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LOCALLY A-PROJECTIVE ABELIAN GROUPS AND GENERALIZATIONS

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Let A be an abelian group. An abelian group G is locally A-projective if every finite subset of G is contained in a direct summand P of G which is isomorphic to a direct summand of $\oplus_I A$ for some index-set I. Locally A-projective groups are discussed without the usual assumption that the endomorphism ring of A is hereditary, a setting in which virtually nothing is known about these groups. The results of this paper generalize structure theorems for homogeneous separable torsion-free groups and locally free modules over principal ideal domains. Furthermore, it is shown that the conditions on A imposed in this paper cannot be relaxed, in general.

1. Introduction and discussion of results. In 1967, Osofsky investigated the projective dimension of torsion-free modules over a valuation domain R. One of her main results in [**O1**] is that a torsion-free R-module M which is generated by $\aleph_n$ many elements has projective dimension at most $n + 1$. [**F2**, Proposition 3.2] emphasizes that it is necessary to assume in this result that R is a valuation domain. However, one of the initial results of this paper shows that these conditions on R are by far too strong (Proposition 2.2).

For this, it is necessary to extend the concept of torsion-freeness of modules over integral domains to modules over arbitrary rings. The obvious way to do this is to call an R-module M *torsion-free* if $mc \neq 0$ for all non-zero $m \in M$ and non-zero-divisors c of R. However, the following approach used in [**G**] proved more successful: An R-module is *non-singular* if $mI \neq 0$ for all $0 \neq m \in M$ and all essential right ideals I of R. The ring R itself is right non-singular if it is non-singular as a right R-module.

A right non-singular ring R is *strongly non-singular*, if the finitely generated non-singular R-modules are exactly the finitely generated submodules of free modules. For instance, every semi-prime ring of finite left and right Goldie-dimension is strongly non-singular [**G**, Theorems 3.10 and 5.17]. Furthermore, these finite dimensional rings are exactly those for which the concepts of torsion-freeness and non-singularity coincide for right and left modules. Moreover, every valuation domain is strongly non-singular and semi-hereditary.

We extend Osofsky's result in Proposition 2.2 in a surprisingly simple and natural way to non-singular modules over strongly non-singular, right semi-hereditary rings.

This natural extension of Osofsky's result indicates that similar results may be available for other classes of modules. It is the main goal of Section 2 to give estimates for the projective dimension of modules M over an arbitrary ring R such that every finite subset of M is contained in a projective direct summand of M (Theorem 2.3). Chase called such a module *locally projective* in **[C]**, where he discussed locally projective modules in the case that R is a principal ideal domain. The author was able to extend Chase's work in **[A3]** to modules over semi-prime, two-sided Noetherian, hereditary rings. However, virtually nothing is known about this class of modules in the case that R is not hereditary.

In the remaining part of this paper, we apply the previously obtained module-theoretic results to the discussion of abelian groups. Before we can start, we have to introduce some further notation: Let A and G be abelian groups. The *A-socle of G*, denoted by $S_A(G)$, is the fully invariant subgroup of G which is generated by $\{\phi(A) | \phi \in \mathrm{Hom}(A, G)\}$. Clearly, $S_A(G)$ is the image of the natural evaluation map $\theta_C\colon \mathrm{Hom}(A, G) \otimes_{E(A)} A \to G$. The group G is *A-solvable* if θ_G is an isomorphism. It is *A-projective* if it is isomorphic to a direct summand of $\bigoplus_I A$ for some index-set I. The smallest cardinality possible for I is the A-rank of G. Finally, Arnold and Murley called an abelian group A *self-small*, if the functor $\mathrm{Hom}(A, -)$ preserves direct sums of copies of A, and showed that A-projective groups are A-solvable in this case. In particular, A is self-small if there is a finite subset X of A such that $\phi(X) \neq 0$ for all $0 \neq \phi \in E(A)$, i.e. $E(A)$ is *discrete in the finite topology*. **[A7**, Theorem 2.8] shows that, for every cotorsion-free ring R, there exists a proper class of abelian groups A which are discrete in the finite topology and flat as $E(A)$-modules such that $E(A) \cong R$.

