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QUASISIMILARITY OF NESTS

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The purpose of this article is to provide a complete classification of nests modulo quasisimilarity. Sample result: if every initial segment of the nest $\mathscr{M}$ is uncountable, then $\mathscr{M}$ is quasisimilar to a continuous nest.

1. Introduction. A chain $\mathscr{M}$ of subspaces is a family of subspaces of a complex, separable, infinite dimensional Hilbert space $\mathscr{H}$, totally ordered by inclusion, and containing $\{0\}$ and $\mathscr{H}$. A *nest* is a chain $\mathscr{M}$ of subspaces which is closed under intersection and closed span.

Two chains of subspaces, $\mathscr{M}$ and $\mathscr{N}$ are *similar* (*unitarily equivalent*, resp.) if there exists an invertible (unitary, resp.) W in $\mathscr{B}(\mathscr{H})$ (:= the algebra of all bounded linear operators acting on $\mathscr{H}$) such that $\mathscr{M} = \{WN\colon N \in \mathscr{N}\}$ (and therefore $\mathscr{M} = \{W^{-1}M\colon M \in \mathscr{M}\}$). It is completely apparent that similarity and unitary equivalence are equivalence relations for chains of subspaces. Furthermore, a chain similar to a nest is also a nest; that is, it is complete. We shall write $\mathscr{M} \sim \mathscr{N}$ ($\mathscr{M} \simeq \mathscr{N}$, resp.) to indicate that $\mathscr{M}$ and $\mathscr{N}$ are similar (unitarily equivalent, resp.).

The classification of nests up to similarity is simply stated. It is necessary and sufficient that there be an order isomorphism of $\mathscr{N}$ onto $\mathscr{M}$ which preserves the dimension of intervals $N_1 \ominus N_2$ for $N_2 < N_1$ in $\mathscr{N}$ **[1]** (see also **[6]**). For the finer relation of unitary equivalence, more invariants are required. Given any nest $\mathscr{N}$, one can choose an order isomorphism γ onto a compact subset Γ of the real interval $[0, 1]$ with $\gamma(\{0\}) = 0$ and $\gamma(\mathscr{H}) = 1$. Moreover, there is a spectral measure $E(\cdot)$ supported on Γ such that $E([0, t]) = \gamma^{-1}(t)$ for all t in Γ. (Here we identify a projection with its range.) As in the case of Hermitian operators, the spectral measure is determined by a scalar measure and a multiplicity function which are unitary invariants for $\mathscr{N}$ **[4]**.

If an element N in $\mathscr{N}$ has an immediate predecessor N_- in $\mathscr{N}$, then $N \ominus N_-$ is an *atom* of $\mathscr{N}$. In this case $\gamma(N_-) = s_0 < t_0 = \gamma(N)$, and Γ is disjoint from (s_0, t_0). Furthermore, $E(\{t_0\}) = P(N \ominus N_-)$. It is not difficult to see that an order isomorphism between two nests preserves dimension if and only if it preserves the dimension of atoms. The reader is referred to [2] for additional information about nests.

Recall that $X \in \mathscr{B}(\mathscr{H})$ is a *quasiaffinity* if X is injective and has dense range. Given a chain of subspaces $\mathscr{N}$ and $X \in \mathscr{B}(\mathscr{H})$, we define

$$X\mathscr{N} = \{(XN)^- : N \in \mathscr{N}\}.$$

1.1. DEFINITION. Two chains of subspaces $\mathscr{M}$ and $\mathscr{N}$ are *quasisimilar* if there exist quasiaffinities X, Y in $\mathscr{B}(\mathscr{H})$ such that

$$\mathscr{M} = X\mathscr{N} \quad \text{and} \quad \mathscr{N} = Y\mathscr{M}$$

(in symbols, $\mathscr{M} \sim_{\text{qs}} \mathscr{N}$).

It is straightforward to check that $\sim_{\text{qs}}$ is an equivalence relation for chains of subspaces. Unfortunately (as we shall see later, in §5), this relation does not preserve completeness. Nevertheless, it is possible to provide a very simple classification for nests, modulo quasisimilarity. Note that we do not require the induced order homomorphisms from $\mathscr{N}$ onto $\mathscr{M}$ and vice versa to be reciprocal. Indeed, they need not be even order isomorphic.

In §2, we obtain a classification of nests up to quasisimilarity. In §3, we try to make this more precise for countable nests by giving examples and theorems in special cases. In §4, we show that a nest is quasisimilar to the Volterra nest precisely when every initial segment is uncountable. The general uncountable case is briefly considered. In the last section, we clarify which chains are quasisimilar to nests. Finally, we give further evidence that our definition of quasisimilarity for nests is the correct one by disposing of the only other likely candidate.

2. The basic construction. The first lemma is routine, and is left to the reader.

2.1. LEMMA. *Let $\mathscr{M}$ and $\mathscr{N}$ be chains of subspaces and let X be a quasiaffinity such that $\mathscr{N} = X\mathscr{M}$. Then* $\operatorname{card}(\mathscr{N}) \leq \operatorname{card}(\mathscr{M})$ *and for each pair of subspaces M_1, M_2 in $\mathscr{M}$ with $M_1 \subset M_2$,*

$$\dim(\overline{XM_2} \ominus \overline{XM_1}) \leq \dim M_2 \ominus M_1.$$

If $\dim M_1 < \infty$, *then this is an equality. Furthermore, if* $\{M_k, k \geq 1\}$ *and* $M = \bigvee_{k\geq 1} M_k$ *belong to $\mathscr{M}$, then*

$$\overline{XM} = \bigvee_{k\geq 1} \overline{XM_k}.$$

Note that the quasiaffinity X determines an order preserving epimorphism θ_X of $\mathscr{M}$ onto $\mathscr{N}$. The lemma shows that θ_X is a dimension reducing and left continuous map which preserves the dimension of intervals $M_2 \ominus M_1$ when M_1 is finite dimensional. The converse is the key to our theorems.

