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A NOTE ON MEASURES ON FOUNDATION SEMIGROUPS WITH WEAKLY COMPACT ORBITS

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For an extensive class of locally compact semigroups S, foundation semigroups with identity element, we prove that two subalgebras of $M(S)$ [the algebra of the bounded Radon measures on S] coincide. Namely, the algebra $L(S)$, generated by the $m \in M(S)^+$ for which the orbits on the compact subsets of S are weakly compact subsets of $M(S)$, or, equivalently, for which the translations are weakly continuous, and the algebra $M_e(S)$, generated by the $m \in M(S)^+$ for which the restrictions of the orbit of m on S to the compact subsets of S are weakly compact. In case S is a group, both these algebras consist of the bounded Radon measures that are absolutely continuous with respect to a Haar measure on S.

1. If S is a locally compact group and $m \in M(S)$ then the orbits F_m of m on all compact subsets F of S [$F_m := \{m*\bar{x} \mid x \in F\} \cup \{\bar{x}*m \mid x \in F\}$, where $\bar{x}$ denotes the point mass at x] are weakly compact subsets of $M(S)$ if and only if the restrictions $S_{m|F}$ of the orbit S_m of m on S to all compact subsets F of S [$S_{m|F} := \{\mu|_F \mid \mu \in S_m\}$] are weakly compact. The proof of this fact follows by observing that $F^{-1}K[:= \{x \in S \mid Fx \cap K \neq \emptyset\}]$ and KF^{-1} are compact as soon as both F and K are compact subsets of the group S. An arbitrary locally compact semigroup S may fail to have this compactness property and may [and actually does] give rise to two different subsets of $M(S)$: namely, to $L(S)$, the collection of all $m \in M(S)$ for which $F_{|m|}$ is weakly compact ($F \subseteqq S$, compact), and to $M_e(S)$, the collection of all $m \in M(S)$ for which $S_{|m||F}$ is weakly compact ($F \subseteqq S$, compact). Elementary properties of $L(S)$ can be found in e.g., [1], [2], [6], and [7] and of $M_e(S)$ in [4]. Although, in some respects $M_e(S)$ has better properties than $L(S)$ [cf. [4], e.g., (5.2) and (5.3)] $L(S)$ is a more obvious analogue of the group algebra than $M_e(S)$: $L(S)$ is a two sided L-ideal in $M(S)$ [cf. [1], (3, 4) and [2], (2.6)], while, in general, $M_e(S)$ is only an L-subalgebra of $M(S)$ [cf. (2.1) and [4], (2.6)].

It is natural to wonder about the relationship between $L(S)$ and $M_e(S)$. In view of the inner regularity of the measures in question, it is clear that $M_e(S) \subseteqq L(S)$. As noted above, $M_e(S)$ may be strictly contained in $L(S)$, but for an important class of semigroups we can show that these collections coincide. We shall prove the following theorem.

THEOREM 1.1. *Let S be a locally compact [topological] semigroup with identity element 1 such that for all compact neighborhoods U of 1 we have that*

(i) $x \in \operatorname{int}(U^{-1}(Ux) \cap (xU)U^{-1})$ $(x \in S)$ *and*

(ii) $1 \in \operatorname{int}(U^{-1}v \cap uU^{-1})$ *for some* $u, v \in U$.

Then $L(S) = M_e(S)$.

The semigroup S, in the theorem, is a so-called *stip* [cf. [7], (2.1)]. The class of stips is extensive, contains the *foundation semigroups* S [i.e., $\cup\{\operatorname{supp}(m) \mid m \in L(S)\}$ is dense in S] with identity element [cf. [7], (2.2)] and, furthermore, contains many semigroups S that contain compact subsets F and K of S such that $F^{-1}K$ is not compact [cf. [6], App. B].

The notations and conventions that are not explained in the text or the introduction are the ones of [7].

2. The failure of $M_e(S)$; an example. Before we proceed to prove the theorem, we give an example of a foundation semigroup S for which $M_e(S)$ is not an ideal in $M(S)$, and, consequently, $M_e(S) \subsetneqq L(S)$. This example solves the question (2.7) in [4].