It is easy to see that an abelian group G is an epimorphic image of an A-projective iff $S_A(G) = G$. Although every abelian group G with $S_A(G) = G$ admits an exact sequence $0 \to U \to \bigoplus_I A \to G \to 0$ with respect to which A is projective, any two A-projective resolutions of such a G can be quite different (Example 3.3). This is primarily due to the fact that there is no general version of Shanuel's Lemma for A-projective resolutions. In particular, there exist abelian groups A and G with $S_A(G) = G$ which admit an A-projective resolution $0 \to U \to \bigoplus_I A \to G \to 0$ in which U is not an epimorphic image of an A-projective group. It is the goal of Section 3 to characterize the abelian groups which are well-behaved with respect to A-projective resolution in the sense that $S_A(U) = U$ in every exact sequence $0 \to U \to \bigoplus_I A \to G \to 0$. In view of [A7, Theorem 2.8], we address this problem in the case that A is self-small and flat as an $E(A)$-module, and show that an abelian group G has the previously stated property exactly if it is A-solvable (Proposition 3.2). This result allows us to extend the definition of projective dimension to A-solvable groups G.

Ulmer first introduced the class $\mathcal{T}_A$ of A-solvable groups in [U] as a tool to investigate abelian groups which are flat as modules over their endomorphism ring. Another application of A-solvable groups was obtained in [A5] and [A7] where the consideration of $\mathcal{T}_A$ yielded partial answers to [F, Problem 84a and c]. The same papers also showed that the restriction that A is flat as an $E(A)$-module is essential in the discussion of A-solvable abelian groups [A7, Theorem 2.2]. Moreover, Hausen used methods similar to the ones used in [A5] and [A7] and some of the results of [A3] to give a partial answer to [F, Problem 9] in [H].

In Section 4, we turn our attention to a class of abelian groups which was first introduced by Arnold and Murley in [AM]: An abelian group G is *locally A-projective* if every finite subset of G is contained in an A-projective direct summand of G. [AM, Theorem III] yields that the categories of locally A-projective abelian groups and locally projective right $E(A)$-modules are equivalent if $E(A)$ is discrete in the finite topology. Since we frequently use the same category equivalence, we assume that A is discrete in the finite topology.

The module-theoretic results of this paper enable us to investigate the structure of locally A-projective groups in the case that $E(A)$ is not hereditary. Our first result shows that a locally A-projective group G has A-projective dimension at most n if there exists an exact sequence $\bigoplus_{\aleph_n} A \to G \to 0$ (Theorem 4.1).

Although the class of locally A-projective groups is not closed with respect to subgroups U which satisfy $S_A(U) = U$, there is a special type of subgroups for which this is true: (Theorem 4.3) A subgroup H of an abelian group G with $S_A(G) = G$ is *A-pure* if $S_A(H) = H$

and $\langle H, \phi_0(A), \ldots, \phi_n(A)\rangle / H$ is isomorphic to a subgroup of an A-projective group of finite A-rank for all $\phi_0, \ldots, \phi_n \in \mathrm{Hom}(A, G)$. A-purity naturally extends the concepts of $\{A\}_*$-purity which have been introduced in [W] and [A3].

We adopt the notations of [F] and [G]. All mappings are written on the left.

2. Locally projective modules. The initial results of this section extend Osofsky's Theorem to strongly non-singular, right semi-hereditary rings. Our discussion is based on the following result by Auslander. Denote the projective dimension of a right R-module M by p.d. M.

LEMMA 2.1 [**Au**]. *Let M be an R-module which is the union of a smooth ascending chain of submodules $\{M_\alpha\}_{\alpha<\kappa}$ whose projective dimension is at most n. Then,* p.d. $M \leq n+1$.

PROPOSITION 2.2. *The following conditions are equivalent for a strongly non-singular ring R:*

(a) *A non-singular R-module M, which is generated by strictly less than $\aleph_n$ many elements for some $n < \omega$, has projective dimension at most n.*

(b) *R is right semi-hereditary.*

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (a): Without loss of generality, we may assume that $n \geq 1$. Suppose that M is countably generated. Since R is a strongly non-singular, semi-hereditary ring, $M = \bigcup_{n<\omega} P_n$ where $0 = P_0 \subseteq P_1 \subseteq \cdots$ is a chain of finitely generated projective submodules of M. By Lemma 2.1, p.d. $M \leq 1$.