2.2. LEMMA. *Let $\mathscr{N}$ be a nest. Then there is a compact quasiaffinity K in $\mathscr{T}(\mathscr{N})$ such that $\overline{KN} = N$ for every N in $\mathscr{N}$. (Here $\mathscr{T}(\mathscr{N}) = \{T \in \mathscr{B}(\mathscr{H})\colon TM \subset M$ for all M in $\mathscr{N}\}$ is the nest algebra associated with $\mathscr{N}$.)*

Proof. Let $\{A_n, n \geq 1\}$ be the atoms of $\mathscr{N}$. Choose compact quasiaffinities K_n in $\mathscr{B}(A_n)$ with $\|K_n\| \leq n^{-1}$. Let P be the projection onto $(\sum_{n\geq 1} \bigoplus A_n)^{\perp}$. Then $P\mathscr{N}|P\mathscr{H}$ is a continuous nest. This is similar to the Volterra nest [**6**]. Use this similarity to produce a compact operator K_0 in $\mathscr{T}(P\mathscr{N})$ similar to the Volterra operator V. Since

$$\overline{VM_t} = M_t = \{f \in L^2(0,1)\colon \operatorname{supp}(f) \subseteq [0,t]\}$$

for all $0 \leq t \leq 1$, it follows that K_0 has the same property in $\mathscr{T}(P\mathscr{N})$. Now

$$K = \sum_{n\geq 0} \bigoplus K_n$$

has the desired properties. □

2.3. THEOREM. *Let $\mathscr{M}$ and $\mathscr{N}$ be nests, and let θ be a dimension reducing, left continuous order epimorphism of $\mathscr{M}$ onto $\mathscr{N}$ such that*

$$\dim(\theta(M_2) \ominus \theta(M_1)) = \dim(M_2 \ominus M_1)$$

whenever $\dim M_1 < \infty$ and M_1, M_2 belong to $\mathscr{M}$. Then there is a compact quasiaffinity X such that $\theta = \theta_X$.

Proof. Since θ is monotone, θ is strictly increasing except for countably many proper disjoint intervals $\theta^{-1}(N_j) = \{M_j', M_j]$ which are closed or half closed. Add to this list the atoms $\theta^{-1}(N_j) = \{M_j\} = (M_j', M_j]$ such that $\dim(N_j \ominus N_j^-) < \dim(M_j - M_j^-)$. In both these circumstances, the hypotheses imply that $\dim(N_j) = \infty$. Let $P_j = P(M_j - M_j')$ and $Q_j = P(N_j \ominus \theta(M_j'))$. Let $P = \sum_j P_j$ and $Q = \sum_j Q_j$. Let $\mathscr{M}_0$ be the restriction to $\mathscr{M}$ to $P^{\perp}\mathscr{H}$, let $\mathscr{N}_0$ be the restriction of $\mathscr{N}$ to $Q_{\mathscr{H}}^{\perp}$, and let $\theta_0(M_0) = Q^{\perp}\theta(M_0)$ for M_0 in $\mathscr{M}_0$. Then θ_0 is a dimension preserving order isomorphism of $\mathscr{M}_0$ onto $\mathscr{N}_0$.

By the Similarity Theorem **[1]**, there is an invertible operator S in $\mathscr{B}(P^\perp\mathscr{H}, Q^\perp\mathscr{H})$ implementing θ_0. For each j and $M_j' < M \leq M_j$,

$$\operatorname{rank} Q_j = \dim \theta(M) \ominus \theta(M_j') \leq \dim M \ominus M_j'.$$

Thus it is possible to choose a projection $R_j \leq P_j$ with $\operatorname{rank} R_j = \operatorname{rank} Q_j$ such that $(R_j M)^- = (\operatorname{Ran} R_j)^-$ for all $M > M_j'$. Let $U_j = U_j R_j$ be a partial isometry carrying $R_j\mathscr{H}$ onto $Q_j\mathscr{H}$. Then the operator $Y = SP^\perp + \sum_j U_j$ is easily seen to have the property that $YM = \theta(M)$ for all M in $\mathscr{M}$. However, Y is generally not injective.

Let K be an injective compact operator in $\mathscr{T}(\mathscr{N})$, provided by Lemma 2.2, so that $\overline{KN} = N$ for every N in $\mathscr{N}$. Since N_j is always infinite dimensional, it is routine to obtain pairwise orthogonal infinite rank projections $E_j \leq P(N_j)$. Let K_j be compact injective operators in $\mathscr{B}(P_j\mathscr{H}, E_j\mathscr{H})$ so that $\operatorname{Ran}(\sum_j E_j K_j P_j) \cap \operatorname{Ran}(K) = \{0\}$ and $\|K_j\| < 2^{-j}$. Define

$$X = KY + \sum_j E_j K_j P_j.$$

It is easy to verify that X is injective and

$$\overline{XM} = \overline{YM} = \theta(M) \quad \text{for all } M \text{ in } \mathscr{M}.$$

Thus X is the desired operator. □

This theorem yields, as an immediate corollary, a necessary and sufficient condition for two nests to be quasisimilar. For any nest $\mathscr{N}$, let

$$N_\infty = \bigwedge\{N \in \mathscr{N} \colon \dim N = \infty\}.$$

2.4. THEOREM. *Two nests $\mathscr{M}$ and $\mathscr{N}$ are quasisimilar if and only if there are dimension decreasing, left continuous order epimorphisms $\varphi\colon \mathscr{M} \to \mathscr{N}$ and $\psi\colon \mathscr{N} \to \mathscr{M}$ such that*

$$\varphi|\{M \in \mathscr{M} \colon M \leq M_\infty\} \quad \text{and} \quad \psi|\{N \in \mathscr{N} \colon N \leq N_\infty\}$$

and reciprocal maps.