EXAMPLE 2.1. Let T_1 be the subset $[0, 1]_d \times \{0\} \cup [0, 1] \times \{1\} \cup \{\theta\}$ of $\boldsymbol{R}^2$, where θ is the point $(1, 2)$ in $\boldsymbol{R}^2$. $[0, 1]_d \times \{0\}$ is endowed with the discrete topology and $[0, 1] \times \{1\} \cup \{\theta\}$ with the restriction topology of the usual topology on $\boldsymbol{R}^2$. Then T_1 is a locally compact semigroup under the multiplication defined as follows:

$$\left.\begin{array}{l}(t, t')\,\theta := \theta(t, t') := \ \theta\theta := \theta \\ (t, t')\,(s, s') := (t, s') \text{ if } t' = 0 \\ (t, t')\,(s, s') := \theta \text{ if } t' \neq 0\,.\end{array}\right\} \quad \text{for all} \quad (t, t'),\ (s, s') \in T_1 \backslash \{\theta\}$$

Now, let a be an element isolated from T_1.
Put

$$T := T_1 \cup \{a\}\,.$$

And let the multiplication on T_1 be extended to T by:

$$\left.\begin{array}{l} aa := a,\ a\theta := \theta a := \theta \\ a(t, t') := (0, t') \\ (t, t')\,a := (t, 0) \text{ if } t' = 0 \\ (t, t')\,a := \theta \text{ if } t' \neq 0\,.\end{array}\right\} \quad \text{for all} \quad (t, t') \in T \backslash \{\theta\}$$

Then one can easily check that T is a locally compact semigroup [only the associativity needs some real verification], and that T is even a foundation semigroup.

Furthermore, one has that

$$\{\bar{a} * \bar{x} \mid x \in T\} = \{\bar{a}, \bar{\theta}, \overline{(0, 1)}, \overline{(0, 0)}\}$$
$$\{\bar{x} * \bar{a} \mid x \in T\} = \{\bar{a}, \bar{\theta}\} \cup \{\overline{(t, 0)} \mid t \in [0, 1]\} .$$

Apparently, $\bar{a} \in M_e(T)$. However, $\bar{a} * \overline{(0, 1)} = \overline{(0, 1)}$ and $\overline{(0, 1)} \notin M_e(T)$; because $\overline{\{(t, 0) * (0, 1) \mid t \in [0, 1]\}} = \overline{\{(t, 1) \mid t \in [0, 1]\}}$. This shows that $M_e(T)$ is not a right ideal in $M(T)$.

Let b be an element, isolated from $T \times \{1, 2\}$. Put $S := T \times \{1, 2\} \cup (b)$ and let the multiplication on S be defined by

$$\left.\begin{array}{l} sb: = bs: = b \quad \text{for all} \quad s \in S \\ (x, 1)(y, 2): = (y, 2)(x, 1): = b \\ (x, 1)(y, 1): = (xy, 1),\ (x, 2)(y, 2): = (yx, 2) \end{array}\right\} \quad \text{for all} \quad x, y \in T .$$

Then S is a foundation semigroup and $M_e(S)$ is neither a right nor a left ideal in $M(S)$.

3. Preliminaries to the proof of the theorem. Before we proceed to the proof of theorem we prove a lemma that can be viewed as a tool to reduce the "underlying space" to a space satisfying the second axiom of countability [cf. (3.4)]. For the convenience of the reader we first summarize some properties concerning the structure of a stip S and its "semigroup algebra" $L(S)$ [cf. (3.1)-(3.3)].

