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ON ARITHMETIC PROPERTIES OF THE TAYLOR SERIES OF RATIONAL FUNCTIONS, II

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Suppose a_n, b_n, and $c_n = a_n b_n$ are sequences of algebraic integers and that all b_n are nonzero. It is easy to verify that if both $a(z) = \sum_{n=0}^{\infty} a_n z^n$ and $b(z) = \sum_{n=0}^{\infty} b_n z^n$ are rational functions, then so is $c(z) = \sum_{n=0}^{\infty} c_n z^n$. We are interested in studying the conjecture that if $b(z)$ and $c(z)$ are rational functions, then so is $a(z)$. We shall prove this in the case that $b(z)$ has no more than three distinct singularities.

Let k be an algebraic number field; denote by M_k the set of valuations of k, normalized so as to satisfy the Artin product-formula. We assume, whenever convenient, that each valuation in M_k has been extended in some fashion to Ω, the algebraic closure of k. Let S be a finite subset of M_k containing all Archimedean valuations. We say that $\alpha \in k$ is an *S-integer* if $|\alpha|_v \leqq 1$ for all $v \in M_k - S$ and that α is an *S-unit* if α and $1/\alpha$ are both S-integers. Let a_n be a sequence of S-integers of k. Suppose there exist rational functions $b(z) = \sum_{n=0}^{\infty} b_n z^n$ and $c(z) = \sum_{n=0}^{\infty} c_n z^n$ whose coefficients lie in an extension field K (possibly transcendental) of k; suppose that none of the b_n are 0 and that $a_n = c_n/b_n$ for $n \geqq 0$. In [1], I showed that if $b(z)$ has only one singularity (possibly a pole of high multiplicity) then $a(z) = \sum_{n=0}^{\infty} a_n z^n$ is a rational function. In [6] G. Pathiaux extended this result by showing that, under the additional assumption that K is algebraic, if $b(z)$ has at most two distinct singularities, then $a(z)$ is rational.

Here we shall study various extensions of these results. In particular we shall show that if $b(z)$ has at most three distinct singularities, then $a(z)$ is rational.

We note that since $b(z)$ and $c(z)$ are rational functions, we may write b_n and c_n as exponential polynomials:

$$b_n = \sum_{i=1}^{r} \lambda_i(n) \theta_i^n \tag{1}$$

$$c_n = \sum_{i=1}^{s} \mu_i(n) \varphi_i^n \tag{2}$$

for all sufficiently large n. Here the $\lambda_i(n)$ and $\mu_i(n)$ are polynomials in n. By appropriately enlarging K, if necessary, we may assume that the θ_i, the φ_i, and the coefficients of the polynomials $\lambda_i(n)$ and $\mu_i(n)$ all lie in K. By omitting a finite number of terms from each of the sequences a_n, b_n, c_n we may assume that (1) and (2) hold for all $n \geqq 0$. The purpose of the first lemma is to show that we may assume that K is algebraic over h.

LEMMA 1. *Suppose a_n, b_n, c_n are sequences as above. There exist sequences $\bar{b}_n$, $\bar{c}_n$ lying in a finite algebraic extension of k with $\bar{b}(z) = \sum_{n=0}^{\infty} \bar{b}_n z^n$ and $\bar{c}(z) = \sum_{n=0}^{\infty} c_n z^n$ rational functions such that $a_n \bar{b}_n = \bar{c}_n$ for all integral $n \geqq 0$ and such that only finitely many $\bar{b}_n$ are 0.*

Proof. As above we may write $b_n = \sum_{i=1}^{r} \lambda_i(n)\theta_i^n$ and $c_n = \sum_{i=1}^{s} \mu_i(n)\varphi_i^n$.

If all the coefficients of the λ_i and the μ_i, and the θ_i and φ_i are in k, then the Lemma is true with the $\bar{b}_n = b_n$ and $\bar{c}_n = c_n$. We henceforth assume this not the case. Let R be the ring generated by adjoining the θ_i, the φ_i, the ratios θ_i/θ_j, and the coefficients of the λ_i and the μ_i to k. By the assumption above the transcendence degree t of R/k is $\geqq 1$. We are going to construct a homomorphism τ of R into a finite algebraic extension of k such that τ, when restricted to k, will be the identity. If $\tau\alpha$ is abbreviated $\bar{\alpha}$ then $\bar{b}_n = \sum_{i=1}^{r} \bar{\lambda}_i(n)\bar{\theta}_i^n$ and thus $\sum \bar{b}_n z^n$ is rational; similarly $\sum \bar{c}_n z^n$ is rational and since $a_n b_n = c_n$ clearly $a_n \bar{b}_n = \bar{c}_n$. The remainder of this proof is devoted to constructing such a homomorphism τ for which only finitely many $\bar{b}_n$ are zero. By the Noether normalization lemma [3], there exists a transcendence basis $x = (x_1, x_2, \cdots, x_t)$ for R/k such that each element of R is integral over $k[x]$. Since $R/k[x]$ is algebraic and finitely generated, its degree d is finite. Each element α in R satisfies a polynomial equation $f(\alpha) = 0$, where

