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OPERATOR-VALUED FEYNMAN INTEGRALS OF FINITE-DIMENSIONAL FUNCTIONALS

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Let $C[a, b]$ denote the space of continuous functions x on $[a, b]$. Let $\{\alpha_1, \cdots, \alpha_n\}$ be an orthonormal set of functions of bounded variation on $[a, b]$. Let

$$F(x) = f\left(\int_a^b \alpha_1(t)dx(t), \cdots, \int_a^b \alpha_n(t)\,dx(t)\right).$$

Recently, Cameron and Storvick defined certain operator-valued function space integrals, and, in particular, an operator-valued Feynman integral. In their setting, we give existence theorems as well as explicit formulas for the function space integrals of functionals F as above. We also study the properties of the operators which arise by "integrating" this type of functional.

Insofar as possible we adopt the definitions and notation of our earlier paper [6]. For a better motivated definition of the operator-valued function space integrals $I_\lambda(F)$ and $J_q(F)$ see [3] and [4]. Throughout the paper we assume that F has the form given above where f is a measurable function on R_n.

Four cases arise: (a) The normalized constant function $\alpha_0(t) \equiv (b-a)^{-1/2}$ is orthogonal to span $\{\alpha_1, \cdots, \alpha_n\}$. (b) $\alpha_0 \in \{\alpha_1, \cdots, \alpha_n\}$, say $\alpha_0 = \alpha_1$ for convenience. (c) $\alpha_0 \notin \{\alpha_1, \cdots, \alpha_n\}$ but $\alpha_0 \in \text{span}\,\{\alpha_1, \cdots, \alpha_n\}$. In this case, one may choose a new orthonormal basis $\{\beta_1, \cdots, \beta_n\}$ for span $\{\alpha_1, \cdots, \alpha_n\}$ such that $\alpha_0 = \beta_1$. Now by an appropriate change in f, one has case (b). (d) $\alpha_0 \notin \text{span}\,\{\alpha_1, \cdots, \alpha_n\}$ and α_0 is not orthogonal to span $\{\alpha_1, \cdots, \alpha_n\}$. In this case one may choose a basis $\{\beta_1, \cdots, \beta_{n+1}\}$ for span $\{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ such that $\alpha_0 = \beta_1$ Again after making an appropriate change in f, we are back to case (b) except that the dimension is raised by one. Examples of cases (a) and (b) are easily given. Choose one of the standard orthonormal sets on $[a, b]$. Pick out a finite subset. If the constant function is not included, we have (a); if it is included, we have (b). Cases (c) and (d) will be illustrated in § 4 of the paper; (c) in connection with the important example of functions of independent increments. Throughout, the hypotheses are made on the function f which arises after the conversion has been made, if necessary, to cases (a) or (b).

To obtain the existence of $I_\lambda(F)$ for $\text{Re}\,\lambda > 0$ we require only that $f(u_1, \cdots, u_n) \exp[-p(u_1^2 + \cdots + u_n^2)]$ be integrable for all $p > 0$. For the existence of $J_q(F)$ we require the integrability of $f(u_1, \cdots, u_n)$. In both cases, the restriction on f is much weaker than in [2] or [9] where Cameron's earlier definition of the Feynman and related integrals was employed to study functionals of the same type. It is perhaps worth mentioning that the existence theorems of this paper are the first results in the theory (see [3], [4], and [6]) in which the functional F is allowed to be unbounded.

In our earlier work [6], the existence of $I_\lambda(F)$ was obtained quite readily but the existence of $J_q(F)$ was more difficult to establish. Here, the situation is reversed. In establishing the existence of $I_\lambda(F)$, a probabilistic interpretation of $I_\lambda^q(F)$ for $\lambda > 0$ allows us to write $I_\lambda^q(F)$ in a more manageable form.

To obtain the existence of $J_q(F)$, one needs to show that it is the weak operator limit of $I_\lambda(F)$ as λ goes to $-iq$ along the line $p - iq$, $p > 0$. We get a stronger result than this; in case (a), as in [6], we show that $J_q(F)$ is the strong operator limit of $I_\lambda(F)$ as $\lambda \to -iq$ through the right half plane. In case (b), we actually get $J_q(F)$ as the limit in operator norm of $I_\lambda(F)$. Also, as in [6], we get the existence of $J_q(F)$ for every $q \neq 0$. In this respect, our results resemble the "deterministic theorem for $J_q(F)$" from [4], an improvement over Theorem 5 of [3] which gave existence of $J_q(F)$ for almost every q. The type of functional dealt with in those two theorems is quite different from ours however. In our case, the operators arising as the function space integrals will turn out to be convolution operators, and so, known results on such operators [5, p. 951–964] can be applied to give information on $I_\lambda(F)$ and $J_q(F)$.