PROPOSITION 3.1 [cf. [7], (2.4), (2.7), and (2.11).] *Let S be a stip. Then S has a smallest dense ideal $\mathring{S}$ [i.e., $S\mathring{S} \cup \mathring{S}S \subseteqq \mathring{S}$, $\operatorname{clo}(\mathring{S}) = S$ and $\mathring{S} \subseteqq J$ for all $J \subseteqq S$ with $\bar{J} = S$ and $JS \subseteqq J$ or $SJ \subseteqq J$].*

For each pair of open subset W and U of S and each $x \in \mathring{S}$ the sets $W^{-1}(Ux)$, $(xU)W^{-1}$, $(W \cap \mathring{S})^{-1}x$ and $x(W \cap \mathring{S})^{-1}$ are open. For each neighborhood U of 1 there exists a neighborhood V of 1 such that

$$\tilde{U}: = \{(x, y) \in S \times S \mid Ux \cap Uy \neq \emptyset\} \supseteqq \tilde{V} \circ \tilde{V}$$

[where $A \circ B: = \{(x, z) \in S \times S \mid (x, y) \in A,\ (y, z) \in B$ for some $y \in S\}$, $(A, B \subseteqq S \times S)$].

PROPOSITION 3.2 [cf. [7]. (3.6) and (3.7)].

Let S be a stip. An idempotent e in S [i.e., $ee = e$] is said to be δ-isolatd if there exists a G_δ-set G in S for which $(G \cap eSe)\backslash\{e\}$ contains no idempotents.

Then for each countable subset A of $\mathring{S}$ and each $\mu \in L(S)$ with compact support there exists a δ-isolated idempotent e such that

$$\operatorname{supp}(\mu) \cup A \subseteq eSe \; .$$

PROPOSITION 3.3 [cf. [1], (3.4), [6], (2.6) and [7], (3.13)].

(1) *For each* $\mu \in M(S)$, *the maps* $x \rightsquigarrow \bar{x} * \mu$ $(x \in S)$ *and* $x \rightsquigarrow \mu * \bar{x}$ $(x \in S)$ *from* S *into* $M(S)$ *are weakly* continuous. Let* S *be a stip.*

(2) *Then* $L(S)$ *is an* L*-ideal in* $M(S)$.

(3) *For* $\mu \in M(S)$, *the following properties are equivalent:*

(i) $\mu \in L(S)$;

(ii) *the map* $x \rightsquigarrow \bar{x} * \mu$ *from* S *into* $M(S)$ *is norm continuous;*

(iii) *the map* $x \rightsquigarrow \mu * \bar{x}$ *from* S *into* $M(S)$ *is norm continuous.*

(4) *If, in addition,* S *is a foundation semigroup then, for all* $\mu \in L(S)$ *and* $\varepsilon > 0$, *there exists a* $\nu \in L(S)$ *such that* $\|\nu\| = \nu(S) = 1$ *and*

$$\|\nu * \mu - \mu\| < \varepsilon \; .$$

Now we can prove the lemma.

LEMMA 3.4. *Let* S *be a stip with a* δ*-isolated identity element* 1, *let* $(V_n)_{n \in N}$ *be a sequence of neighborhoods of* 1 *and let* M *be a compact subset of* S.

Then there exists a decreasing sequence $(U_n)_{n \in N}$ *of open relatively-compact neighborhoods of* 1 *and a semi-metric* ρ *on* S *such that*

(1) $U_{n+1}^{-1}(U_{n+1}x) \subseteq B_\rho(x, 1/n) \subseteq U_n^{-1}(U_n x)$ $\quad (n \in N,\ x \in S)$, *where* $B_\rho(x, \varepsilon) := \{y \in S \mid \rho(x, y) < \varepsilon\}$ $(x \in S, \varepsilon > 0)$;

(2) $G := B_\rho(1, 0)$ *is a compact group with identity element* 1 *contained in* V_n *for all* $n \in N$.

Let π *be the normalized Haar measure on* G. *Then*

(3) *for each* σ*-compact subset* A *of* S, $P := \bigcap_{m=1}^{\infty} U_m A$ *is a Borel set and* $|\pi * \mu|(\bar{P} \backslash P) = 0$ *for all* $\mu \in L(S)$.

Finally, there exists a countable subset T *of* S *such that*

(4) *for all* $x \in M$ *and all neighborhood* W *of* 1 *we have that* $GWx \cap T \neq \varnothing$.

