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REFINEMENTS OF WALLIS'S ESTIMATE AND THEIR GENERALIZATIONS

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Some refinements of Wallis's estimate for π noticed in the recent literature are pointed out as already contained in a certain continued fraction expansion due to Stieltjes. A property of the approximants to this continued fraction is established which yields a simple proof of the expansion and furnishes, in particular, interesting monotone sequences of rational numbers with limit π. Two estimates of the Wallis type involving quotients of gamma functions are derived. They include estimates for $\Gamma(\alpha)$ and π csc $\pi\alpha$ $(0 < \alpha < 1)$ both of which reduce for $\alpha = 1/2$ to one of the known refinements of the Wallis estimate.

0. Introduction.

Let

$$g_0 = 1, \qquad g_n = \frac{1.3 \cdots (2n-1)}{2.4 \cdots 2n}, \qquad n = 1, 2, \cdots .$$

We have the well-known Wallis estimate

$$ng_n^2 < \frac{1}{\pi} < \left(n + \frac{1}{2}\right)g_n^2 .$$

Obtaining the case $x = n + 1/2$ of the inequalities

$$(1) \qquad x - \frac{1}{4} < \left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x)}\right]^2 < \frac{x^2}{x+\frac{1}{4}}, \qquad x > 0$$

by an application of a theorem in mathematical statistics, John Gurland [3] notes that

$$\left(n + \frac{1}{4}\right)g_n^2 < \frac{1}{\pi} < \frac{(n+\frac{1}{2})^2}{n+\frac{3}{4}}g_n^2 .$$

The first inequality here has been found earlier by D. K. Kazarinoff [4]. On the basis of a result of G. N. Watson, A. V. Boyd [1] has shown that one cannot have

$$\left(n + \frac{1}{4} + 1/(an + b)\right)g_n^2 < \frac{1}{\pi}, \qquad a > 0, b > 0$$

for all n if $a < 32$ and asserts that

$$\left(n + \frac{1}{4} + 1/(32n + b_1)\right)g_n^2 < \frac{1}{\pi} < \frac{(n+\frac{1}{2})^2}{n + \frac{3}{4} + 1/(32n + b_2)}g_n^2$$

for all $n \geqq 1$ with $b_1 = 32$ and $b_2 = 48$. All these facts are, however, overshadowed by the following continued fraction expansion due to Stieltjes [5]:

$$\text{(I)} \qquad 4\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2 = 4x+1+\frac{1^2}{2(4x+1)}\,+\,\frac{3^2}{2(4x+1)}\,+\cdots\,, \qquad x > -\frac{1}{4}\,.$$

Indeed, this result, together with its obvious transformation

$$4\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2 = \frac{(4x+2)^2}{4x+3}\,+\,\frac{1^2}{2(4x+3)}\,+\,\frac{3^2}{2(4x+3)}\,+\cdots\,, \qquad x > -\frac{1}{2}\,,$$

suffices to dispose of (1) and the two observations made in [1], the second of which is seen to hold even with $b_1 = 12$ and $b_2 = 27$. We wish to point out a simple and informative proof of (I) which shows, in particular, that

$$(4n+1)g_n^2 \uparrow \frac{4}{\pi}\,, \qquad \left(4n+1+\frac{1^2}{2(4n+1)}\right)g_n^2 \downarrow \frac{4}{\pi}, \cdots.$$

A direct proof of (1) is easy. In fact, assuming throughout that $0 < \alpha < 1$, we prove the two generalizations

$$\text{(II)} \qquad x - \frac{1-\alpha}{2} < \left[\frac{\Gamma(x+\alpha)}{\Gamma(x)}\right]^{1/\alpha} < \frac{1}{(1+\alpha/x)^{1/\alpha}-1}\,, \qquad x > 0\,,$$

$$\text{(III)} \qquad \begin{aligned} x - \alpha(1-\alpha) &< \frac{\Gamma(x+\alpha)\Gamma(x+1-\alpha)}{\Gamma^2(x)} \\ &< \frac{x^2}{x+\alpha(1-\alpha)}\,, \qquad x > 0\,. \end{aligned}$$