Finally we mention that the class of functionals studied here neither includes nor is included in the class studied earlier [6]. The most obvious difference is that, in the present case, $F(x)$ may depend upon the values of x throughout $[a, b]$ whereas a functional F of the form $F(x) = f_1(x(t_1)) \cdots f_n(x(t_n))$ depends only on the values of x at $t_1, \cdots, t_n$.

2. The operator $I_\lambda(F)$.

We let $\alpha_0, \alpha_1, \cdots, \alpha_n$ and F be as before. For convenience we let $e_\lambda(u) = \lambda^{1/2}[2\pi(b-a)]^{-1/2} \exp(-\lambda u^2/2(b-a))$ and let $*$ denote the operation of convolution. The following theorem establishes the existence of $I_\lambda(F)$.

THEOREM 1. *Let $f(u_1, \cdots, u_n)$ be such that*

$$f(u_1, \cdots, u_n) \exp[-p(u_1^2 + \cdots + u_n^2)]$$

is integrable on R_n for all $p > 0$. Then the operator $I_\lambda(F)$ exists for all λ such that $\operatorname{Re}\lambda > 0$. (a) *If α_0 is orthogonal to span $\{\alpha_1, \cdots, \alpha_n\}$, then $I_\lambda(F)$ is given by the formula*

$$(1)\qquad \begin{aligned}(I_\lambda(F)\psi)(\xi) &= d_\lambda \int_{-\infty}^{\infty} e_\lambda(v-\xi)\psi(v)dv \\ &= [d_\lambda e_\lambda(v)] * [\psi(v)](\xi)\end{aligned}$$

where

$$d_\lambda = (\lambda/2\pi)^{n/2} \int_{-\infty}^{\infty} \cdot (n) \cdot \int_{-\infty}^{\infty} f(v_1, \cdots, v_n) \exp[-\lambda(v_1^2 + \cdots + v_n^2)/2] dv_1 \cdots dv_n \, ,$$

$\psi \in L_2$ *and* $-\infty < \xi < \infty$. (b) *If* $\alpha_0 = \alpha_1$, *then* $I_\lambda(F)$ *is given by the formula*

$$(2)\qquad \begin{aligned}(I_\lambda(F)\psi)(\xi) &= \int_{-\infty}^{\infty} h_\lambda(v-\xi)e_\lambda(v-\xi)\psi(v)dv \\ &= [h_\lambda(v)e_\lambda(v)] * [\psi(v)](\xi)\end{aligned}$$

where

$$h_\lambda(u) = (\lambda/2\pi)^{(n-1)/2} \int_{-\infty}^{\infty} \cdot (n-1) \cdot \int_{-\infty}^{\infty} f(u(b-a)^{-1/2}, v_2, \cdots, v_n) \exp[-\lambda(v_2^2 + \cdots + v_n^2)/2] dv_2 \cdots dv_n \, ,$$

$\psi \in L_2$ *and* $-\infty < \xi < \infty$.

Proof. (b) We first establish the existence of the operator $I_\lambda^{an}(F)$. Let $\psi \in L_2$. For $\lambda > 0$ the following Wiener integral exists and is given by

$$\begin{aligned}
&\int_{C_0[a,b]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(b) + \xi)dx \\
&= \int_{C_0[a,b]} f\Big(\lambda^{-1/2}\int_a^b \alpha_1(t)dx(t), \cdots, \lambda^{-1/2}\int_a^b \alpha_n(t)dx(t)\Big) \\
&\qquad \psi\Big(\lambda^{-1/2}(b-a)^{1/2}\int_a^b \alpha_1(t)dx(t) + \xi\Big)dx \\
&= (2\pi)^{-n/2}\int_{-\infty}^{\infty} \cdot (n) \cdot \int_{-\infty}^{\infty} f(\lambda^{-1/2}u_1, \cdots, \lambda^{-1/2}u_n)\psi(\lambda^{-1/2}(b-a)^{1/2}u_1 + \xi) \\
&\qquad \exp[-\tfrac{1}{2}(u_1^2 + \cdots + u_n^2)]du_1 \cdots du_n \\
&= [h_\lambda(v)e_\lambda(v)] * [\psi(v)](\xi)
\end{aligned}$$