We proceed by induction and assume that M is generated by elements $\{x_\nu | \nu < \omega_n\}$. Define $M_0 = \{0\}$, $M_{\alpha+1} = \langle M_\alpha, x_\alpha\rangle$, and $M_\lambda = \bigcup_{\nu<\lambda} M_\nu$ if λ is a limit ordinal. The projective dimension of M_α is at most n for all $\alpha < \omega_n$ since M_α is generated by at most ω_{n-1} many elements. By Lemma 2.1, p.d. $M \leq n+1$.

We now turn to locally projective modules over arbitrary rings.

THEOREM 2.3. *A locally projective module M over a ring R has projective dimension at most n if it is generated by at most $\aleph_n$ many elements.*

Proof. If $n = 0$, choose a countable generating set $\{x_n | n < \omega\}$ for M with $x_0 = 0$. For every projective direct summand P of M, there is a free R-module $Q(P)$ such that $P \oplus Q(P)$ is free. We set

$$\widehat{M} = M \oplus \left[\bigoplus\{Q(P) | P \text{ is a projective summand of } M\}\right].$$

To see that every finite subset $\{w_1, \ldots, w_n\}$ of $\widehat{M}$ is contained in a finitely generated free direct summand of $\widehat{M}$, write $w_i = a_i + b_i$ for $i = 1, \ldots, n$, where $a_i \in M$ and $b_i \in \bigoplus\{Q(P) | P$ a projective direct summand of $M\}$. There are projective direct summands $P_0, \ldots, P_m$ of M such that $a_0, \ldots, a_n \in P_0$ and $b_1, \ldots, b_n \in Q(P_1) \oplus \cdots \oplus Q(P_m)$. If $P_0 \in \{P_1, \ldots, P_m\}$, then we may assume $P_0 = P_1$. The module $P_0 \oplus Q(P_1) \oplus \cdots \oplus Q(P_m)$ is a free direct summand of $\widehat{M}$ which contains $\{w_1, \ldots, w_n\}$. In the other case, consider the direct summand $P_0 \oplus Q(P_0) \oplus Q(P_1) \oplus \cdots \oplus Q(P_m)$ of $\widehat{M}$. In either case, there exists a finitely generated free direct summand V of $\widehat{M}$ which contains $\{w_1, \ldots, w_n\}$.

Set $U_0 = \{0\}$, and suppose that we have constructed an ascending chain $U_0 \subseteq \cdots \subseteq U_n$ of finitely generated free direct summands of $\widehat{M}$ such that $\{x_0, \ldots, x_i\} \subseteq U_i$ for $i = 0, \ldots, n$. By the results of the previous paragraph, there exists a finitely generated free direct summand U_{n+1} of $\widehat{M}$ which contains U_n and x_{n+1}. If we write $U_{n+1} = U_n \oplus V_n$, then $M \subseteq \bigcup_{n<\omega} U_n = \bigoplus_{n<\omega} V_n$. Since M is a direct summand of $\widehat{M}$, it also is one of the projective module $\bigcup_{n<\omega} U_n$. This completes the proof in the case $n = 0$.

Assume that the result fails for a minimal positive integer n. Let $\{x_\nu | \nu < \omega_n\}$ be a generating set of M, and suppose that we have constructed a smooth ascending chain $\{M_\nu\}_{\nu<\alpha}$ of submodules of M for some $\alpha < \omega_n$ which are generated by less than $\aleph_n$ many elements and have the following two properties: Every finite subset of M_ν is contained in a projective direct summand N of M which satisfies $N \subseteq M_\nu$; and $x_\nu \in M_{\nu+1}$ for all $\nu < \alpha$.