Proof. From Lemma 2.1, it follows that a quasisimilarity between $\mathscr{M}$ and $\mathscr{N}$ yields the desired epimorphisms φ and ψ which in particular effect a dimension preserving order isomorphism between $\{M \in \mathscr{M} \colon M \leq M_\infty\}$ and $\{N \in \mathscr{N} \colon N \leq N_\infty\}$. Conversely, the fact that $\varphi|\{M \leq M_\infty\}$ and $\psi|\{N \leq N_\infty\}$ are reciprocal and dimension reducing means that they are dimension preserving isomorphisms. Now the quasiaffinities are produced by Theorem 2.3, showing that $\mathscr{M}$ and $\mathscr{N}$ are quasisimilar. □

3. The countable case. It is immediate from Lemma 2.1 that quasisimilar nests have the same cardinality. In particular, one is countable if and only if the other is. Although Theorem 2.4 is, in a certain sense, a complete classification up to quasisimilarity, it is not always easy to recognize when the conditions are met. The following results and examples illustrate some of the phenomena involved.

3.1. COROLLARY. *If $\mathscr{M}$ is a finite nest and $\mathscr{M} \sim_{\mathrm{qs}} \mathscr{N}$, then $\mathscr{M} \simeq \mathscr{N}$.*

3.2. COROLLARY. *Suppose that $\mathscr{M}$ is order isomorphic to $\omega + 1$ and all its atoms are finite dimensional, that is*

$$\mathscr{M} = \{M_0 = \{0\} \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots \subset \mathscr{H}\},$$

where $\{\dim M_j\}_{j=0}^{\infty}$ is a strictly increasing sequence in $\mathbf{N} \cup \{0\}$. If $\mathscr{M} \sim_{\mathrm{qs}} \mathscr{N}$, then $\mathscr{M} \simeq \mathscr{N}$.

However, the presence of a single infinite dimensional atom can produce a quite different picture. Indeed, we have

3.3. EXAMPLE. Let

$$\mathscr{M} = \{M_0 = \{0\} \subset M_1 \subset M_2 \subset M_3 \subset \cdots \subset M_n \subset \cdots \subset \mathscr{H}\}$$

and

$$\mathscr{N} = \{N_0 = \{0\} \subset N_1 \subset N_2 \subset N_3 \subset \cdots \subset N_n \subset \cdots \subset \mathscr{H}\}$$

($\mathscr{M}, \mathscr{N} \simeq \omega + 1$), where M_1 and N_1 are infinite dimensional,

$$\dim\{M_{2j+1} \ominus M_{2j}) = 1$$

and

$$\dim(M_{2j} \ominus M_{2j-1}) = \dim(N_{j+1} \ominus N_j) = 2$$

for all $j = 1, 2, \ldots$; then $\varphi(\{0\}) = \psi(\{0\}) = \{0\}, \varphi(\mathscr{H}) = \psi(\mathscr{H}) = \mathscr{H}, \varphi(M_{2j+1}) = \varphi(M_{2j}) = N_j, \psi(N_j) = M_j$ for $j = 2, 3, \ldots$, and $\varphi(M_1) = N_1, \psi(N_1) = M_1$ define epimorphisms satisfying the hypotheses of Theorem 2.4.

Therefore, $\mathscr{M} \sim_{\mathrm{qs}} \mathscr{N}$. However, $\mathscr{M}$ and $\mathscr{N}$ are not similar. □

3.4. EXAMPLE. Two quasisimilar countable nests need not be order isomorphic. In fact, they need not have initial intervals which are isomorphic. Let

$$\mathscr{M} = 1 + \cdots + \mathscr{Q}_2 + \mathscr{P}_2 + \mathscr{Q}_1 + \mathscr{P}_1 \quad \text{(ordinal sum)}$$

and

$$\mathscr{N} = 1 + \cdots + \mathscr{Q}_2 + \mathscr{P}_4 + \mathscr{P}_3 + \mathscr{Q}_1 + \mathscr{P}_2 + \mathscr{P}_1 \quad \text{(ordinal sum)},$$

where $\mathscr{P}_k$ have order type $\omega + 1$ with one dimensional atoms and $\mathscr{Q}_k$ have order type $1 + \omega^*$ with one dimensional atoms.

Define φ from $\mathscr{M}$ onto $\mathscr{N}$ by sending the atoms of $\mathscr{P}_k$ onto the corresponding atoms of $\mathscr{P}_k$ in $\mathscr{N}$, likewise sending $\mathscr{Q}_{2k}$ onto $\mathscr{Q}_k$, and annihilating $\mathscr{Q}_{2k-1}$ for $k \geq 1$. This is easily seen to produce an order surjection. To define an epimorphism ψ, of $\mathscr{N}$ onto $\mathscr{M}$, we proceed in a similar way. Map $\mathscr{Q}_k$ onto $\mathscr{Q}_k$ in the natural way; and map $\mathscr{P}_{2k}$ onto $\mathscr{P}_k$ and annihilate $\mathscr{P}_{2k-1}$ for $k \geq 1$. Again this is seen to be a dimension reducing order epimorphism. By Theorem 2.4, $\mathscr{M} \sim_{\text{qs}} \mathscr{N}$.

On the other hand, it is apparent that no initial segments of $\mathscr{M}$ and $\mathscr{N}$ are order isomorphic.

Now, let us consider nests order isomorphic to $1 + \omega^*$. Let $\mathscr{N}^{(p)}_{1+\omega^*}$ denote the nest order isomorphic to $1 + \omega^*$ with all atoms of rank p, $1 \leq p \leq \infty$; and let $\mathscr{N}^{(\omega)}_{1+\omega^*}$ denote the nest with atoms A_n satisfying rank $A_n = n$ for $n \geq 1$.