which is in L_2 since $h_\lambda e_\lambda$ is in L_1. Now for $\operatorname{Re}\lambda > 0$ let

$$A(\lambda; \psi)(\xi) = [h_\lambda(v)e_\lambda(v)] * [\psi(v)](\xi) \, .$$

Then $A(\lambda; \psi)$ is in L_2 for $\operatorname{Re}\lambda > 0$. Furthermore for any $\phi \in L_2$, an application of Morera's theorem (together with the Fubini theorem and the Cauchy Integral theorem) to $(A(\lambda; \psi), \phi)$, as in [3, p. 533] enables us to conclude that $A(\lambda; \psi)$ is analytic (as a vector valued function) in λ for $\operatorname{Re}\lambda > 0$. But for $\lambda > 0$,

$$A(\lambda; \psi)(\xi) = \int_{C_0[a,b]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(b) + \xi)dx ,$$

and so $I_\lambda^{an}(F)$ exists for $\operatorname{Re}\lambda > 0$ and is given by

$$(I_\lambda^{an}(F)\psi)(\xi) = [h_\lambda(v)e_\lambda(v)] * [\psi(v)](\xi) .$$

Let $\sigma: [a = t_0 < t_1 < \cdots < t_m = b]$ be a partition of $[a, b]$ and let $I_\lambda^\sigma(F)$ be defined by (4.7) of [3] or (2) of [6]. We must show that $I_\lambda^\sigma(F) \to I_\lambda^{an}(F)$ in the weak operator topology as $\|\sigma\| \to 0$. This will establish the existence of $I_\lambda^{seq}(F)$ (and hence of $I_\lambda(F)$, the common value of $I_\lambda^{seq}(F)$ and $I_\lambda^{an}(F)$) and verify (2). We begin with an outline of the proof. Using the general multivariate normal probability density function, we obtain an alternate expression for $I_\lambda^\sigma(F)$ for $\lambda > 0$. This expression and the old expression agree on the real axis and are both analytic throughout the right half-plane; hence they agree for all λ such that $\operatorname{Re}\lambda > 0$. Using the new expression for $I_\lambda^\sigma(F)$ we are able to prove the necessary limit statement; the key here is showing that the covariance matrix associated with the multivariate normal density function converges to the identity matrix.

As is pointed out in [3, p. 530], for $\lambda > 0$,

$$(I_\lambda^\sigma(F)\psi)(\xi) = \int_{C_0[a,b]} f\Big(\int_a^b \alpha_1(t)d(\lambda^{-1/2}x_\sigma + \xi), \cdots, \int_a^b \alpha_n(t)d(\lambda^{-1/2}x_\sigma + \xi)\Big) \cdot\psi(\lambda^{-1/2}x(b) + \xi)dx .$$

But, as $\alpha_1(t) \equiv (b-a)^{-1/2}$, we have

$$(I_\lambda^\sigma(F)\psi)(\xi) = \int_{C_0[a,b]} f\Big(\lambda^{-1/2}\int_a^b \alpha_1(t)dx(t), \lambda^{-1/2}\sum_{j=1}^m \alpha_2(t_j)[x(t_j) - x(t_{j-1})], \cdots, \lambda^{-1/2}\sum_{j=1}^m \alpha_n(t_j)[x(t_j) - x(t_{j-1})]\Big)\psi\Big((b-a)^{1/2}\lambda^{-1/2}\int_a^b \alpha_1(t)dx(t) + \xi\Big)dx .$$

Now let X_1^σ denote the random variable on the Wiener space $C_0[a, b]$ defined by $X_1^\sigma(x) = \int_a^b \alpha_1(t)dx(t)$. It is well known [7] that X_1^σ is distributed normally with mean 0 and variance 1; i.e., $X_1^\sigma \sim N(0, 1)$. Also $[x(t_j) - x(t_{j-1})] \sim N(0, t_j - t_{j-1})$ and, for $j \neq k$, $[x(t_j) - x(t_{j-1})]$ and $[x(t_k) - x(t_{k-1})]$ are independent. Hence for $i = 2, \cdots, n$, the random variable X_i^σ defined by $X_i^\sigma(x) = \sum_{j=1}^m \alpha_i(t_j)[x(t_j) - x(t_{j-1})]$ satisfies