As special cases of interest, we have estimates for $\Gamma(\alpha)$ and $\pi\csc\pi\alpha$ generalizing Gurland's estimate for π:

$$(n+\alpha/2)^{1-\alpha}g_n(\alpha) < \frac{1}{\Gamma(\alpha)} < \frac{n+\alpha}{(n+(1+\alpha)/2)^\alpha}g_n(\alpha)\,,$$

$$\left(1-\frac{\alpha^2}{n+\alpha}\right)G_n(\alpha) < \frac{\sin\pi\alpha}{\pi} < \left(1+\frac{\alpha^2}{n+1-\alpha}\right)^{-1}G_n(\alpha)\,,$$

where

$$g_n(\alpha) = \binom{\alpha+n-1}{n}\,, \qquad G_n(\alpha) = \alpha\prod_{k=1}^{n}\left(1-\frac{\alpha^2}{k^2}\right).$$

One should compare (II), (III) and the inequalities

$$x - 1 + \alpha < \left[\frac{\Gamma(x+\alpha)}{\Gamma(x)}\right]^{1/\alpha} < x\,, \qquad x > 0\,, \tag{2}$$

which follow at once from the log-convexity of the gamma function. Wallis's estimate is the special case of (2) in which $\alpha = 1/2$ and $x = n + 1/2$ — the two together actually yield $\Gamma(1/2) = \sqrt{\pi}$. This is a simple evaluation of $\Gamma(1/2)$ that goes back to Stieltjes [2]; it is simple because (2) for $\alpha = 1/2$ requires only Schwarz's inequality for integrals.

The proofs of (I), (II) and (III) all utilize this familiar asymptotic formula implied by (2):

$$\Gamma(x+\alpha) \backsim x^{\alpha}\Gamma(x)\,, \qquad x \to \infty\,. \tag{3}$$

1. The expansion (I). We have

$$C_k(x) \equiv x + \frac{1^2}{2x+}\,\frac{3^2}{2x+}\cdots+\frac{(2k-1)^2}{2x} = \frac{A_k(x)}{B_k(x)}\,,$$
$$k = 0, 1, \cdots,$$

$W_k = A_k(x)$ and $W_k = B_k(x)$ being the two solutions of the recursion

$$W_{k+1} = 2xW_k + (2k+1)^2 W_{k-1}$$

defined by the initial values

$$A_{-2}(x) = -x,\ A_{-1}(x) = 1;\ B_{-2}(x) = 1,\ B_{-1}(x) = 0\,.$$

It is easily verified that the above recursion is equivalent to

$$W'_{k+1} = 2(x + 2\varepsilon)W'_k + (2k+1)^2 W'_{k-1}\,,$$

where

$$W'_k = (x + (2k+2)\varepsilon)W_k + (2k+1)^2 W_{k-1}\,, \qquad \varepsilon = \pm 1\,.$$

This establishes the matrix identity

$$\begin{bmatrix} (x+1)^2 B_k(x+2) & A_k(x+2) \\ (x-1)^2 B_k(x-2) & A_k(x-2) \end{bmatrix} = \begin{bmatrix} x+2k+2 & (2k+1)^2 \\ x-2k-2 & (2k+1)^2 \end{bmatrix}\cdot$$
$$\cdot\begin{bmatrix} A_k(x) & B_k(x) \\ A_{k-1}(x) & B_{k-1}(x) \end{bmatrix}$$

by an induction from the cases $k-1$ and $k(\geqq 0)$ to the case $k+1$. Passing to determinants, we at once see that

$$\operatorname{sgn}\{(x-1)^2 C_k(x+2) - (x+1)^2 C_k(x-2)\} = (-1)^k\,, \qquad x > 2\,,$$

which, on replacing x by $4x+3$ and introducing

$$\gamma_k(x) = \left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2 C_k(4x+1)\,, \qquad x > -\frac{1}{4}\,,$$

may be written

$$\operatorname{sgn}\{\gamma_k(x+1) - \gamma_k(x)\} = (-1)^k\,.$$

By (3), this yields

$$(*) \qquad \gamma_{2k}(x+n) \uparrow 4\,, \qquad \gamma_{2k+1}(x+n) \downarrow 4\,, \qquad n \uparrow \infty\,.$$

Hence $\gamma_{2k}(x) < 4 < \gamma_{2k+1}(x)$ and so we obtain (I):

$$\lim_{k\to\infty} \gamma_k(x) = 4\,.$$

The existence of this limit is assured by a known theorem [5, p.239] on the convergence of an infinite continued fraction with positive elements.