3.5. Theorem. *A nest $\mathscr{M}$ is quasisimilar to a nest of order type $1 + \omega^*$ only if $\mathscr{M} \cong 1 + \omega^*$. Furthermore, if A_n are the atoms of $\mathscr{M}$,*

(i) $\mathscr{M} \sim_{\text{qs}} \mathscr{N}^{(p)}_{1+\omega^*}$, $1 \leq p < \infty$ *if and only if*

$$\sup_n \dim A_n = \limsup_{n\to\infty} \dim A_n = p;$$

(ii) $\mathscr{M} \sim_{\text{qs}} \mathscr{N}^{(\omega)}_{1+\omega^*}$ *if and only if* $\dim A_n < \infty$ *for all n and*

$$\sup \dim A_n = \infty;$$

(iii) $\mathscr{M} \sim_{\text{qs}} \mathscr{N}^{(\infty)}_{1+\omega^*}$ *if and only if* $\dim A_n = \infty$ *infinitely often.*

In general, suppose that m_0 is the integer such that

$$\dim A_{m_0} > \sup_{n>m_0} \dim A_n = \limsup_{n\to\infty} \dim A_n = p.$$

Let $\mathscr{L}$ be the finite nest with atoms $A_{m_0}, A_{m_0-1}, \ldots, A_1$. Then $\mathscr{M} \sim_{\text{qs}} \mathscr{N}^{(p)}_{1+\omega^*} + \mathscr{L}$. Similarly, if A_{m_0} is infinite rank and A_n is finite rank for $n > m_0$, but $\sup \dim A_n = \infty$, *then $\mathscr{M} \sim_{\text{qs}} \mathscr{N}^{(\omega)}_{1+\omega^*} + \mathscr{L}$.*

Proof. The only homomorphic images of $1 + \omega^*$ are finite or isomorphic to $1 + \omega^*$. Furthermore, if

$$\mathscr{M} = \{M_0 = \{0\} \subset \cdots \subset M_n \subset \cdots \subset M_2 \subset M_1 = \mathscr{H}\},$$

then any quasiaffinity X such that $X\mathscr{M}$ is a nest must satisfy

$$\bigcap_{n\geq 1} \overline{XM}_n = \overline{XM}_0 = \{0\}.$$

Thus if $\mathscr{M} \sim_{\mathrm{qs}} \mathscr{N}$, then $\mathscr{N}$ is order isomorphic to $1+\omega^*$.

Suppose $\mathscr{M} \sim_{\mathrm{qs}} \mathscr{N}^{(p)}_{1+\omega^*}$. From Theorem 2.4, it is clear that the atoms $A_n = M_n \ominus M_{n+1}$ of $\mathscr{M}$ cannot exceed p in dimension, and $\dim A_n = p$ must occur infinitely often. Whence

$$\sup_n \dim A_n = \lim_{n\to\infty} \sup \dim A_n = p.$$

Conversely, suppose $\dim A_{n_k} = p$ for an infinite sequence $n_1 < n_2 < \cdots$. Define $\varphi\colon \mathscr{M} \to \mathscr{N}^{(p)}_{1+\omega^*}$ by $\varphi(\{0\}) = \{0\}$,

$$\varphi(M) = N_1 \quad \text{for all } M \text{ in } [M_{n_1}, \mathscr{H}]$$

and

$$\varphi(M) = N_k \quad \text{for all } M \text{ in } [M_{n_k}, M_{n_{k-1}}), \qquad k \geq 2.$$

Likewise, define $\psi\colon \mathscr{N}^{(p)}_{1+\omega^*} \to \mathscr{M}$ by $\psi(N_k) = M_k$ for $k \geq 0$. It is not difficult to verify the hypotheses of Theorem 2.4. Hence $\mathscr{M} \sim_{\mathrm{qs}} \mathscr{N}^{(p)}_{1+\omega^*}$.

The remaining cases are similar and are left to the reader. □

3.6. Remark. A similar analysis can be made of nests order isomorphic to $\omega+1$ which contain an infinite dimensional atom to permit absorption of atoms as in Example 3.3. For example, let $\mathscr{N}$ have order type $\omega+1$ and atoms of dimension p except for the first which is infinite dimensional. Then $\mathscr{M} \sim_{\mathrm{qs}} \mathscr{N}$ if and only if $\dim M_1 = \infty$ and

$$\sup_{n\geq 2} \dim A_n = \lim_{n\to\infty} \sup \dim A_n = p.$$

3.7. Example. If $\mathscr{M}$ and $\mathscr{N}$ are order-isomorphic to $1+\omega^*+\omega^*+\omega^*+\cdots+1$, and $M_{1+\omega^*}$ and $N_{1+\omega^*}$ are the subspaces of $\mathscr{M}$ and, respectively, of $\mathscr{N}$, corresponding to $1+\omega^*$ in this order, then the dimensions of the atoms of $\mathscr{N}|N_{1+\omega^*} := \{N \in \mathscr{N} : N \subset N_{1+\omega^*}\}$ (thought of as a nest on $N_{1+\omega^*}$) are related to the dimensions of the atoms of $\mathscr{M}|M_{1+\omega^*}$ according to the same rules as in Theorem 3.5.

But this is not necessarily true for the atoms in the segments corresponding to $(1+\omega^*, 1+\omega^*+\omega^*]$, etc.

For instance, if all the atoms of $\mathscr{M}$ have dimension one, and all the atoms of $\mathscr{N}$ have dimension one, except for the one corresponding to $1+\omega^*+\omega^*$, which has infinite dimension, then we still have $\mathscr{M} \sim_{\mathrm{qs}} \mathscr{N}$!

To see this, define

$$\varphi(\{0\}) = \psi(\{0\}) = \{0\}, \quad \varphi(\mathscr{H}) = \psi(\mathscr{H}) = \mathscr{H},$$

$$\varphi(M_{1+\omega^*k+(\omega^*-n)}) = \begin{cases} N_{1+(\omega^*-n)}, & \text{if } k=0,\ m=0,1,2,\ldots, \\ N_{1+\omega^*+(\omega^*-n-1)}, & \text{if } k=1,\ n=1,2,\ldots, \\ N_{1+\omega^*+\omega^*}, & \text{if } k=2,\ n=0,1,2,\ldots, \\ N_{1+\omega^*(k-1)+(\omega^*-n)}, & \text{if } k\geq 3,\ n=0,1,2,\ldots, \end{cases}$$

and

$$\psi(N_{1+\omega^*k+(\omega^*-n)}) = M_{1+\omega^*k+(\omega^*-n)} \quad \text{for } k,n=0,1,2,\ldots.$$

Now the result follows from Theorem 2.4. Indeed, $\mathscr{M} \sim_{\mathrm{qs}} \mathscr{N}$ provided all but finite atoms of each of the segments $(N_{1+\omega^*k}, N_{1+\omega^*(k+1)}]$ for $k \geq 1$, as well as all the atoms of the initial segment of $\mathscr{N}$ have dimension one.