$$X_i^\sigma \sim N\Big(0, \sum_{j=1}^m \alpha_i^2(t_j)(t_j - t_{j-1})\Big) .$$

Now let us consider the convariance matrix C_σ associated with the random variables $X_1^\sigma, \cdots, X_n^\sigma$. Since each X_i^σ has mean 0, the ik^{th} entry, a_{ik}^σ, of C_σ is given by $\int_{C_0[a,b]} X_i^\sigma X_k^\sigma dx$. We will now show that $C_\sigma \to I$ in operator norm as $\|\sigma\| \to 0$ where I denotes the n by n identity matrix. It suffices to show that $a_{ik}^\sigma \to \delta_{ik}$ (the Kronecker δ) as $\|\sigma\| \to 0$. Since $\{(t_j - t_{j-1})^{-1/2}[x(t_j) - x(t_{j-1})]\}_{j=1}^m$ is a family of independent random variables each distributed $N(0, 1)$ we obtain

$$a_{ik}^\sigma = \sum_{j=1}^m \alpha_i(t_j)\alpha_k(t_j)(t_j - t_{j-1})$$

for all $i = 1, 2, \cdots, n$ and $k = 1, 2, \cdots, n$. Thus $a_{11}^\sigma = 1$ for all σ. For $i = 2, 3, \cdots, n$, a_{ii}^σ is an approximating sum for the Riemann integral $\int_a^b \alpha_i^2(t)dt = 1$ and so $a_{ii}^\sigma \to 1$ as $\|\sigma\| \to 0$. For $i \neq k$, a_{ik}^σ is an approximating sum for the Riemann integral $\int_a^b \alpha_i(t)\alpha_k(t)dt = 0$ and so for $i \neq k$, $a_{ik}^\sigma \to 0$ as $\|\sigma\| \to 0$. Hence $C_\sigma \to I$ in operator norm as $\|\sigma\| \to 0$. Thus for $\|\sigma\|$ sufficiently small, C_σ is a positive-definite matrix and is invertible and has a positive determinant [1]; we assume throughout the remainder of the proof that $\|\sigma\|$ is small enough so that C_σ has these properties. Thus $C_\sigma \to I$, $C_\sigma^{-1} \to I$ and $|C_\sigma|^{-1/2} \to 1$ as $\|\sigma\| \to 0$. Now let $\phi(v_1, \cdots, v_n) = (2\pi)^{-n/2}|C_\sigma|^{-1/2} \exp\{-\frac{1}{2}((v_1, \cdots, v_n), C^{-1}(v_1, \cdots, v_n))\}$ be the multivariate normal density function associated with $X_1^\sigma, \cdots, X_n^\sigma$ [1]. Here (,) refers to the inner product. Then we can write [8, p. 41],

$$(I_\lambda^\sigma(F)\psi)(\xi) = \int_{-\infty}^\infty \cdot(n)\cdot \int_{-\infty}^\infty f(\lambda^{-1/2}v_1, \cdots, \lambda^{-1/2}v_n)\phi(v_1, \cdots, v_n) \cdot\psi((b-a)^{1/2}\lambda^{-1/2}v_1 + \xi)dv_1 \cdots dv_n \tag{3}$$

which upon a change of variables becomes

$$\begin{aligned}(I_\lambda^\sigma(F)\psi)(\xi) &= \lambda^{n/2}(2\pi)^{-n/2}(b-a)^{-1/2}|C_\sigma|^{-1/2} \\ &\int_{-\infty}^\infty \cdot(n)\cdot \int_{-\infty}^\infty f((u_1 - \xi)(b-a)^{-1/2}, u_2, \cdots, u_n)\psi(u_1) \\ &\cdot\exp[-\lambda(((u_1-\xi)(b-a)^{-1/2}, u_2, \cdots, u_n), \\ &\quad C_\sigma^{-1}((u_1-\xi)(b-a)^{-1/2}, u_2, \cdots, u_n))/2]du_1 \cdots du_n .\end{aligned} \tag{4}$$

We now have our alternate expression for $I_\lambda^\sigma(F)$ for $\lambda > 0$. This formula defines an operator-valued analytic function of λ for $\operatorname{Re}\lambda > 0$ as can be shown in the usual manner [3, p. 533] by applying Morera's theorem. To check the details of this, one should keep in mind the properties of C_σ and may also wish to consult the remainder of this proof.