Proof. Since 1 is δ-isolated there exists a sequence $(U'_n)_{n \in N}$ of neighborhoods of 1 such that $\bigcap_{n=1}^{\infty} U'_n \backslash \{1\}$ contains no idempotents. Now, by induction and using (3.1), one can construct a decreasing sequence $(U_n)_{n \in N}$ of open relatively-compact neighborhoods of 1 and a sequence $(r_n)_{n \in N}$ of elements of $\mathring{S}$ such that for all $n \in N$ we have

$r_n \in U_n$

(i) $\bar{U}_{n+1} \subseteq U_n \cap U'_n \cap V_n \cap U_n^{-1} r_n$

(ii) $U_{n+1}^2 \subseteq U_n$

(iii) $\tilde{U}_{n+1} \circ \tilde{U}_{n+1} \circ \tilde{U}_{n+1} \subseteq \tilde{U}_n$.

From (iii) and the result on p. 184-186 in [5] [see also [7], (2.13)] it follows that there exists a semi-metric ρ with property (1). By using (i) and (ii), one can easily conclude that (2) holds.

To prove (3), let $(F_n)_{n\in N}$ be a sequence of compact subsets of S. Put $A:=\bigcup_{n=1}^{\infty} F_n$ and $P:=\bigcap_{m=1}^{\infty} U_m A$. Then by (i) we have

$$P=\bigcap_{m=1}^{\infty} U_m A \subseteqq \bigcap_{m=1}^{\infty} \bar{U}_m A \subseteqq \bigcap_{m=2}^{\infty} U_{m-1} A = P .$$

Since $\bar{U}_m A = \bigcup_{n=1}^{\infty} \bar{U}_m F_n$ is σ-compact $(m \in N)$, this proves that P is a Borel set. Furthermore, by (1), (i) and (ii), we have that

$$\begin{aligned}\bar{P} &\subseteqq B_\rho(P, 1/(n+4)) \subseteqq U_{n+3}^{-1}(U_{n+3}P) \\ &\subseteqq r_{n+2}^{-1}(U_{n+1}(U_{n+1}A)) \subseteqq r_{n+2}^{-1}(U_n A) \subseteqq r_{n+2}^{-1}(\bar{U}_n A) \quad (n \in N) .\end{aligned}$$

Now let $\varepsilon > 0$ and $\mu \in L(S)$. Then, by (3.3.3), we have that

$$V:=\{x \in S \,|\, \|\bar{x}*\pi*|\mu| - \pi*|\mu|\| < \varepsilon\}$$

is an open neighborhood of G. Since G is a compact group, (2) implies that there exists a $k \in N$ such that

$$U_k \subseteqq V .$$

We may assume that

$$|\pi*|\mu|(P) - \pi*|\mu|(\bar{U}_k A)| < \varepsilon .$$

Obviously, $r_{k+2} \in U_k \subseteqq V$, which leads to

$$\begin{aligned}&\pi*|\mu|(\bar{U}_k A) - \varepsilon \leqq \pi*|\mu|(P) \leqq \pi*|\mu|(\bar{P}) \\ &\bar{r}_{k+2}*\pi*|\mu|(\bar{U}_k A) \leqq \pi*|\mu|(\bar{U}_k A) + \varepsilon .\end{aligned}$$

This shows that $|\pi*\mu|(\bar{P}\backslash P) \leqq \pi*|\mu|(\bar{P}\backslash P) \leqq 2\varepsilon$, and completes the proof of (3).

To prove (4), define the equivalence relation $\approx$ on S by

$$x \approx y \quad \text{if and only if} \quad \rho(x, y) = 0 \quad (x, y \in S) .$$

Put $\tilde{\tilde{x}}:=\{y \in S \,|\, \rho(x, y) = 0\}$, $\tilde{\tilde{B}}:=\{\tilde{\tilde{b}} \,|\, b \in B\}$ $(x \in S, B \subseteqq S)$ and let Σ be the quotient space $S/\approx$. Then Σ is a metric space and $\tilde{\tilde{M}}$ is a compact subset of Σ. Therefore we can find a countable subset A of S such that

$$\operatorname{clo}(\tilde{\tilde{A}}) = \tilde{\tilde{M}} \quad \text{in} \quad \Sigma .$$