4. Uncountable nests. In some ways, things become simpler for uncountable nests. The prototypical continuous nest is the Volterra nest $\mathscr{N} = \{N_t : 0 \leq t \leq 1\}$ where

$$N_t = \{f \in L^2(0,1) : \operatorname{supp}(f) \subseteq [0,t]\}.$$

The fundamental step in our analysis is the classification of nests quasisimilar to $\mathscr{N}$.

4.1. THEOREM. *A nest $\mathscr{M}$ is quasisimilar to the Volterra nest if and only if every initial segment of $\mathscr{M}$ is uncountable.*

Proof. First, we note that there is a quasiaffinity X such that $X\mathscr{N}$ is a nest $\mathscr{L}$ if and only if $\mathscr{L}$ contains no non-zero finite dimensional elements. Necessity follows from Lemma 2.1. Conversely, suppose $\mathscr{L}$ has order type Γ, a subset of $[0,1]$ containing $\{0,1\}$ such that either 0 is a limit point of $\Gamma\backslash\{0\}$ or $\inf_{\Gamma\backslash\{0\}} \gamma = \gamma_0$ corresponds to an infinite dimensional atom. Define a map $\varphi: [0,1] \to \Gamma$ by $\varphi(s) = \inf\{\gamma \in \Gamma : \gamma \geq s\}$ for $s \geq \gamma_0$; if $\gamma_0 > 0$, set $\varphi(s) = \gamma_0$ for s in $(0, \gamma_0]$. Then φ is a left continuous order epimorphism. Moreover, every nontrivial interval of $\mathscr{N}$ has infinite dimension, so φ is dimension reducing. By Theorem 2.3, $\mathscr{L}$ is a quasiaffinity of $\mathscr{N}$.

Now, suppose X is a quasiaffinity such that $X\mathscr{M} = \mathscr{N}$. For any nonzero M_0 in $\mathscr{M}, \overline{XM_0}$ is infinite dimensional and thus θ_X takes $\{M \in \mathscr{M} : M \leq M_0\}$ onto a nontrivial initial segment of $\mathscr{N}$. As the image is uncountable, the initial segment of $\mathscr{M}$ is necessarily uncountable as well.

Conversely, if every initial segment of $\mathscr{M}$ is uncountable, one can choose $M_1 = \mathscr{H} \supset M_2 \supset \cdots \supset M_n \ldots$ so that $\bigcap M_n = \{0\}$ and $[M_{k+1}, M_k]$ is uncountable for every $k \geq 1$. Every uncountable compact subset of $\mathbf{R}$ contains a Cantor set, from which we deduce that there exists an order epimorphism of the set onto an interval. Let φ be such a map which carries the interval $[M_{k+1}, M_k]$ onto $[(k+1)^{-1}, k^{-1}]$ for each $k \geq 1$, and takes $\{0\}$ to 0. It is clear that the image of an interval of $\mathscr{M}$ is nonzero only if the interval is uncountable, and thus is infinite dimensional. Therefore, φ is dimension reducing. By Theorem 2.3, φ is implemented by a quasiaffinity. Hence $\mathscr{M} \sim_{\mathrm{qs}} \mathscr{N}$.□

This result and its proof yield the following generalization.

4.2. Corollary. *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be quasisimilar nests on an infinite dimensional space. Let $\mathscr{M}$ be any uncountable nest, and let $\mathscr{N}$ be the Volterra nest. Then*

$$\mathscr{L}_1 + \mathscr{M} \sim_{\mathrm{qs}} \mathscr{L}_2 + \mathscr{N}.$$

Proof. Let $\varphi\colon \mathscr{L}_1 \to \mathscr{L}_2$ and $\psi\colon \mathscr{L}_2 \to \mathscr{L}_1$ be the dimension reducing epimorphisms given by Theorem 2.4. As in the previous proof, it is possible to extend ψ to a map Ψ of $\mathscr{L}_2 + \mathscr{N}$ onto $\mathscr{L}_1 + \mathscr{M}$ by taking any surjection of $\mathscr{N}$ onto $\mathscr{M}$. On the other hand, $\mathscr{M}$ contains an element M_0 so that every proper interval $[M_0, M)$ is uncountable. Thus, extend φ to Φ by setting $\Phi(M) = \mathscr{H} \oplus 0 = \varphi(1_{\mathscr{L}_1})$ for M in $[0, M_0]$ and mapping $[M_0, \mathscr{H}]$ onto $\mathscr{N}$ as in the previous theorem. Now Φ and Ψ satisfy the requirements of Theorem 2.4. So $\mathscr{L}_1 + \mathscr{M} \sim_{\mathrm{qs}} \mathscr{L}_2 + \mathscr{N}$. □

For example, if $\mathscr{M}$ and $\mathscr{N}$ are uncountable nests with initial segments $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n$ and $\{0\} = N_0 \subset N_1 \subset \cdots \subset N_n$ such that $\dim M_k = \dim N_k < \infty$ for $1 \leq k \leq n-1$ and $\dim M_n = \dim N_n = \infty$, then $\mathscr{M} \sim_{\mathrm{qs}} \mathscr{N}$.

4.3. Remark. It should be noted that for the proof of Corollary 4.2 to work, it is only necessary that there be quasiaffinities X and Y so that $X\mathscr{L}_1$ is an initial segment of $\mathscr{L}_2$, and $Y\mathscr{L}_2$ is an initial segment of $\mathscr{L}_1$. However, we have been unable to construct an example in which this additional generality is required.

The analysis that we have made of certain special cases leads us to

4.4. Conjecture. Let $\mathscr{L}$ and $\mathscr{M}$ be countable nests, and let $\mathscr{N}$ be the Volterra nest. If $\mathscr{L} + \mathscr{N}$ and $\mathscr{M} + \mathscr{N}$ are quasisimilar, then there exist quasisimilar initial segments $\mathscr{L}_1$ and $\mathscr{M}_1$ of $\mathscr{L}$ and $\mathscr{M}$ respectively acting on infinite dimensional spaces.