Now (4) and the defining expression for $I_\lambda^\sigma(F)$ are equal to the same Wiener integral for $\lambda > 0$ and both expressions are analytic for $\operatorname{Re}\lambda > 0$. Thus (4) gives $I_\lambda^\sigma(F)$ whenever $\operatorname{Re}\lambda > 0$.

Now let λ be fixed ($\operatorname{Re}\lambda > 0$) and let $\psi, \psi_0 \in L_2$ We finish by showing $(I_\lambda^\sigma(F)\psi, \psi_0) \to (I_\lambda^{an}(F)\psi, \psi_0)$ as $\|\sigma\| \to 0$. It will suffice to show this for an arbitrary sequence of partitions $\{\sigma_k\}$ such that $\|\sigma_k\| \to 0$. Comparing (4) and (2) carefully and recalling that $C_{\sigma_k}^{-1} \to I$ and $|C_{\sigma_k}|^{-1/2} \to 1$, we see that the proof of (b) will be finished if we can justify an application of the dominated convergence theorem to

$$(5) \qquad (2\pi)^{n/2}\lambda^{-n/2}(b-a)^{1/2}|C_{\sigma_k}|^{1/2}(I_\lambda^{\sigma_k}(F)\psi, \psi_0) .$$

Now since $C_{\sigma_k}^{-1} \to I$, it is easy to see that there exists N such that $k \geqq N$ implies

$$((w_1, \cdots, w_n), C_{\sigma_k}^{-1}(w_1, \cdots, w_n)) \geqq \tfrac{1}{2}((w_1, \cdots, w_n), (w_1, \cdots, w_n))$$

for all vectors $(w_1, \cdots, w_n)$. Hence for $k \geqq N$, a dominating function is given by

$$|f((u_1 - \xi)(b-a)^{-1/2}, u_2, \cdots, u_n)||\psi(u_1)||\psi_0(\xi)| \cdot \exp\left\{-\frac{\operatorname{Re}\lambda}{4}[(u_1-\xi)^2(b-a)^{-1} + u_2^2 + \cdots + u_n^2]\right\}$$

which is integrable by our hypotheses. Thus the proof of (b) is finally complete.

(a) In this case we note that for $\lambda > 0$ and $\psi \in L_2$ the following Wiener integral exists and is given by

$$\begin{aligned}
&\int_{C_0[a,b]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(b) + \xi)dx \\
&= \int_{C_0[a,b]} f\Big(\lambda^{-1/2}\int_a^b \alpha_1(t)dx(t), \cdots, \lambda^{-1/2}\int_a^b \alpha_n(t)dx(t)\Big) \\
&\qquad \cdot \psi\Big(\lambda^{-1/2}(b-a)^{1/2}\int_a^b \alpha_0(t)dx(t)\Big)dx \\
&= (2\pi)^{-(n+1)/2}\int_{-\infty}^{\infty} \cdot (n+1) \cdot \int_{-\infty}^{\infty} f(\lambda^{-1/2}u_1, \cdots, \lambda^{-1/2}u_n)\psi(\lambda^{-1/2}u + \xi) \\
&\qquad \cdot \exp\{-\tfrac{1}{2}(u^2 + u_1^2 + \cdots + u_n^2)\}du du_1 \cdots du_n \\
&= [d_\lambda e_\lambda(v)] * [\psi(v)](\xi) .
\end{aligned}$$

The remainder of the proof in this case is similar to the proof of the above case and is omitted.

Using the lemma from [6] and results on convolution operators found in [5, p. 951–964], we easily obtain the following corollary. $I_\lambda(F)^\#$ will denote the adjoint of $I_\lambda(F)$.