2. The inequalities (II). Consider

$$f(p, x) = (x - p)\left[\frac{\Gamma(x)}{\Gamma(x+\alpha)}\right]^{1/\alpha}, \qquad x > 0,\ -\infty < p < +\infty\,.$$

We have

$$\operatorname{sgn}\{f(p, x+1) - f(p, x)\} = \operatorname{sgn}\{p - p(x)\}\,,$$

$$p(x) \equiv x - \frac{1}{(1+\alpha/x)^{1/\alpha} - 1} \uparrow \frac{1-\alpha}{2}\,, \qquad (0 <)\ x \uparrow \infty\,,$$

$$f(p(x), x) = f(p(x), x+1) > f(p(x+1), x+1)\,.$$

The first of these assertions is easily checked and the last is obvious from the first two. The second, restated in the more convenient form

$$\chi(u) \equiv p\Big(\frac{\alpha}{e^{2\alpha u} - 1}\Big) = \frac{\alpha}{e^{2\alpha u} - 1} - \frac{1}{e^{2u} - 1} \uparrow \frac{1-\alpha}{2}\,, \qquad u \downarrow 0\,,$$

follows on observing that

$$2\chi'(u) = \frac{1}{\operatorname{sh}^2 u} - \frac{\alpha^2}{\operatorname{sh}^2 \alpha u} < 0\,,$$

$(\operatorname{sh} u)/u$ being increasing in $(0, \infty)$, while

$$\lim_{u\to 0} \chi(u) = \lim_{h\to 0} \frac{\alpha(e^h - 1) - (e^{\alpha h} - 1)}{\alpha h \cdot h} = \frac{1-\alpha}{2}\,.$$

Hence, by (3), we have the following limit relations which contain more than (II):

$$(**)\qquad f((1-\alpha)/2,\, x+n)\uparrow 1\,, \qquad f(p(x+n),\, x+n)\downarrow 1\,, \qquad n\uparrow\infty\,.$$

3. **The inequalities (III).** Proceeding as before, let

$$g(q, x) = (x-q)\frac{\Gamma^2(x)}{\Gamma(x+\alpha)\Gamma(x+1-\alpha)}\,,$$
$$x > 0,\ -\infty < q < +\infty\,.$$

The readily verified facts

$$\operatorname{sgn}\{g(q, x+1) - g(q, x)\} = \operatorname{sgn}\{q - q(x)\}\,,$$
$$q(x) \equiv \frac{\alpha(1-\alpha)x}{x+\alpha(1-\alpha)}\uparrow \alpha(1-\alpha)\,, \qquad (0<)\ x\uparrow\infty\,,$$
$$g(q(x), x) = g(q(x), x+1) > g(q(x+1), x+1)\,,$$

together with (3), prove more than (III):

$$(***)\qquad g(\alpha(1-\alpha), x+n)\uparrow 1\,, \qquad g(q(x+n), x+n)\downarrow 1\,, \qquad n\uparrow\infty\,.$$

An alternative proof is given by the product expansion

$$G(x) \equiv \frac{x\Gamma^2(x)}{\Gamma(x+\alpha)\Gamma(x+1-\alpha)} = \prod_{n=0}^{\infty}\left(1 + \frac{\alpha(1-\alpha)}{(x+n)(x+n+1)}\right),$$

which is evident from

$$\frac{G(x)}{G(x+1)} = 1 + \frac{\alpha(1-\alpha)}{x(x+1)}\,, \qquad \lim_{x\to\infty} G(x) = 1\,,$$

where the limit relation is a consequence of (3). The case $x=1$ of the above expansion occurs in [6].

References

1. A. V. Boyd, *Note on a paper by Uppuluri*, Pacific J. Math., **22** (1967), 9-10.
2. E. Cesàro, *Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung*, B. G. Teubner, Leipzig und Berlin, (1904), 719-720.
3. John Gurland, *On Wallis' formula*, Amer. Math. Monthly, **63** (1956), 643-645.
4. N. D. Kazarinoff. *Analytic Inequalities*, Holt, Reinhart and Winston, New York, (1961), 65-67.
5. O. Perron, *Die Lehre von den Kettenbrüchen*, B. G. Teubner, Leipzig und Berlin, (1929), 226.
6. G. Pólya und G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Bd. I, Springer-Verlag, Berlin, (1954), 128.

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