We set $M_0 = \{0\}$ and $M_\alpha = \bigcup_{\nu<\alpha} M_\nu$ if α is a limit ordinal. In the case $\alpha = \nu + 1$, we choose a generating set Z_α of $M_\alpha^0 = \langle M_\nu, x_\nu \rangle$ whose cardinality is less than $\aleph_n$. For every finite subset Y of Z_α, there is a projective direct summand N_Y of M which contains Y. Since N_Y is a direct sum of countably generated R-modules by Kaplansky's Theorem [**K**, Proposition 1.1], we may assume that N_Y itself is countably generated. Consequently, the submodule $M_\alpha^1 = \langle N_Y | Y \subseteq Z_\alpha, |Y| < \infty \rangle$ of M is generated by less than $\aleph_n$ many elements. Inductively, we construct an ascending chain $M_\alpha^0 \subseteq M_\alpha^1 \subseteq \cdots \subseteq M_\alpha^m$ $(m < \omega)$ such that $M_\alpha = \bigcup_{m<\omega} M_\alpha^m$ is generated by less than $\aleph_n$ many elements and has the properties required for the M_ν's. Therefore, p.d. $M_\alpha \leq n-1$. Since the chain $\{M_\nu\}_{\nu<\omega_n}$ is smooth, Lemma 2.1 yields p.d. $M \leq n$.

Following **[G]**, we call a submodule U of an R-module M *$\mathcal{S}$-closed* if M/U is non-singular. Let M be a non-singular R-module. The *$\mathcal{S}$-closure of a submodule* U of M is the smallest submodule W of M which contains U and has the property that M/W is non-singular.

PROPOSITION 2.4. *Let R be a strongly non-singular, right semi-hereditary ring.*

(a) *The class of locally projective R-modules is closed under $\mathcal{S}$-closed submodules.*

(b) *A non-singular R-module M is locally projective if and only if the $\mathcal{S}$-closure of a finitely generated submodule of M is a direct summand of M.*

Proof. (a) Suppose that U is an $\mathcal{S}$-closed submodule of the locally projective module M. For every finite subset X of U, there exists a projective direct summand V of M which contains X. Since R is right semi-hereditary, V is a direct sum of finitely generated modules by **[Ab]**. Consequently, we may assume that V itself is finitely generated. The module $V/(V \cap U) \cong \langle V, U\rangle/U$ is finitely generated and non-singular. Because R is strongly non-singular and right semi-hereditary, $V \cap U$ is a projective direct summand of V which contains X. Finally, if we observe that V is a direct summand of M, then $V \cap U$ is a direct summand of U. Therefore, U is locally projective.

(b) It remains to show that a locally projective module M has the splitting property for $\mathcal{S}$-closures of finitely generated submodules. If U is a finitely generated submodule of M, then there exists a finitely generated projective direct summand V of M which contains U by (a). As in the proof of that result, the $\mathcal{S}$-closure of U is a direct summand of V.

Furthermore, this characterization of locally projective modules may fail if R is not right semi-hereditary:

COROLLARY 2.5. *Let R be a semi-prime, right and left finite dimensional ring. The following conditions are equivalent*:

(a) *R is right semi-hereditary.*

(b) *If M is locally projective, then the $\mathcal{S}$-closure of a finitely generated submodule is a direct summand of M.*

Proof. By [G, Theorem 3.10], a semi-prime right and left finite dimensional ring is strongly non-singular. Therefore, it suffices to show (b) $\Rightarrow$ (a): Let I be a finitely generated right ideal of R. There exists an exact sequence $0 \to U \to \bigoplus_n R \to I \to 0$ for some $n < \omega$. Since $(\bigoplus_n R)/U$ is non-singular, U is a direct summand of $(\bigoplus_n R)$ by (b). Consequently, I is projective.

COROLLARY 2.6. *Let R be a strongly non-singular, right semi-hereditary ring. A $\mathcal{S}$-closed countably generated submodule U of a locally projective module is projective.*

3. A-Projective resolutions. Let A and G be abelian groups, M a right $E(A)$-module. The functors H_A and T_A between the categories of abelian groups and right $E(A)$-modules, which are defined by $H_A(G) = \mathrm{Hom}(A, G)$ and $T_A(M) = M \otimes_{E(A)} A$, are an adjoint pair where the $E(A)$-module structure of $H_A(G)$ is induced by composition of maps. Associated with them are the natural maps $\theta_G\colon T_AH_A(G) \to G$ and $\phi_M\colon M \to H_AT_A(M)$ which are given by the rules $\theta_G(\phi\otimes a) = \phi(a)$ and $[\phi_M(m)](a) = m\otimes a$ for all $a \in A$, $\phi \in H_A(G)$ and $m \in M$. Finally, an exact sequence $0 \to B \to C \to G \to 0$ of abelian groups is *A-balanced* if the induced sequence $0 \to H_A(B) \to H_A(C) \to H_A(G) \to 0$ is exact.