COROLLARY. *For all* λ *such that* $\operatorname{Re}\lambda > 0$, $I_\lambda(F)$ *is a normal operator. In case* (a): (i) $I_\lambda(F)^\#$ *is given by the formula*

$$(I_\lambda(F)^\#\psi)(\xi) = [\bar{d}_\lambda \bar{e}_\lambda(u)] * [\psi(u)](\xi) .$$

(ii) $||I_\lambda(F)|| = |d_\lambda|$. (iii) *The spectrum of* $I_\lambda(F)$ *consists entirely of continuous spectrum and is* $\{0\} \cup \{d_\lambda e^{-(b-a)y^2/2\lambda} : -\infty < y < \infty\}$. (iv) *The range of* $I_\lambda(F)$ *is contained in the set of equivalence classes of* L_2 *which contain a continuous function.* (v) *If* $d_\lambda \neq 0$, $I_\lambda(F)$ *is one-to-one. In case* (b): (i) $I_\lambda(F)^\#$ *is given by the formula*

$$(I_\lambda(F)^\#\psi)(\xi) = [\bar{h}_\lambda(-u)\bar{e}_\lambda(u)] * [\psi(u)](\xi) .$$

(ii) $||I_\lambda(F)|| = \sup\{|\mathscr{F}(h_\lambda e_\lambda)(y)| : -\infty < y < \infty\}$ *where* $\mathscr{F}$ *denotes the Fourier transform.* (iii) *The spectrum of* $I_\lambda(F)$ *is the closure of the range of* $\mathscr{F}(h_\lambda e_\lambda)$.

3. The Operator $J_q(F)$.

THEOREM 2. *Assume* $f(u_1, \cdots, u_n)$ *is integrable on* R_n. *Then the operator* $J_q(F)$ *exists for all* $q \neq 0$. (a) *If* α_0 *is orthogonal to span* $\{\alpha_1, \cdots, \alpha_n\}$, *then* $J_q(F)$ *is given by*

$$(6)\qquad \begin{aligned}(J_q(F)\psi)(\xi) &= d_{-iq}\int_{-\infty}^{\infty} e_{-iq}(v-\xi)\psi(v)dv \\ &= [d_{-iq}e_{-iq}(v)] * [\psi(v)](\xi)\end{aligned}$$

for $\psi \in L_2$, *where the integral is interpreted in the mean* [3, p. 521]. *Furthermore* $J_q(F)$ *is the strong operator limit of* $I_\lambda(F)$ *as* $\lambda \to -iq$ *in the right half plane.* (*b*) If $\alpha_0 = \alpha_1$, *then* $J_q(F)$ *is given by*

$$(7)\qquad \begin{aligned}(J_q(F)\psi)(\xi) &= \int_{-\infty}^{\infty} h_{-iq}(v-\xi)e_{-iq}(v-\xi)\psi(v)dv \\ &= [h_{-iq}(v)e_{-iq}(v)] * [\psi(v)](\xi)\end{aligned}$$

for $\psi \in L_2$. *In this case* $J_q(F)$ *is the limit in operator norm of* $I_\lambda(F)$ *as* $\lambda \to -iq$ *in the right half plane.*

REMARK. (i) In case (a), $J_q(F)$ is not the limit in operator norm of $I_\lambda(F)$ since, if $d_{-iq} \neq 0$, $J_q(F)$ lies in the open set of invertible operators [5, p. 862] while by the corollary above we see that $I_\lambda(F)$ is never invertible. (ii) The integrability of

$$f(u_1, \cdots, u_n)\exp\{-p(u_1^2 + \cdots + u_n^2)\}$$

for all $p > 0$ is not sufficient to insure the existence of $J_q(F)$, in fact the boundedness of $f(u_1, \cdots, u_n)$ is not sufficient.

Proof. (a) The proof of this case follows from the theorem in [6]. (b) Let $q \neq 0$ be given. Let $K_q(F)$ denote the map defined by $(K_q(F)\psi)(\xi) = [h_{-iq}(v)e_{-iq}(v)] * [\psi(v)](\xi)$. $K_q(F)$ is an operator since $h_{-iq}e_{-iq}$ is in L_1. It will suffice to show that $K_q(F)$ is the operator norm limit of $I_\lambda(F)$ as $\lambda \to -iq$. However, by [5, p. 953], it suffices to show that $h_\lambda e_\lambda$ converges in L_1 norm to $h_{-iq}e_{-iq}$ as $\lambda \to -iq$. But for all λ such that $|\lambda| \leqq 2|q|$ and $\operatorname{Re}\lambda > 0$, $|h_\lambda(v)e_\lambda(v)|$ is dominated by the L_1 function

$$g(v) = |q|^{n/2}\pi^{-n/2}\int_{-\infty}^{\infty}\cdot(n-1)\cdot\int_{-\infty}^{\infty}|f(v, v_2, \cdots, v_n)|\,dv_2\cdots dv_n\,.$$

Thus the result follows upon application of the dominated convergence theorem.