5. Quasisimilarity does not preserve completeness. First, let us look at two examples.

5.1. EXAMPLE. Let $\mathscr{N}$ be the nest $\{0\} = N_0 \subset \cdots \subset N_n \subset \cdots \subset N_1 = \mathscr{H}$ where $\dim N_k^\perp < \infty$ for $k \geq 1$, and $\bigcap_{k\geq 1} N_k = N_0 = \{0\}$. If X is any quasiaffinity, let $M_n = \overline{XN_n}$; and $\mathscr{M} = \{M_n,\ n \geq 0\}$. Then either $\mathscr{M}$ is finite (whence $\mathscr{M}$ is a nest but is not quasisimilar to $\mathscr{N}$) or $\mathscr{M}$ is infinite. Then $M_\omega = \bigcap_{n\geq 1} M_n$ belongs to the complete chain generated by $\mathscr{M}$. Indeed $\mathscr{M}$ is a nest if and only if $M = \{0\}$.

However, if $M_\omega \neq \{0\}$, $\mathscr{M}$ cannot be quasisimilar to $\mathscr{N}$. For if Y is any quasiaffinity,

$$\bigcap_{n\geq 1} \overline{YM_n} \supseteq \overline{Y\left(\bigcap_{n\geq 1} M_n\right)} = \overline{YM_\omega} \neq \{0\}.$$

Thus it is not possible that $Y(\mathscr{M}\backslash\{0\}) = \mathscr{N}\backslash\{0\}$ as this latter set has zero intersection.

Note that a similar argument holds if $\mathscr{N}$ has subspaces N_n with finite dimensional intersection. □

5.2. EXAMPLE. Let $\mathscr{N}_0$ be the nest on $\mathscr{H} \oplus \mathscr{H}$ given by $\{0\}, \mathscr{H} \oplus 0$, and $\mathscr{H} \oplus N_n$, where $N_n \in \mathscr{N}$ of Example 5.1. If $\mathscr{M}_0 = X\mathscr{N}_0$ is a quasiaffinity of $\mathscr{N}_0$, then $\overline{X(\mathscr{H} \oplus 0)}$ is infinite dimensional. If it is of finite codimension, then $\mathscr{M}_0$ is finite and is not quasisimilar to $\mathscr{N}_0$. So after a suitable unitary equivalence, we may assume that $\overline{X(\mathscr{H} \oplus 0)} = \mathscr{H} \oplus 0$. Following the reasoning of the previous example, $\mathscr{M}_0$ is finite or of the form $\{0\}$, $\mathscr{H} \oplus 0, \mathscr{H} \oplus M_n, n \geq 1$ where $M_1 = \mathscr{H}$. The completion of $\mathscr{M}_0$ contains $\mathscr{H} \oplus M_\omega$ where $M_\omega = \bigcap_{n\geq 1} M_n$.

For $\mathscr{M}_0$ to be quasisimilar to $\mathscr{N}_0$, it is necessary that the dimensions of $\dim N_n \ominus N_{n+1}$ and $\dim M_n \ominus M_{n+1}$ be related as in Theorem 3.5. However, it is not necessary (as it was in Example 5.1) that $M_\omega = \{0\}$. To illustrate this, let us suppose that

$$\dim N_n \ominus N_{n+1} = 1 = \dim M_n \ominus M_{n+1} \quad \text{for all } n \geq 1.$$

Think of $\mathscr{M}_0$ as the chain of subspaces of $\mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$ given by $\{0\}$, $\mathscr{H} \oplus 0 \oplus 0, \mathscr{H} \oplus \mathscr{H} \oplus N_n, n \geq 1$.

Let K be an injective compact operator on $\mathscr{H}$ such that $\overline{KN_n} = \mathscr{H}$ for all $n \geq 1$. For example, the Volterra operator on $L^2(0,1)$ has this property with respect to $N_n = \operatorname{span}\{1, x, \ldots, x^{n-2}\}^\perp$. Fix a basis e_n for $\mathscr{H}$, $n \geq 0$. Choose integers $1 = k_1 \leq k_i$ and vectors x_i in $N_{k_i} \ominus N_{k_{i+1}}$ such that

$$\lim_{i\to\infty} Kx_{(2i+1)2^n} = e_n.$$

Now let $d_i = (i\|x_i\| + 1)^{-1}$, and define an injective compact operator

$$D = \sum_{i\geq 1} d_i P(N_{k_i} \ominus N_{k_{i+1}}).$$

Now define $X\colon \mathscr{H} \oplus \mathscr{H} \to \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$ by

$$X = \begin{bmatrix} I & 0 \\ 0 & K \\ 0 & D \end{bmatrix}.$$

Then $\overline{X(\mathscr{H} \oplus 0)} = \mathscr{H} \oplus 0$, and $\overline{X(\mathscr{H} \oplus N_n)} = \mathscr{H} \oplus \mathscr{H} \oplus N_n$ for $n \geq 1$. To see this latter identity, note that

$$\lim_{k\to\infty} X \begin{bmatrix} 0 \\ x_{(2i+1)2^k} \end{bmatrix} = \lim_{k\to\infty} \begin{bmatrix} 0 \\ Kx_{(2i+1)2^k} \\ Dx_{(2i+1)2^k} \end{bmatrix} = \begin{bmatrix} 0 \\ e_k \\ 0 \end{bmatrix}.$$

Thus it is apparent that $\overline{X(\mathscr{H} \oplus N_n)}$ contains $\mathscr{H} \oplus \mathscr{H} \oplus 0$. Since $\overline{DN_n} = N_n$, it follows that $\overline{X(\mathscr{H} \oplus N_n)} = \mathscr{H} \oplus \mathscr{H} \oplus N_n$. So $X\mathscr{N}_0 = \mathscr{M}_0$.