$$f(Y) = \sum_{i=0}^{e} p_i(x) Y^{e-i}$$

is a polynomial with coefficients $p_i(x)$ in $k[x]$, of degree $e \leqq d$, and monic ($p_0(x) \equiv 1$). Any homomorphism τ of $k[x]$ into k, which is the identity on k, has the form $p(x) \rightarrow p(u)$ where $u = (u_1, u_2, \cdots, u_t)$ is a t-tuple of elements of k and $p(x)$ is in $k[x]$. Such a homomorphism τ can be extended to a homomorphism of R into Ω, the algebraic closure of k [3]. The image $\bar{\alpha}$ of α will satisfy the monic polynomial $\sum_{i=0}^{e} p_i(u) Y^{e-i}$ and hence have degree $\leqq e$ over k. Since $e \leqq d$, every element in τR will have degree $\leqq d$ over k and hence τR will be contained in a finite algebraic extension of k. Moreover if $p_e(u) \neq 0$, then $\bar{\alpha} \neq 0$. Denote by $\Phi_k(h)$ the degree of the field generated by the primitive h^{th} roots of unity over k. It is easy to verify that $\Phi_k(h) \geqq \Phi_Q(h)/[k\colon Q]$ where Q is the field of rational numbers and $\Phi_Q(h)$ is, of course, Euler's phi-function. Since $\Phi_Q(h) \rightarrow \infty$ as $h \rightarrow \infty$, so does $\Phi_k(h)$. Let h be the largest integer for which $\Phi_k(h) \leqq d$. Let m be the least common multiple of all of the orders of all of the roots of unity which can be written in the form θ_i/θ_j. We can write

$$b_{mn+s} = \sum_{i=1}^{q} \eta_{is}(n)\sigma_i^n$$

where the σ_i are the distinct m^{th} powers of the θ_i, and the $\eta_{is}(n)$ are polynomials, not all 0 (for each value of s). Let α be the product of all the nonzero coefficients of the $\eta_{is}(n)$ and the elements $(\sigma_i/\sigma_j)^{h!} - 1$ for $i \neq j$ (the latter quantities are not 0 since the ratios σ_i/σ_j cannot be roots of unity). Now let $u = (u_1, u_2, \cdots, u_t)$ be elements of h for which $p_e(u) \neq 0$. Then under the homomorphism τ, defined above, $\bar{\alpha} = \tau\alpha$ will be nonzero, and $\bar{\eta}_{is}(n)$ (the polynomial obtained by applying τ to each coefficient of the polynomial $\eta_{is}(n)$) will be the zero-polynomial if and only if $\eta_{is}(n)$ is the zero-polynomial. None of the ratios $\bar{\sigma}_i/\bar{\sigma}_j$, with $i \neq j$, are roots of unity, for since $(\bar{\sigma}_i/\bar{\sigma}_j)^{h!} \neq 1$, if $\bar{\sigma}_i/\bar{\sigma}_j$ were a root of unity, it would have to have order $>h$ and hence degree $>d$ over k; but the latter is not the case. If any of the m sequences $\bar{b}_{mn+s}$ had infinitely many zeros then either all of the polynomials $\bar{\eta}_{is}(n)$ would be zero or by a theorem of Mähler [4] and Lech [5] the zeros would be periodic, and two of the $\bar{\sigma}_i$ would have ratio a root of unity. Thus the sequence b_n has only finitely many zeros.