Again using [5, p. 951-964] and the lemma from [6], we easily obtain the following corollary.

COROLLARY. *In case* (a): (i) *For* $d_{-iq} \neq 0$, $(d_{-iq})^{-1}J_q(F)$ *is a unitary operator and so* $J_q(F)$ *is a normal operator and* $\|J_q(F)\| = |d_{-iq}|$. (ii) $J_q(F)^\#$ *is given by the formula*

$$(J_q(F)^\#\psi)(\xi) = \bar{d}_{-iq}\int_{-\infty}^{\infty}e_{iq}(v-\xi)\psi(v)dv$$

where the integral is interpreted in the mean. (iii) *If* $d_{-iq} \neq 0$, $J_q(F)$ *is invertible as an element of* $\mathscr{L}(L_2)$, *and* $J_q(F)^{-1} = |d_{-iq}|^{-2}J_q(F)^\#$. *In case* (b): (i) $J_q(F)$ *is a normal operator.* (ii) $J_q(F)^\#$ *is given by the formula* $(J_q(F)^\#\psi)(\xi) = [\bar{h}_{-iq}(-u)e_{iq}(u)] * [\psi(u)](\xi)$.

(iii) $$\|J_q(F)\| = \sup\{|\mathscr{F}(h_{-iq}e_{-iq})(y)| : -\infty < y < \infty\}\,.$$

(iv) *The spectrum of* $J_q(F)$ *is the closure of the range of* $\mathscr{F}(h_{-iq}e_{-iq})$.

4. Examples.

EXAMPLE 1. Let

$$\begin{aligned} F(x) &= \exp\left\{i\int_a^b[x(t)-x(a)]dt\right\} = \exp\left\{i\int_a^b(b-t)dx(t)\right\} \\ &= \exp\left\{i(b-a)^{3/2}3^{-1/2}\int_a^b(b-a)^{-3/2}3^{-1/2}(b-t)dx(t)\right\}. \end{aligned}$$

Now this is a functional of the type we are considering where

$$\alpha_1(t) = 3^{1/2}(b-a)^{-3/2}(b-t)$$

and $f_0(u_1) = \exp\{i(b-a)^{3/2}3^{-1/2}u_1\}$. This illustrates case (d) of the introduction. If we let $\beta_2(t) = 3^{1/2}(b-a)^{-3/2}(a+b-2t)$ then $\{\beta_1 = \alpha_0, \beta_2\}$ is an orthonormal basis for span $\{\alpha_0, \alpha_1\}$. Also $\alpha_1(t) = \frac{1}{2}(\beta_2(t) + 3^{1/2}\beta_1(t))$. Thus

$$F(x) = \exp\left\{i(b-a)^{3/2}2^{-1}3^{-1/2}\int_a^b\beta_2(t)dx(t) + i(b-a)^{3/2}2^{-1}\int_a^b\beta_1(t)dx(t)\right\}.$$

Thus the appropriate function is

$$f(u_1, u_2) = \exp(i(b-a)^{3/2}2^{-1}3^{-1/2}(u_2 + 3^{1/2}u_1))\}$$

and so case (b) of Theorem 1 is applicable. We obtain

$$h_\lambda(u) = \exp\{i(b-a)u/2 - (b-a)^3/24\lambda\}$$

and $\mathscr{F}(h_\lambda e_\lambda)(y) = \exp\{-(b-a)^3/24\lambda - (b-a-2y)^2(b-a)/8\lambda\}$. Thus, by the corollary, we see for example, that

$$\|I_\lambda(F)\| = \exp\left\{-\frac{(b-a)^3 \operatorname{Re}\lambda}{24|\lambda|^2}\right\}.$$

In [3] the functional

$$F_0(x) = \exp\left\{i\int_a^b x(t)dt\right\}$$

is considered. But $(I_\lambda(F_0)\psi)(\xi) = e^{i\xi(b-a)}(I_\lambda(F)\psi)(\xi)$, so that $I_\lambda(F_0)$ is simply $I_\lambda(F)$ followed by the unitary operator of multiplication by $e^{i\xi(b-a)}$. In particular, $\|I_\lambda(F_0)\| = \|I_\lambda(F)\|$.