Define $T:=\{r_n a \,|\, n \in N, a \in A\}$. We shall prove that this set T has property (4). Obviously T is countable. Let W be an open neighborhood of 1 and let $x \in M$. Since G is a compact group, GW is an open neighborhood of G and, by (2), there exists a $k \in N$ such that $U_k x \subseteqq GWx$. Furthermore we have that

$$B_\rho(\tilde{\tilde{x}}, 1/(k+2)) = B_\rho(x, 1/(k+2)) \\ \subseteqq U_{k+2}^{-1}(U_{k+2}x) \subseteqq r_{k+2}^{-1}(U_k x) .$$

Since $\tilde{\tilde{A}}$ is dense in $\tilde{\tilde{M}}$ it follows that $B_\rho(\tilde{\tilde{x}}, 1/(k+2)) \cap \tilde{\tilde{A}} \neq \varnothing$ and therefore $GWx \cap T \neq \varnothing$.

4. The proof of theorem. Initially, we shall prove the theorem for a foundation semigroup S with identity element. Then we shall indicate how to prove the theorem for stips.

(A) Let S be a foundation semigroup with identity element 1 and let $m \in L(S)$.

To prove that $m \in M_e(S)$, we may assume without loss of generality that $M := \operatorname{supp}(m)$ is compact, $m \geqq 0$ and $\|m\| = 1$. Take a compact subset K of S. In view of (3.3.1), it is sufficient to show that $\{(m * \bar{x})|_K \mid x \in S\}$ is a relatively weakly compact subset of $M(S)$. A combination of the theorem of Eberlein and Smulian [cf. e.g., [3], V. 6.1], (3.3.1) and the denseness of $\mathring{S}$ in S tells us now that $m \in M_e(S)$ as soon as

$$(1) \qquad \begin{cases} \{(m * \bar{x}_n)|_K \mid n \in N\} & \text{has a weak limit point in } M(S) \\ \text{for all sequences } (x_n)_{n \in N} & \text{in } \mathring{S} . \end{cases}$$

Let $(x_n)_{n \in N}$ be a sequence in $\mathring{S}$. By (3.2), there exists a δ-isolated idempotent e such that $M \cup \{x_n \mid n \in N\} \subseteqq eSe$. Therefore, again without loss of generality, we may assume that 1 is δ-isolated.

For each $n \in N$ let $V_n := \{x \in S \mid \|\bar{x} * m - m\| < 1/n\}$. Then $(V_n)_{n \in N}$ is a sequence of neighborhoods of 1 [cf. (3.3.3)] and we can apply our lemma. In the sequel, we shall use the same notations as in the lemma and its proof.

Since $G \subseteqq V_n$ $(n \in N)$ it follows that

$$\bar{g} * m = m \ (g \in G) ,$$

and hence that

$$(2) \qquad \pi * m = m .$$

Let $\langle t_1, t_2, t_3, \cdots \rangle$ be an enumeration of T.

Now, by induction, one can construct sequences

$x_1^0, x_2^0, x_3^0, x_4^0, \cdots$
$x_1^1, x_2^1, x_3^1, x_4^1, \cdots$
$x_1^2, x_2^2, x_3^2, x_4^2, \cdots$
$\cdots\cdots$
$\cdots$

with the following properties

(a) $x_m^0 = x_m$ $(m \in \boldsymbol{N})$,

(b) $(x_m^{n+1})_{m \in N}$ is a subsequence of $(x_m^n)_{m \in N}$ $(n \in \boldsymbol{N})$,

(c) $(t_m x_m^n)_{m \in N}$ is a ρ-Cauchy sequence or
$x_m^n = x_m^{n-1}$ $(m \in \boldsymbol{N})$ and $(t_n x_m^n)_{m \in N}$ does not contain any ρ-Cauchy sequence $(n \in \boldsymbol{N})$.

Put $y_m := x_m^m$ $(m \in \boldsymbol{N})$, $T_1 := \{t \in T \mid (ty_m)_{m \in N}$ is a ρ-Cauchy sequence$\}$, $P := \bigcap_{m=1}^{\infty} U_m T_1$, $m_1 := m|_P$ and $m_2 := m - m_1$.