LEMMA 2. *Suppose a_n is a sequence of S-integers of k, that $a_n = c_n/b_n$ where $b_n = \sum \lambda_i(n)\theta_i^n$ is never 0 and $c_n = \sum \mu_i(n)\varphi_i^n$; suppose the θ_i, φ_i and the coefficients of the $\lambda_i(n)$ and the $\mu_i(n)$ are integers of k. Suppose there exists a valuation $v_0 \in S$ such that $|\theta_1|_{v_0} > |\theta_i|_{v_0}$ for $i \geqq 2$. Then $\sum_{n=0}^{\infty} a_n z^n$ is rational.*

Proof. Elementary estimates show there exist $M > 0$ and $R > 0$ such that $|b_n|_v$ and $|c_n|_v$ are $\leqq MR^n$ for all $v \in S$ and $n \geqq 0$, and that $|b_n|_v \leqq 1$ for all $v \notin S$ and $n \geqq 0$. Since $\prod_{v \in S} |b_n|_v \geqq 1$, if $w \in S$, then

$$\left|\frac{1}{b_n}\right|_w \leqq \prod_{\substack{v \in S \\ v \neq w}} |b_n|_v \leqq M^{s-1}R^{(s-1)n}$$

where s is the cardinality of S. Then $|a_n|_w = |c_n/b_n|_w \leqq M^s R^{sn}$. It follows that $\sum_{n=0}^{\infty} a_n z^n$ has positive radius of convergence in k_w, the completion of k under the valuation w. Let $\tilde{k}_w$ be the algebraic closure of k_w and assume that w has been extended to $\tilde{k}_w$. Let R_w be the radius convergence of $a(z) = \sum_{n=0}^{\infty} a_n z^n$ in k_w. Then $a(z)$ is analytic in $\tilde{k}_w$ for $|z|_w < R_w$. Now

$$\lambda_1(n)\theta_1^n a_n = c_n - \sum_{i=2}^{r} \lambda_i(n)\theta_i^n a_n$$

or

$$\sum_{n=0}^{\infty} \lambda_1(n) a_n z^n = c\left(\frac{z}{\theta}\right) - \sum_{i=2}^{r} \lambda_i(n)\left(\frac{\theta_i}{\theta_1}\right)^n a_n z^n .$$

In the field $\tilde{k}_{v_0}$, the algebraic closure of k_{v_0}, the last equation expresses

$\sum_{n=0}^{\infty} \lambda_1(n) a_n z^n$ as a rational function plus a sum of functions each meromorphic for $|z|_{v_0} \leqq \delta R_{v_0}$ where $\delta = \min_{i \geqq 2} |\theta_1/\theta_i|_{v_0}$ is >1. Thus by analytic extension $\sum_{n=0}^{\infty} \lambda_1(n) a_n z^n$ is meromorphic for $|z|_{v_0} < \delta R_{v_0}$. Repeated applications of the above transformation show that $\sum_{n=0}^{\infty} \lambda_1(n)^j a_n z_u$ is meromorphic for $|z|_{v_0} < \delta^j R_{v_0}$. Elementary estimates show that $\sum_{n=0}^{\infty} \lambda_1(n)^j a_n z^n$ has radius of convergence R_v for all $v \in S$. Choosing j so large that $\delta^j \prod_{v \in S} R_v$ is >1, we find, by a theorem of Dwork [2], that $\sum_{n=0}^{\infty} \lambda_1(n)^j a_n z^n$ is a rational function. By [1] so is $\sum_{n=0}^{\infty} a_n z^n$.

LEMMA 3. *Suppose a_n is a sequence of S-integers of k, that $a_n = c_n/b_n$ where $b_n = \sum \lambda_i(n)\theta_i^n$ is never zero and $c_n = \sum \mu_i(n)\varphi_i^n$, suppose the θ_i, φ_i and the nonzero coefficients of the $\lambda_i(n)$ and the $\mu_i(n)$ are S-units of h. Suppose there exists a valuation v_0 of h such that $|\theta_1|_{v_0} < |\theta_i|_{v_0}$ for $i \geqq 2$. Then $\sum_{n=0}^{\infty} a_n z^n$ is rational.*

Poof. Extend the definition of c_n and b_n to negative n by their formulas. If infinitely many such b_n were zero, then by a theorem of Lech [4] and Mähler [5], b_n would be zero for all n in a doubly infinite arithmetic progression, contradicting the hypotheses. Extend the definition of a_n to negative n by putting $a_n = c_n/b_n$ if $b_n \neq 0$ and otherwise put $a_n = 0$. Now let v be any valuation of M_k not in S. Then v is an extension of a p-adic valuation $|\ |_p$ of Q. There exists an integer f such that if $\alpha \in k$ and $|\alpha|_v = 1$ then $|\alpha^{p^f} - 1|_v < 1$. Letting m be an integer of the form $p^h(p^f - 1)$, where h is large, we find that $|\alpha^m - 1|_v$ can be made very small. In particular we can choose m so large that if $b_n \neq 0$ then $|b_{n+m}|_v = |b_n|_v$ and that $|c_{n+m} - c_n|_v < |b_n|_v$. We can choose m so large that $m + n \geqq 0$ and then $|c_{m+n}/b_{m+n}|_v \leqq 1$. Thus $|a_n|_v \leqq 1$. Restating all this, we have shown that there exists n_0 such that if $n \leqq n_0$ then $b_n \neq 0$ and if $v \in S$ then $|a_n|_v \leqq 1$. We apply Lemma 2 to the sequences $a'_n = a_{n_0-n}$, $b'_n = b_{n_0-n}$ and $c'_n = c_{n_0-n}$, to conclude that $\sum_{n=0}^{\infty} a'_n z^n$ is rational. It follows that a_n can be written in the form

