} \supset B_{j_0,\cdots,j_n}$.

Define $S_n = \bigcup_{j_0,\cdots,j_n} C_{j_0,\cdots,j_n} \cap \theta^{-1}(E_{j_0,\cdots,j_n})$, so that

$$S_n = \bigcup_{j_0,\cdots,j_n} (A_{j_0,\cdots,j_n}\backslash B_{j_0,\cdots,j_n}) .$$

Note that if $fL \in E_{j_0,\cdots,j_n}$ then also $fL \in E_{j_0,\cdots,j_{n-1}}$. Thus there is a sequence $j_0(f), j_1(f), \cdots$ with the property that $fL \in E_{j_0(f),\cdots,j_n(f)}$ for each n. Then $S_n \cap fL = C_{j_0(f),\cdots,j_n(f)} \cap fL$. Thus S_n and $S = \bigcap_{n=0}^{\infty} S_n$ have the asserted properties.

Now we return to the big group G and its subgroup M. Let $R = \{g_\alpha: \alpha < \alpha_0\}$ be a well-ordered family of representatives for the cosets of F in G, and π the canonical map $\pi: G \to G/M$.

Notice that S is also a transversal for the cosets of M in the M saturated open set FM. To see this, we must show that $SM = FM$, and that $S \cap fM$ consists of only one point for each $f \in F$. Now: $SM = SM^2 \supset (SL)M = FM$, so $SM = FM$. Also, if fm_1 and fm_2 are both in S with m_1 and m_2 in M, then m_1 and m_2 are in $f^{-1}S \subset F$, so m_1 and m_2 are in $F \cap M = L$, and since $S \cap fF$ is a single point, $m_1 = m_2$.

Now let $\tilde{S} = \bigcup_{g\in R} gS$. We easily see that $\tilde{S}$ is in $\boldsymbol{B}(G)$; indeed, let Q be $\{X \subset F: \bigcup_{g\in R} gX \in \boldsymbol{B}(G)\}$. Then clearly Q contains all open sets, and it is easy to check that Q is closed under countable union and complementation. Thus Q contains all sets of $\boldsymbol{B}(F)$; in particular $S \in Q$, so $\tilde{S} \in \boldsymbol{B}(G)$. Obviously we have $\pi(\tilde{S}) = G/M$, but $\tilde{S}$ is not yet necessarily a transversal, because $gM \cap \tilde{S}$ may contain more than one point. We shall correct this by intersecting $\tilde{S}$ with a certain closed set so that the intersection will be a transversal. Let $U = \bigcup_{\alpha<\alpha_0} (g_\alpha F \cap \bigcup_{\beta<\alpha} g_\beta FM)$. Then U is open, and we set $T = \tilde{S}\backslash U$. Thus $T = \bigcup_{\alpha>\alpha_0}(g_\alpha S\backslash \bigcup_{\beta<\alpha} g_\beta FM)$.

To see that $\pi(T) = G/M$, fix $\bar{g} \in G$ and let γ be the first ordinal for which $g_\gamma F \cap \bar{g}M$ is nonempty. Since $SM = FM$, it follows that $g_\gamma S \cap \bar{g}M$ is also nonempty, containing some g. If $\alpha \neq \gamma$, then $g_\alpha F \cap g_\gamma F = \phi$, so $g \notin g_\alpha F$. On the other hand, if $\alpha = \gamma$, then for

$\beta < \gamma$ we have $g_\beta F \cap \bar{g}M = \phi$, by minimality of γ, so also $g_\beta FM \cap \bar{g}M = \phi$, and $g \notin g_\beta FM$. Then $g \notin \bigcup_{\beta<\gamma} g_\beta FM$. Thus

$$g \notin \bigcup_{\alpha<\alpha_0} (g_\alpha F \cap \bigcup_{\beta<\alpha} g_\beta FM) = U ,$$

but $g \in g_\gamma S \subset \tilde{S}$, so $g \in T$, and $\pi(g) = \pi(\bar{g})$.

To see that $T \cap \bar{g}M$ consists of exactly one point, let g and γ be as above, and suppose $gm \in T \cap \bar{g}M$ for some $m \in M$. Then $gm \in g_\delta S$ for some δ. By minimality of γ, we have $\gamma \leqq \delta$. If $\gamma = \delta$, then $g_\gamma^{-1}g$ and $g_\gamma^{-1}gm$ are both in S, so are equal, since S is a transversal for the left cosets of M in FM. Thus $m = e$. On the other hand, if $\gamma \neq \delta$, then gm lies in both $g_\delta F$ and $\bigcup_{\beta<\delta} g_\beta FM$, hence is *not* in T, contradiction.