We shall prove that

(α) $(\pi * m_1 * \bar{y}_n)_{n \in N}$ is a norm-Cauchy sequence in $M(S)$, and

(β) $((\pi * m_2 * \bar{y}_n)|_K)_{n \in N}$ converges in norm to 0;

then obviously $((\pi * m * \bar{y}_n)|_K)_{n \in N}$ is a Cauchy sequence and consequently $\{(\pi * m * \bar{x}_n)|_K \mid n \in \boldsymbol{N}\}$ has a weak limit point. Therefore, in view of (2), this proves (1) and completes the proof of the theorem in this case.

(α) Let $\varepsilon > 0$. By (3.3.4), there exists a $\nu \in L(S)^+$ such that

$$(3) \qquad \|\nu * \pi * m_1 - \pi * m_1\| < \varepsilon/4 .$$

Since $\{x \in S \mid \|\nu * \pi * \bar{x} - \nu * \pi\| < \varepsilon/4\}$ is a neighborhood of G [cf. (3.3)], there exists a $k \in \boldsymbol{N}$ such that

$$(4) \qquad \bar{U}_k \subseteqq \{x \in S \mid \|\nu * \pi * \bar{x} - \nu * \pi\| < \varepsilon/4\}$$

and

$$(5) \qquad \|m_1 - m|_{\bar{U}_k T_1}\| < \varepsilon/8 .$$

Furthermore, there exists a finite subset E of T_1 such that

$$(6) \qquad \|m|_{\bar{U}_k T_1} - m|_{\bar{U}_k E}\| < \varepsilon/8 .$$

Since $(ty_n)_{n \in N}$ is a ρ-Cauchy sequence for all $t \in E$, we can find an $l \in \boldsymbol{N}$ such that

$$U_k t y_n \cap U_k t y_m \neq \emptyset \quad (n, m \geqq l, t \in E) \quad [\text{cf. } (3.4.1)] .$$

Now, by (4), we have that

$$\|\nu * \pi * (\overline{xt}) * \bar{y}_n - \nu * \pi * (\overline{xt}) * \bar{y}_m\| < \varepsilon \quad (n, E > l, x \in \bar{U}_k, t \in E) .$$

Therefore,

$$\|\nu * \pi * m|_{\bar{U}_k} U * \bar{y}_n - \nu * \pi * E|_{\bar{U}_k E} * \bar{y}_m\| < \varepsilon \quad (n, m \geqq l) .$$

Now, an application of (3), (5), and (6) leads to

$$\|\pi * m_1 * \bar{y}_n - \pi * m_1 * \bar{y}_m\| < 3\varepsilon/2 \quad (n, m \geqq l) .$$

This shows that

$(\pi * m_1 * \bar{y}_n)_{n \in N}$ is a Cauchy sequence in $M(S)$.

(β) For each $n \in N$, let $P_n := \{x \in S \mid xy_m \notin GK \ (m \in N, m \geqq n)\}$. We shall show that

$$(7) \qquad M \backslash \bar{P} \subseteqq \bigcup_{n=1}^{\infty} P_n .$$

For this purpose, suppose these exists an $x \in M \backslash \bar{P}$ such that

$$x \notin P_n \quad \text{for all} \quad n \in N .$$

Then there is an open neighborhood W of x such that $W \cap P = \varnothing$. Since Wx^{-1} is a neighborhood of 1, (3.4.4) tells us that $G(Wx^{-1})x \cap T \neq \varnothing$. Therefore, there are $g \in G$, $v \in Wx^{-1}$ and $t \in T$ such that $g^{-1}t = vx \in W$. Hence $g^{-1}t \notin P$ or $t \notin gP = g(\bigcap_{m=1}^{\infty} U_m T_1) = \bigcap_{m=1}^{\infty} gU_m T_1 \subseteqq \bigcap_{m=2}^{\infty} U_{m-1} T_1 = P$. In particular, we have that $t \notin T_1$. Since $x \notin P_n$ $(n \in N)$, one can find a subsequence $(z_n)_{n \in N}$ of $(y_n)_{n \in N}$ such that $xz_n \in GK$ $(n \in N)$. This implies that $tz_n = gvxz_n \in gvGK$ $(n \in N)$, or in other words, $(tz_n)^{\approx} \in (gvGK)^{\approx}$ $(n \in N)$. However, this violates the fact that $t \notin T_1$; because $(gvGK)^{\approx}$ is a compact subset of the metric space Σ. Apparently, (7) holds.