The full subcategory of the category of abelian groups whose elements G have the property that θ_G is an isomorphism is denoted by $\mathcal{G}_A$. Similarly, $\mathcal{M}_A$ indicates the category of all right $E(A)$-modules M for which ϕ_M is an isomorphism. The functors H_A and T_A define a category-equivalence between $\mathcal{G}_A$ and $\mathcal{M}_A$ [A5].

LEMMA 3.1 [A7, LEMMA 2.1]. *Let A be an abelian group. An exact sequence $0 \to B \xrightarrow{\alpha} C \xrightarrow{\beta} G \to 0$ of abelian groups such that C is A-solvable induces an exact sequence*

$$\mathrm{Tor}^1_{E(A)}(M, A) \xrightarrow{A} T_AH_A(B) \xrightarrow{\theta_B} B \xrightarrow{\delta} T_A(M) \xrightarrow{\theta(\beta)} G \to 0$$

where $M = \mathrm{im}\, H_A(\beta)$ and $[\theta(\beta)](m \otimes a) = m(a)$ for all $m \in M$ and $a \in A$.

In particular, the last result shows that a subgroup B of an A-solvable group which satisfies $S_A(B) = B$ is A-solvable if A is flat over $E(A)$. Under the same conditions on A, every A-balanced exact sequence $0 \to B \to C \to G \to 0$ such that C is A-solvable satisfies $M = H_A(G)$ and $\theta(\beta) = \theta_G$. Therefore, we obtain an exact sequence

$$0 \to T_AH_A(B) \xrightarrow{\theta_B} B \xrightarrow{\delta} T_AH_A(G) \xrightarrow{\theta_G} G \to 0.$$

PROPOSITION 3.2. *Let A be a self-small abelian group which is flat as an $E(A)$-module. The following conditions are equivalent for an abelian group G with $S_A(G) = G$:*

(a) *G is A-solvable.*

(b) *If $0 \to U \xrightarrow{\alpha} P \xrightarrow{\beta} G \to 0$ is an exact sequence such that P is A-projective, then $S_A(U) = U$.*

(c) *Every epimorphism $\phi_0 \colon P_0 \to G$ where P_0 is A-projective extends to a long exact sequence*

$$\cdots \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_1} P_0 \xrightarrow{\phi_0} G \to 0$$

in which P_n is A-projective for all $n < \omega$.

(d) *There exists an exact sequence $\cdots \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_1} P_0 \xrightarrow{\phi_0} G \to 0$ such that, for all $n < \omega$, P_n is A-projective and the induced sequence*

$$0 \to \operatorname{im} \phi_{n+1} \to P_n \xrightarrow{\phi_n} \operatorname{im} \phi_n \to 0$$

is A-balanced.

Proof. (a) $\Rightarrow$ (b): The exact sequence $0 \to U \xrightarrow{\alpha} P \xrightarrow{\beta} G \to 0$ induces a projective resolution

$$0 \to H_A(U) \xrightarrow{H_A(\alpha)} H_A(P) \xrightarrow{H_A(\beta)} M \to 0$$

of the right $E(A)$-module $M = \operatorname{im} H_A(\beta)$. By Lemma 3.1, there is a map $\theta(\beta) \colon T_A(M) \to G$ which fits into the following commutative diagram whose rows are exact:

$$\begin{array}{ccc} 0 \to T_A(M) & \to & T_A H_A(G) \\ \| & & \wr \downarrow \theta_G \\ T_A(M) & \xrightarrow{\theta(\beta)} & G \to 0 \end{array}$$

Thus, $\theta(\beta)$ is an isomorphism, and $S_A(U) = U$ by Lemma 3.1.

(b) $\Rightarrow$ (c): The map ϕ_0 induces an exact sequence

$$0 \to U \to P_0 \xrightarrow{\phi_0} G \to 0,$$

where $S_A(U) = U$ because of (b). Lemma 3.1 yields that U is A-solvable. The long exact sequence is now constructed inductively.