To reverse the process, it suffices to "bury" $0 \oplus \mathscr{H} \oplus 0$ in $\mathscr{H} \oplus 0$. Let J be a quasiaffinity with $\operatorname{Ran} J \cap \operatorname{Ran} D = \{0\}$. Define

$$Y = \begin{bmatrix} J & D & 0 \\ 0 & 0 & D \end{bmatrix}.$$

Then $Y\mathscr{H} \oplus 0 \oplus 0 = \mathscr{H} \oplus 0$ and $Y\mathscr{H} \oplus \mathscr{H} \oplus 0 = \mathscr{H} \oplus 0$ also. It is now clear that $Y\mathscr{H} \oplus \mathscr{H} \oplus N_n = \mathscr{H} \oplus N_n$, whence $Y\mathscr{M}_0 = \mathscr{N}_0$. So $\mathscr{M}_0 \sim_{\text{qs}} \mathscr{N}_0$.$\square$

The lesson of these two examples is that as long as there are infinite dimensions available, a "gap" can be created and swallowed up again. A modification of this example yields a complete characterization of chains quasisimilar to nests.

5.3. THEOREM. *A chain $\mathscr{M}$ of subspaces of a separable Hilbert space is quasisimilar to a nest if and only if $\mathscr{M}$ is closed under spans, and for each finite dimensional element M in $\mathscr{M}$,*

$$M^+ = \bigwedge\{M' \in \mathscr{M} : M' > M\} \quad \textit{belongs to } \mathscr{M}.$$

Proof. If $\mathscr{M}$ is quasisimilar to a nest $\mathscr{N}$, then the epimorphisms $\varphi\colon \mathscr{M} \to \mathscr{N}$ and $\psi\colon \mathscr{N} \to \mathscr{M}$ are left continuous by Lemma 2.1. Thus $\mathscr{M}$ must be closed under spans. The argument of Example 5.1 shows that if M is finite dimensional, then $\mathscr{M}$ contains

$$\bigwedge\{M' \in \mathscr{M} : M' > M\}.$$

So we consider the converse.

Let $\mathscr{M}'$ be the completion of $\mathscr{M}$. Since $\mathscr{M}$ is closed under spans, $\mathscr{M}'\backslash\mathscr{M}$ is a countable set $\{M_k'\}$ and each M_k' has an immediate predecessor M_k in $\mathscr{M}$. Let $\mathscr{H}_0 = (\bigvee_k M_k' \ominus M_k)^\perp$, and let $\mathscr{N} = \mathscr{M}|\mathscr{H}_0$. Then $\mathscr{N}$ is a nest. The map $\varphi(M) = M \cap \mathscr{H}_0$ is an order isomorphism of $\mathscr{M}$ onto $\mathscr{N}$ which preserves dimension. It will be shown that $\mathscr{M} \sim_{\mathrm{qs}} \mathscr{N}$.

By hypothesis, each M_k has infinite dimension, so $N_k = \varphi(M_k)$ does also. Thus it is possible to choose pairwise orthogonal infinite dimensional subspaces E_k in $\mathscr{H}_0$ so that $E_k \leq N_k$. By Lemma 2.2, there is a compact injective operator K on $\mathscr{H}_0$ so that $\overline{KN} = N$ for every N in $\mathscr{N}$. Choose compact injective operators K_k from $M_k' \ominus M_k$ into E_k such that $\sum \bigoplus K_k$ is a compact operator with range disjoint from $\operatorname{Ran}(K)$. Define

$$X = KP(\mathscr{H}_0) + \sum_k \bigoplus K_k P(M_k' \ominus M_k).$$

It is routine to verify that X is a quasiaffinity such that $X\mathscr{M} = \mathscr{N}$.

To reverse the process, note that each N_k is also infinite dimensional as $\mathscr{N}$ contains a sequence $N_{k,n} > N_k$ such that $\bigwedge_{n\geq 1} N_{k,n} = N_k$. One can choose pairwise orthogonal projections R_k in $\mathscr{N}'$ so that $R_k P(N_k) = 0$ but $R_k P(N) \neq 0$ for all $N > N_k$. (The details are left to the reader.) Now (by dropping to a subsequence) choose the sequence $N_{k,n}$ described above so that $R_k(N_{k,n} \ominus N_{k,n+1})$ contains a unit vector $x_{k,n}$. Let $F_k = \operatorname{span}\{x_{k,n}; n \geq 1\}$. The spaces F_k are pairwise orthogonal, and $N \cap F_k$ has finite codimension in F_k for every $N > N_k$ in $\mathscr{N}$. As in Example 5.2, we can choose a norm one compact injective operator C_k in $\mathscr{B}(F_k, M_k' \ominus M_i)$ such that $\overline{C_k(N \cap F_k)} = M_k' \ominus M_k$ for every $N > N_k$.

Further mimicry of Example 5.2 suggests that we choose vectors $y_{k,j}$ in $R_k(N_{k,n_j} \ominus N_{k,n_{j+1}})$ so that

$$\lim_{j\to\infty} C_k y_{k,(2j+1)2^i} = e_{k,i}$$

where $\{e_{k,i}: i > 1\}$ is an orthonormal basis of $M_k' \ominus M_k$. Define positive constants $d_{k,j} = (j\|y_{k,j}\| + 1)^{-1}$, and the positive injective operator

$$D = \left(\sum_{k\geq 1} R_k P(N_{k,1})\right)^\perp + \sum_{k\geq 1}\sum_{j\geq 1} d_{k,j} R_k P(N_{k,n_j} \ominus N_{k,n_{j+1}}).$$

As in Example 5.2, we see that

$$\lim_{j\to\infty} \begin{bmatrix} C_k y_{k,(2j+1)2^i} \\ D y_{k,(2j+1)2^i} \end{bmatrix} = \begin{bmatrix} e_{k,i} \\ 0 \end{bmatrix}.$$

Now define an operator Y in $\mathscr{B}(\mathscr{H}_0, \mathscr{H})$ by $Y = D + \sum_{k\geq 1} C_k R_k$. This is clearly injective. Moreover, the previous computation shows that whenever $N > N_k$, $\overline{YN}$ contains $M_k' \ominus M_k$. On the other hand, since $\overline{DN} = N$ for all N in $\mathscr{N}$, it follows that $\overline{YN} = \varphi^{-1}(N)$ for all N in $\mathscr{N}$. Hence $\mathscr{M} \sim_{\text{qs}} \mathscr{N}$. We also note that Y can be replaced by YK, where K is obtained by Lemma 2.2. So Y can be taken to be compact if desired. □

Since $\sim_{\text{qs}}$ does not preserve completeness, perhaps we should consider the possibility of replacing this relation by the following relation for *nests*:

> $\mathscr{M} \equiv \mathscr{N}$ if there exist quasiaffinities X, Y such that $\mathscr{N} =$ strong closure $(X\mathscr{M})$ and $\mathscr{M} =$ strong closure $(Y\mathscr{M})$.