Now let $\varepsilon > 0$. Recall that $m(\bar{P}) = m(P)$ [cf. (3.4.3)]. Hence, by (7), we can find a $k \in N$ such that

$$\|m_2 - m_2|_{P_k}\| < \varepsilon .$$

Since $gxy_n \notin K$ for all $g \in G$, $n \geqq k$, $x \in P_k$, we also have that

$$\pi * m_2|_{P_k} * \bar{y}_n(K) = 0 \quad \text{for all} \quad n \geqq k .$$

A combination of this property with the previous one leads to

$$\begin{aligned} &\|(\pi * m_2 * \bar{y}_n)|_K\| \\ &\quad \leqq \|(\pi * m_2 * \bar{y}_n)|_K - (\pi * m_2|_{P_k} * \bar{y}_n)|_K\| + (\pi * m_2|_{P_k} * \bar{y}_n)(K) \leqq \varepsilon \quad (n \geqq k) . \end{aligned}$$

This completes the proof of (β) and hence the proof of the theorem for foundation stips.

(B) Finally, let S be a stip.

Note that, to prove the theorem in this case, it is sufficient to adjust the above proof of (α). One can do this by using techniques as developed in [7], (3.4)-(3.8). Since the difficulties, by using these methods, are even more technical than the ones in the already presented proofs, we shall restrict ourselves to an indication of this proof.

For each $x \in S$, let $\cdot_x$ be the map from $Sx \times xS$ into S, defined by

$$px \cdot_x xq := \cdot_x(px, xq) := pxq \quad (p, q \in S) .$$

These maps $\cdot_x$ $(x \in S)$ induce "convolution products" $*_x$ between measures $\nu \in L(S)$ with $\nu \in M(Sx)$ [i.e., $|v|(S\backslash A) = 0$ for a certain σ-compact subset A of Sx] and $\mu \in L(S)$ with $\mu \in M(xS)$ by

$$\nu *_x \mu(f) := \iint f(y \cdot_x z) d\nu(y) d\mu(z) \quad (f \in C_\infty(S))\ ;$$

note that the fact that S is a stip implies that $|\nu|(S\backslash Ax) = |\mu|(S\backslash xA) = 0$ for a certain σ-compact subset $A \subseteqq S$ [see also [7], (3.4)]. As in (3.3.2) one has for the above ν and μ that $\nu *_x \mu \in L(S)$ and also that $\nu *_x (\mu * \rho) = (\nu *_x \mu) * \rho$ $(\rho \in M(S))$. Furthermore, one has that for each $x \in d(S) := \operatorname{clo} \cup \{\operatorname{supp}(\mu) \mid \mu \in L(S)\}$, $\varepsilon > 0$ and $\mu \in L(S)$ there exists a $\nu \in L(S)^+$ with $||\nu|| = \nu(Sx) = 1$ such that

$$||\mu|_{xS} - \nu *_x \mu|_{xS}|| = ||\bar{x} *_x \mu|_{xS} - \nu *_x \mu|_{xS}|| < \varepsilon\ .$$

Using these results and an adjusted group G, one obtain, by changing the proof of (α) in the obvious ways, that

$$((\pi * m_1)|_{xS} * \bar{y}_n)_{n \in N} \text{ is a Cauchy sequence in } M(S) \quad (x \in d(S))\ .$$

As is proved implicitly in §3 of [7], there exists a sequence $(x_n)_{n \in N}$ in $d(S)$ such that

$$\pi * m_1 \ll \sum_{n=1}^{\infty} 2^{-n} (\pi * m_1)|_{x_n S}\ .$$

Using (3.5) of [6], this leads to the relatively weak compactness of $\{\pi * m_1 * \bar{y}_n \mid n \in N\}$.

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