(c) $\Rightarrow$ (d): There exists an A-balanced exact sequence

$$0 \to U \to P_0 \xrightarrow{\phi_0} G \to 0 \ (1)$$

where P_0 is A-projective, and $U \subseteq P_0$ satisfies $S_A(U) = U$ because of (c). Since U is A-solvable by Lemma 3.1, and we have already verified the implication (a) $\Rightarrow$ (c), we obtain an A-balanced exact sequence

$$0 \to V \to P_1 \xrightarrow{\phi_1} U \to 0$$

with P_1 A-projective. Because of the validity of the implication (a) $\Rightarrow$ (b), $S_A(V) = V$. Consequently, an inductive argument completes the proof.

(d) $\Rightarrow$ (a): The sequence in (d) induces an exact sequence

$$0 \to T_A H_A(U) \xrightarrow{\theta_U} U \xrightarrow{\delta} T_A H_A(G) \xrightarrow{\theta_G} G \to 0$$

by Lemma 3.1 where $U = \operatorname{im} \phi_1$. Because of $S_A(U) = U$, the map θ_U is onto; and θ_G is an isomorphism.

EXAMPLE 3.3. Let A be a countable abelian group of infinite rank with $E(A) \cong \mathbf{Z}$. Every free subgroup F of A which has infinite rank contains a subgroup F_1 such that $F/F_1 \cong T_A(\mathbb{Q}) \cong \bigoplus_\omega \mathbb{Q}$. Thus, F/F_1 is a direct summand of A/F_1, and there exists a non-zero proper subgroup U of A with $A/U \cong \bigoplus_\omega \mathbb{Q}$.

We now show that $S_A(U) = 0$. If this is not the case, then $H_A(U) \neq 0$; and $H_A(A)/H_A(U)$ is a bounded abelian group. However, since the latter is isomorphic to a subgroup of the torsion-free group $H_A(\bigoplus_\omega \mathbb{Q})$, this is only possible if $H_A(A) = H_A(U)$. Because this contradicts the condition $A \neq U$, we obtain $S_A(U) = 0$. On the other hand, every free resolution $0 \to \bigoplus_\omega \mathbb{Z} \to \bigoplus_\omega \mathbb{Z} \to \mathbb{Q} \to 0$ yields an exact sequence

$$0 \to \bigoplus_\omega A \to \bigoplus_\omega A \to \bigoplus_\omega \mathbb{Q} \to 0.$$

Finally, we construct an A-balanced exact sequence $0 \to V \to \bigoplus_I A \to \bigoplus_\omega \mathbb{Q} \to 0$ such that $S_A(V) \neq V$: For this, we observe $H_A(\mathbb{Q}) \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q}$. Hence, there exists an A-balanced exact sequence $0 \to V \to \bigoplus_{2^{\aleph_0}} A \to \bigoplus_\omega \mathbb{Q} \to 0$. By Proposition 3.2, $S_A(V) \neq V$.

In the next part of this section, we introduce the concept of an A-projective dimension for A-solvable groups. In view of Proposition 3.2, we assume, that A is self-small and flat as an $E(A)$-module, and consider two A-balanced A-projective resolutions $0 \to U_i \to P_i G \to 0$ $(i = 1, 2)$ of an A-solvable group G. They induce exact sequences $0 \to H_A(U_i) \to H_A(P_i) \to H_A(G) \to 0$ of right $E(A)$-modules for $i = 1, 2$. By Shanuel's Lemma [**R**, Theorem 3.62], $H_A(P_1) \oplus H_A(U_2) \cong H_A(P_2) \oplus H_A(U_1)$. Since both, the U_i's and the P_i's, are A-solvable, we obtain

$$U_1 \oplus P_2 \cong T_A(H_A(U_1) \oplus H_A(P_2)) \cong T_A(H_A(P_1) \oplus H_A(U_2)) \cong P_1 \oplus U_2.$$

As in the case of modules this suffices to show that the following is well-defined:

An A-solvable group G has *A-projective dimension at most n* if there exists an exact sequence $0 \to P_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_1} P_0 \xrightarrow{\phi_0} G \to 0$ such that $P_0, \ldots, P_n$ are A-projective, and the induced sequences

$$0 \to \operatorname{im} \phi_{i+1} \to P_i \xrightarrow{\phi_i} \operatorname{im} \phi_i \to 0$$

are A-balanced for $i = 0, \ldots, n-1$. We write A-p.d. $G \leq n$ in this case. Otherwise, we say that G has *infinite A-projective* dimension and write A-p.d. $G = \infty$.