$$a_n = \sum_{i=1}^{t} \eta_i(n)\sigma_i^n$$

for $n \leqq n_0$. Then the exponential polynomial

$$\sum_{i=1}^{s} \mu_i(n)\varphi_i^n - \sum_{i=1}^{r} \lambda_i \theta_i^n \sum_{i=1}^{t} \eta_i(n)\sigma_i^n$$

is 0 for $n \leqq n_0$. By the theorem of Mähler [5] and Lech [4], it is identically 0. Thus $a_n = \sum_{i=1}^{t} \eta_i(n)\sigma_i^n$ for $n \geqq 0$ and $a(z) = \sum_{n=0}^{\infty} a_n z^n$ is a rational function.

We now come to the result mentioned at the beginning of this paper.

THEOREM 4. *Suppose a_n is a sequence of S-integers of k and that b_n and c_n are sequences of elements of an extension field K of k such that $\sum_{n=0}^{\infty} b_n z^n$ and $\sum_{n=0}^{\infty} c_n z^n$ are rational functions and b_n is never zero. If $a_n = c_n/b_n$ and the rational function $\sum_{n=0}^{\infty} b_n z^n$ has at most 3 distinct singularities then $\sum_{n=0}^{\infty} a_n z^n$ is rational.*

Proof. By Lemma 1, we may assume K is algebraic over k and that $b_n = \sum_{i=1}^{r} \lambda_i(n)\theta_i^n$ and that $c_n = \sum_{i=1}^{s} \mu_i(n)\varphi_i^n$ where the θ_i, φ_i and all coefficients of the λ_i and μ_i are algebraic over k. By replacing k by a larger field and S by the set of extensions of the valuations in S to this new field, we may assume that the above quantities are, in fact, in k. By increasing S appropriately, we may assume that those of the above quantities which are not zero are S-units. Now if $r = 1$, the theorem follows immediately from [1]. If $r = 2$ then either θ_1/θ_2 is a root of unity, in which case the theorem follows from the case $r = 1$ or there is a valuation v such that $|\theta_1|_v > |\theta_2|_v$, and the theorem follows from Lemma 2. If $r = 3$ then either $|\theta_1|_v = |\theta_2|_v = |\theta_3|_v$, for all $v \in S$ and θ_1/θ_2 and θ_1/θ_3 are roots of unity, so the theorem follows from the case $r = 1$, or there is a valuation $v_0 \in S$ for which not all of the three values are equal. In the latter case we may assume that $|\theta_1|_{v_0} \leqq |\theta_2|_{v_0} \leqq |\theta_3|_{v_0}$ and $|\theta_1|_{v_0} < |\theta_3|_{v_0}$. If $|\theta_2|_{v_0} = |\theta_3|_{v_0}$ then the theorem follows from Lemma 3, and otherwise from Lemma 2.

It is worth noting that the method of the theorem cannot be extended to the case where $b(z)$ has 4 singularities. In fact, consider the case where k is the field $Q(i)$ where $i = \sqrt{-1}$ and $\theta_1 = (1 + 2i) \times (1 + 4i)$, $\theta_2 = (1 + 2i)(1 - 4i)$, $\theta_3 = (1 - 2i)(1 + 4i)$, $\theta_4 = (1 - 2i)(1 - 4i)$. The ideals generated by $(1 + 2i)$, $(1 - 2i)$, $(1 + 4i)$, $(1 - 4i)$ are prime and give rise to 4 valuations of $Q(i)$. At each of these valuations, two of the θ_j take one value and two another. For example at the valuation corresponding to the prime ideal generated by $1 - 2i$, θ_1 and θ_2 both have value 1, while θ_3 and θ_4 both have the same value which is less than 1. All 4 θ_j take the same value at all other valuations. Thus the hypotheses of Lemma 2 or Lemma 3 cannot be met.

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