Having constructed a transversal $T \in \boldsymbol{B}(G)$ for the M, cosets we now examine the measurability properties of the corresponding cross-section τ. For $X \subset G$, we have $\tau^{-1}(X) = \pi(T \cap X)$. Let $\boldsymbol{T} = \{X \subset G: \tau^{-1}(X) \in \boldsymbol{C}(G/M)\}$. Then $\boldsymbol{T}$ is closed under countable unions and differences. We shall show that $\boldsymbol{T}$ contains all compact X, which will complete part (a). If X is compact, then it meets only finitely many left cosets of F, say $g_{\alpha_1}F, \cdots, g_{\alpha_k}F$, so that

$$\pi(T \cap X) = \bigcup_{i=1}^{k} \left(\pi(g_{\alpha_i}S \cap X) \Big\backslash \bigcup_{\beta<\alpha_i} \pi(g_\beta FM)\right) .$$

Here $\pi(gS \cap X)$ may be written $g\pi(S \cap g^{-1}X)$ if we let G act on G/M in the usual way.

Now: for any compact set $Y \subset G$,

$$S_n \cap Y = \bigcup_{j_0,\cdots,j_n} ((A_{j_0,\dots,j_n} \cap Y) \backslash B_{j_0,\dots,j_n}L) ,$$

and since $A_{j_0,\dots,j_n}$ and $B_{j_0,\dots,j_n}$ are subsets of F, this may be written $\bigcup_{j_0,\dots,j_n} ((A_{j_0,\dots,j_n}) \cap Y) \backslash B_{j_0,\dots,j_n}M)$. Then

$$\pi(S_n \cap Y) = \bigcup_{j_0,\cdots,j_n} (\pi(A_{j_0,\dots,j_n} \cap Y) \backslash \pi(B_{j_0,\dots,j_n})) \cdot$$

This is clearly in $\boldsymbol{C}(G/M)$ in view of the compactness of Y and continuity of π. Next, we show that $\pi(S \cap Y) = \bigcap_{n=0}^{\infty} \pi(S_n \cap Y)$. Clearly $\pi(S \cap Y) \subset \bigcap_{n=0}^{\infty} \pi(S_n \cap Y)$. Now, $S_n \cap fM = S_n \cap fL = C_{j_0(f),\dots,j_n(f)} \cap fL$, so $S_n \cap Y \cap fM = C_{j_0(f),\dots,j_n(f)} \cap Y \cap fL$. If this is nonempty for each n, then, since $C_{j_0(f),\dots,j_{n+1}(f)} \subset C_{j_0(f),\dots,j_n(f)}$, we have $S \cap Y \cap fM = \bigcap_{n=0}^{\infty} S_n \cap Y \cap fM \neq \phi$. Thus, $\pi(f) \in \pi(S \cap Y)$, and $\bigcap_{n=0}^{\infty} \pi(S_n \cap Y) = \pi(S \cap Y)$.

Applying this to $Y = g^{-1}X$, we get $\pi(S \cap g^{-1}X) \in \boldsymbol{C}(G/M)$, so $g\pi(S \cap g^{-1}X) \in \boldsymbol{C}(G/M)$, hence $\bigcup_{i=1}^{k} \pi(g_{\alpha_i}S \cap X)$ is in $\boldsymbol{C}(G/M)$, and $X \in \boldsymbol{T}$. Part (a) is completed.

Next, suppose the added assumption of (b) is satisfied: then F may be chosen to contain M, so that $L = M$. In this case the set U is empty, and $T = \tilde{S} = \bigcup_{\alpha<\alpha_0} g_\alpha S$. Thus $T = \bigcap_{n=0}^{\infty} T_n$, where $T_n = \bigcup_{\alpha<\alpha_0} g_\alpha S_n$. Since $S_n = \bigcup_{j_0,\dots,j_n} (A_{j_0,\dots,j_n} \backslash B_{j_0,\dots,j_n} M)$, $\pi(gS_n \cap X) = g\pi(S_n \cap g^{-1}X) = \bigcup_{j_0,\dots,j_n} g(\pi(A_{j_0,\dots,j_n} \cap g^{-1}X)\backslash \pi(B_{j_0,\dots,j_n}))$. Then

$$\begin{aligned}\pi(T_n \cap X) &= \bigcup_{\alpha<\alpha_0} g_\alpha \pi(S_n \cap g_\alpha^{-1} X) \\ &= \bigcup_{j_0 \cdots j_n} \bigcup_{\alpha<\alpha_0} g_\alpha(\pi(A_{j_0,\dots,j_n} \cap g_\alpha^{-1}X))\backslash \pi(B_{j_0,\dots,j_n})) \\ &= \bigcup_{j_0,\dots,j_n} \Big(\bigcup_{\alpha<\alpha_0} g_\alpha \pi(A_{j_0,\dots,j_n} \cap g_\alpha^{-1}X)\Big)\Big\backslash\Big(\bigcup_{\alpha<\alpha_0} g_\alpha \pi(B_{j_0,\dots,j_n})\Big)\end{aligned}$$