Our next result relates the A-projective dimension of an A-solvable group G to the projective dimension of the $E(A)$-module $H_A(G)$.

Proposition 3.4. *Let A be a self-small abelian group which is flat as an $E(A)$-module. If G is an A-solvable abelian group, then*

$$A\text{-p.d. } G = \text{p.d. } H_A(G).$$

Proof. Since an A-solvable group G is A-projective iff $H_A(G)$ is projective, it suffices to consider the case A-p.d. $G > 0$. Suppose that there exists an exact sequence

$$0 \to P_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_1} P_0 \xrightarrow{\phi_0} G \to 0$$

such that $P_0, \ldots, P_n$ are A-projective, and the induced sequences

$$0 \to \operatorname{im} \phi_{i+1} \to P_i \xrightarrow{\phi_i} \operatorname{im} \phi_i \to 0$$

are A-balanced exact for every $i = 0, \ldots, n-1$. Therefore the induced sequences of right $E(A)$-modules,

$$0 \to H_A(\operatorname{im} \phi_{i+1}) \to H_A(P_i) \xrightarrow{H_A(\phi_i)} H_A(\operatorname{im} \phi_i) \to 0,$$

are also exact for $i = 0, \ldots, n-1$; and we obtain $\operatorname{im} H_A(\phi_i) = H_A(\operatorname{im} \phi_i)$. Consequently, there is a projective resolution

$$0 \to H_A(P_n) \xrightarrow{H_A(\phi_n)} \cdots \xrightarrow{H_A(\phi_1)} H_A(P_0) \to H_A(G) \to 0$$

of $H_A(G)$, and p.d. $H_A(G) \leq n$.

Conversely, suppose that the right $E(A)$-module $H_A(G)$ has projective dimension at most n for some positive integer n. There exists an exact sequence $0 \to U \xrightarrow{\alpha} P \xrightarrow{\beta} H_A(G) \to 0$ of right $E(A)$-modules, where P is projective, and p.d. $U \leq n-1$. It induces the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_AT_A(U) & \xrightarrow{H_AT_A(\alpha)} & H_AT_A(P) & \xrightarrow{H_AT_A(\beta)} & H_AT_AH_A(G) & & \\
 & & \uparrow \phi_U & & \wr\uparrow \phi_P & & \uparrow \phi_{H_A(G)} & & \\
0 & \longrightarrow & U & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & H_A(G) & \longrightarrow & 0.
\end{array}$$

with exact rows. The maps ϕ_P and $\phi_{H_A(G)}$ are isomorphisms since H_A and T_A are a category equivalence between $\mathcal{T}_A$ and $\mathcal{M}_A$. Therefore, the map $H_AT_A(\beta)$ is onto. This shows that the sequence

$$0 \to T_A(U) \to T_A(P) \to T_AH_A(G) \to 0$$

is A-balanced exact, and $U \cong H_AT_A(U)$. Consequently, A-p.d. $T_A(U) \leq n-1$. Hence, G has A-projective dimension at most n.

As a first application of the last result, we give an upper estimate for the A-projective dimension of an A-torsion-free group provided that A is self-small and flat as an $E(A)$-module and has a strongly non-singular right semi-hereditary endomorphism ring: Following **[A8]**, we call an abelian group G with $S_A(G) = G$ *A-torsion-free* if $\langle \phi_0(A), \ldots, \phi_n(A) \rangle$ is isomorphic to a subgroup of an A-projective group of finite A-rank for all $\phi_0, \ldots, \phi_n \in H_A(G)$. If A is flat as $E(A)$-module, and if $E(A)$ is strongly non-singular, then G is A-torsion-free, iff G is A-solvable, and $H_A(G)$ is non-singular.