Clearly, $\equiv$ is reflexive and symmetric, but unfortunately is not transitive (and therefore $\equiv$ *is not an equivalence relation*).

5.4. Example. Consider the following nests

$$\mathscr{N} = \{\{0\} \subset l^2 \subset \cdots \subset l^2 \oplus M_2 \subset l^2 \oplus M_1 \subset l^2 \oplus l^2\}$$

on $l^2 \oplus l^2$, where $M_n = \{e_k\}_{k>n}$, $\{e_k\}_{k=1}^{\infty}$ an orthonormal basis of l^2,

$$\mathscr{M} = \{\{0\} \subset l^2 \subset l^2 \oplus \mathbf{C} \subset \cdots \subset l^2 \oplus \mathbf{C} \oplus M_2 \subset l^2 \oplus \mathbf{C} \oplus M_1 \subset l^2 \oplus \mathbf{C} \oplus l^2\}$$

on $l^2 \oplus \mathbf{C} \oplus l^2$, and

$$\mathscr{L} = \Big\{\{0\} \subset l^2 \subset l^2 \oplus \mathbf{C} \subset l^2 \oplus \mathbf{C}^2 \subset \cdots \\ \subset l^2 \oplus \mathbf{C}^2 \oplus M_2 \subset l^2 \oplus \mathbf{C}^2 \oplus M_1 \subset l^2 \oplus \mathbf{C}^2 \oplus l^2\Big\}$$

on $l^2 \oplus \mathbf{C}^2 \oplus l^2$.

Let $De_k = e_k/k$ $(k = 1, 2, \dots)$ and pick any $u \notin \operatorname{Ran} D$. Define

$$X(x \oplus y) = x \oplus (y, u) \oplus D^2 y,$$
$$Y(x \oplus \lambda \oplus y) = (Dx + \lambda u) \oplus y.$$

It is straightforward to check that X and Y are quasiaffinities, $Y\mathscr{M} = \mathscr{N}$, and $X\mathscr{N} \neq \mathscr{M}$, but

$$\text{strong closure } (X\mathscr{N}) = \mathscr{M}.$$

Hence, $\mathscr{M} \equiv \mathscr{N}$. Define

$$W(x \oplus \lambda \oplus y) = x \oplus \lambda \oplus (y, u) \oplus D^2 y,$$
$$Z(x \oplus \lambda \oplus \mu \oplus y) = (Dx + \mu u) \oplus \lambda \oplus y;$$

then W and Z are quasiaffinities, $Z\mathscr{L} = \mathscr{M}$, $W\mathscr{M} \neq \mathscr{L}$,

$$\text{strong closure } (W\mathscr{M}) = \mathscr{L},$$

and $YZ\mathscr{L} = \mathscr{N}$, but

$$\begin{aligned} &\text{strong closure } (WX\mathscr{N}) \\ &\quad = \{\{0\} \subset l^2 \subset l^2 \oplus \mathbf{C}^2 \subset \cdots \subset l^2 \oplus \mathbf{C}^2 \oplus M_2 \\ &\qquad\qquad \subset l^2 \oplus \mathbf{C}^2 \oplus M_1 \subset l^2 \oplus \mathbf{C}^2 \oplus l^2\} \subsetneq \mathscr{L}. \end{aligned}$$

This indicates that the obvious quasiaffinities, YZ and WX, do not implement the relation $\mathscr{N} \equiv \mathscr{L}$.

Indeed, $\mathscr{N}$ and $\mathscr{L}$ are not related at all! For, if $R\colon l^2 \oplus l^2 \to l^2 \oplus \mathbf{C}^2 \oplus l^2$ is any quasiaffinity such that

$$\begin{aligned} &\text{strong closure } (R\mathscr{N}) \\ &\quad \supset \{\{0\} \subset l^2 \subset \cdots \subset l^2 \oplus \mathbf{C}^2 \oplus M_2 \\ &\qquad \subset l^2 \oplus \mathbf{C}^2 \oplus M_1 \subset l^2 \oplus \mathbf{C}^2 \oplus l^2\}, \end{aligned}$$

then

$$(Rl^2)^- = l^2 \quad \text{and} \quad [R(l^2 \oplus M_n)]^- = l^2 \oplus \mathbf{C}^2 \oplus M_{h(n)}$$

for a suitable non-decreasing function $h(n)$ of n such that $h(n+1) - h(n) \leq 1$ and $h(n) \to \infty$ $(n \to \infty)$.

Therefore,

$$\begin{aligned} \text{strong closure } (R\mathscr{N}) &= (R\mathscr{N}) \cup \{\text{“one point”}\} \\ &= \{R\mathscr{N}\} \cup \left[\bigcap \{l^2 \oplus \mathbf{C}^2 \oplus M_{h(n)}\}_{n=1}^{\infty}\right] \\ &= \{R\mathscr{N}\} \cup \left[\bigcap \{l^2 \oplus \mathbf{C}^2 \oplus M_n\}_{n=1}^{\infty}\right] \\ &= \{R\mathscr{N}\} \cup \{l^2 \oplus \mathbf{C}^2\} \end{aligned}$$

cannot contain $l^2 \oplus \mathbf{C}$.

We conclude that $\equiv$ is not an equivalence relation.

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