with the last equality valid since $F \supset A_{j_0,\dots,j_n} \supset B_{j_0,\dots,j_n}$ and the sets $\pi(g_\alpha F) = g_\alpha \pi(F)$ are disjoint for all $\alpha < \alpha_0$. Now assume X is closed. Then $\bigcup_{\alpha<\alpha_0} g_\alpha \pi(A_{j_0,\dots,j_n} \cap g_\alpha^{-1}X)$ and $\bigcup_{\alpha>\alpha_0} g_\alpha \pi(B_{j_0,\dots,j_n})$ are sets in G/M whose intersections with each of the disjoint open/closed sets $g_\alpha \pi(F)$ are closed, hence they are themselves closed sets in G/M. Thus $\pi(T_n \cap X) \in \boldsymbol{B}(G/M)$. A similar argument to one used in showing that τ is measurable for $\boldsymbol{C}(G/M)$ and $\boldsymbol{C}(G)$ may now be used to show $\pi(T \cap X) = \bigcap_{n=0}^{\infty} (T_n \cap X)$. Thus $\pi(T \cap X) \in \boldsymbol{B}(G/M)$. As before, we consider the family of sets for which $\tau^{-1}(X) = \pi(T \cap X) \in \boldsymbol{B}(G/M)$. It is closed under countable union and complementation, and contains all closed sets; thus it contains all of $\boldsymbol{B}(G)$. Thus (b) is proved.

REMARK 1. If G/M is σ-compact, then the argument evidently becomes much simpler.

REMARK 2. Openness of the map π was nowhere used. Thus, the theorem can actually be stated somewhat more generally: if the locally compact group G acts as a transitive group of homeomorphisms of the locally compact space X, $G \times X \to X$ is separately continuous, and the isotropy group $M = \{g: g(x_0) = x_0\}$ is metrizable, then there is a measurable transversal T (this part is actually no improvement on Theorem 1), and the cross-section map $\tau: X \to G$, given by $\tau(x) =$ that point g in T for which $gx_0 = x$, is measurable, in the various senses described in the theorem (repeat the above measurability arguments taking $\pi : G \to X$, with $\pi(g) = g(x_0)$, in place of $\pi : G \to G/M$) .

REMARK 3. The hypothesis of (b) in Theorem 1 will be satisfied if any of the following three conditions hold:

(i) G/M is σ-compact

(ii) M is normal

(iii) M is σ-compact.

For (ii), take any σ-compact open subgroup F_0 of G. Then $F = F_0 M$ is also a group, since $(f_1 m_1)(f_2 m_2) = (f_1(m_1 f_2 m_1^{-1}))(m_1 m_2)$, $F_0 M$ is open in G since F_0 is, and the image of F in G/M is the same as the image of F_0; but F_0 is σ-compact, therefore so is its image.

For (iii), again choose a σ-compact open subgroup F_0 of G, set $H = F_0M \cap MF_0 \supset F_0 \cup M$, and let $F = \bigcup_{n=1}^{\infty} H^n$. H is open and $H = H^{-1}$, so F is an open subgroup. Furthermore if J_n and K_n are ascending compact sets with $\bigcup_{n=1}^{\infty} J_n = F_0$ and $\bigcup_{n=1}^{\infty} K_n = M$, then the compact sets $(J_nK_n \cap K_nJ_n)^n$ have F as their union, so F is σ-compact, and therefore its image in G/M is σ-compact.

REMARK 4. Under the hypothesis of (b), if we choose the $C_{j_0,\cdots,j_n}$ and $D_{j_0,\cdots,j_n}$ to be G_δ sets, then T will also be $\boldsymbol{B}_0(G)$-measurable, and τ will be measurable with respect to $\boldsymbol{C}_0(G/M)$ and $\boldsymbol{C}_0(G)$, and with respect to $\boldsymbol{B}_0(G/M)$ and $\boldsymbol{B}_0(G)$.

REMARK 5. To some extent, it is possible to replace local compactness by the Lindelöf property. Thus, for example, it may be shown that if G is a topological group, M a closed metrizable subgroup which is complete in the left uniformity, and G/M is Lindelöf, then there exists a $\boldsymbol{B}(G)$-measurable transversal. However, we will neither prove this here, nor pursue this type of generalization. Moreover, it does not seem possible to prove measurability of the associated cross-section map in this context.

REMARK 6. It is apparent from our construction that if KCG/M is compact, then the closure $(\tau(K))^-$ is compact in G; this uses openness of $\pi : G \to G/M$.

REFERENCES

1. P. Halmos, *Measure Theory*, D. Van Nostrand, Princeton, 1952.
2. H. Leptin, *Verallgemeinerte L^1-Algebren*, Math. Ann. **159** (1965), 51-76.
3. G. W. Mackey, *Induced representation of locally compact groups I*, Ann. of Math. **55** (1952), 101-139.
4. ———, *Les ensembles Boreliens et les extensions des groupes*, Journal de Math. **36** (1957), 171-178.
5. ———, *Unitary representations of group extensions*, Acta Math. **99** (1958), 265-311.
6. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York, 1955.
7. M. Rieffel, *On extensions of locally compact groups*, Amer. J. Math. **88** (1966), 871-880.

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