EXAMPLE 2. (Functions of independent increments.) Let

$$\sigma: [a = t_0 < t_1 < \cdots < t_n = b]$$

be a partition of $[a, b]$. Let

$$F(x) \equiv g(x(t_1) - x(a), x(t_2) - x(t_1), \cdots, x(b) - x(t_{n-1})).$$

We wish to illustrate how such functionals may be treated in the framework of our theorems. We consider the case where $n = 3$. Now

$$\begin{aligned} &g(x(t_1) - x(a), x(t_2) - x(t_1), x(b) - x(t_2)) \\ &= g\Big((t_1-a)^{1/2}\int_a^b \alpha_1(t)dx(t), (t_2-t_1)^{1/2}\int_a^b \alpha_2(t)dx(t), (b-t_2)^{1/2}\int_a^b \alpha_3(t)dx(t)\Big) \end{aligned}$$

where

$$\alpha_1(t) \equiv (t_1-a)^{-1/2}\chi_{[a,t_1)}(t),\ \alpha_2(t) \equiv (t_2-t_1)^{-1/2}\chi_{[t_1,t_2)}(t),$$

and $\alpha_3(t) \equiv (b-t_2)^{-1/2}\chi_{[t_2,b]}(t)$ is an orthonormal set. This situation illustrates case (c) of the introduction. Accordingly, we seek another orthonormal basis $\{\beta_1, \beta_2, \beta_3\}$ for span $\{\alpha_1, \alpha_2, \alpha_3\}$ with $\beta_1 = \alpha_0$. Routine but tedious computations show that we may take

$$\begin{aligned} \beta_2(t) = (b-t_1)^{1/2}(t_1-a)^{-1/2}(b-a)^{-1/2}\chi_{[a,t_1)}(t) \\ - (t_1-a)^{1/2}(b-t_1)^{-1/2}(b-a)^{-1/2}\chi_{[t_1,b]}(t) \end{aligned}$$

and

$$\beta_3(t) = (b - t_2)^{1/2}(t_2 - t_1)^{-1/2}(b - t_1)^{-1/2}\chi_{[t_1,t_2)}(t) \\ - (t_2 - t_1)^{1/2}(b - t_2)^{-1/2}(b - t_1)^{-1/2}\chi_{[t_2,b]}(t) .$$

Then writing the α_j's in terms of the β_j's and letting

$$\begin{aligned} & f(u_1, u_2, u_3) \\ \equiv\ & g((t_1 - a)^{1/2}(b - a)^{-1/2}[(b - t_1)^{1/2}u_2 \\ & + (t_1 - a)^{1/2}u_1], (t_2 - t_1)^{1/2}[-(t_2 - t_1)^{1/2}(t_1 - a)^{1/2}(b - t_1)^{-1/2}(b - a)^{-1/2}u_2 \\ & + (b - t_2)^{1/2}(b - t_1)^{-1/2}u_3 \\ & + (t_2 - t_1)^{1/2}(b - a)^{-1/2}u_1], (b - t_2)^{1/2}[-(b - t_2)^{1/2}(t_1 - a)^{1/2}(b - t_1)^{-1/2}(b - a)^{-1/2}u_2 \\ & - (t_2 - t_1)^{1/2}(b - t_1)^{-1/2}u_3 + (b - t_2)^{1/2}(b - a)^{-1/2}u_1]) , \end{aligned}$$

we obtain

$$F(x) = f\left(\int_a^b \beta_1(t)dx(t), \int_a^b \beta_2(t)dx(t), \int_a^b \beta_3(t)dx(t)\right)$$

which is case (b). In connection with Theorem 2 we mention that if g is integrable, so also is f.

EXAMPLE 3. Let $f(u_1, u_2) = e^{-(u_1^2+u_2^2)}$ and let

$$F(x) = f\left(\int_a^b \alpha_1(t)dx(t), \int_a^b \alpha_2(t)dx(t)\right)$$

where $\alpha_1(t) = (b - a)^{-1/2}$ and $\{\alpha_1, \alpha_2\}$ are orthonormal. In this case, $h_\lambda(u) = \lambda^{1/2}(\lambda + 2)^{-1/2}e^{-u^2/(b-a)}$ and $\mathscr{F}(h_\lambda e_\lambda)(y) = \lambda(\lambda + 2)^{-1}e^{-(b-a)y^2/2(\lambda+2)}$. Thus for example $\|I_\lambda(F)\| = |\lambda|\,|\lambda + 2|^{-1}$ and $\|J_q(F)\| = |q|\,|2 - iq|^{-1}$.

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