In **[A5**, Satz 5.9], we showed that every exact sequence $\bigoplus_I A \to G \to 0$ such that G is A-solvable is A-balanced if A is *faithfully flat* as an $E(A)$-module; i.e. A is a flat $E(A)$-module, and $IA \neq A$ for all proper right ideals I of $E(A)$. **[A7**, Theorem 2.8] shows that for every cotorsion-free ring R there is a proper class of abelian groups A with $E(A) \cong R$, which are faithfully flat as an $E(A)$-module, and whose endomorphism ring is discrete in the finite topology. On the other hand, the group $\mathbb{Z} \oplus \mathbb{Z}_p$ is not faithful although it is self-small and flat as an $E(A)$-module, **[Ar2**, Example 5.10].

COROLLARY 3.5. *Let A be a self-small abelian group which is faithfully flat as an $E(A)$-module and has a strongly non-singular endomorphism ring. The following conditions are equivalent*:

(a) *$E(A)$ is right semi-hereditary.*

(b) *If G is an A-torsion-free abelian group which admits an exact sequence $\bigoplus_I A \to G \to 0$ where $|I| < \aleph_n$ for some $n < \omega$, then A*-p.d. $G \leq n$.

Proof. (a) $\Rightarrow$ (b): Since A is faithfully flat, the sequence $\bigoplus_I A \to G \to 0$ is A-balanced exact. Hence, $H_A(G)$ is generated by less than $\aleph_n$-many elements. Because of Proposition 2.2, $H_A(G)$ has projective dimension at most n. By Proposition 3.4, A-p.d. $G \leq n$.

(b) $\Rightarrow$ (a): Let I be a finitely generated right ideal of $E(A)$. There exists an exact sequence $0 \to U \to \bigoplus_n E(A) \to I \to 0$ for some $n < \omega$. It induces the exact sequence $0 \to T_A(U) \to T_A(\bigoplus_n E(A)) \to T_A(I) \to 0$ which is A-balanced exact since A is faithfully flat as an $E(A)$-module. Thus, A-p.d. $T_A(I) \leq 0$ by (b). In particular, $T_A(I)$ is A-projective. Consider the induced diagram

$$\begin{tikzcd}
H_A T_A\left(\bigoplus_n E(A)\right) \arrow[r] & H_A T_A(I) \arrow[r] & 0 \\
\bigoplus_n E(A) \arrow[u, "\phi_{\oplus_n E(A)}"', "\wr"] \arrow[r] & I \arrow[u, "\phi_I"'] \arrow[r] & 0
\end{tikzcd}$$

It yields that ϕ_I is onto. Furthermore, the natural isomorphism δ_I: $T_A(I) \to IA$ which is defined by $\delta_I(i \otimes a) = i(a)$ for all $i \in I$ and $a \in A$ yields $[H_A(\delta_I)\phi_I(i)](a) = i(a)$ for all $i \in I$ and $a \in A$. Thus, $\phi_I(i) = 0$ implies $i = 0$. Thus, ϕ_I is an isomorphism, and $I \cong H_A T_A(I)$ is projective.

We conclude this section with another application of Proposition 3.4. Denote the right global dimension of a ring R by $\operatorname{gl.dim} R$.

COROLLARY 3.6. *Let A be a self-small abelian group which is faithfully flat as an $E(A)$-module.*

(a) *If $E(A)$ has right global dimension $n < \infty$, then there exist a subgroup U of an A-projective group with $S_A(U) = U$ and A-p.d. $U = n - 1$.*

(b) *If $E(A)$ has infinite global dimension, then there exists a subgroup U of an A-projective group with $S_A(U) = U$ and A-p.d. $U = \infty$.*

Proof. The global dimension of $E(A)$ is the supremum of the projective dimensions of modules of the form $E(A)/I$ where I is a right ideal of $E(A)$. Hence, there exists a family $\{I_n\}_{n<\omega}$ of right ideals of I such that p.d. $(\bigoplus_{n<\omega} I_n) = \text{gl.}\dim R - 1$ where $\infty - 1$ is defined to be ∞. The inclusion map $i\colon \bigoplus_{n<\omega} I_n \to \bigoplus_\omega E(A)$ yields the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_A T_A\left(\bigoplus_{n<\omega} I_n\right) \arrow[r, "H_A T_A(i)"] & H_A T_A\left(\bigoplus_\omega E(A)\right) \\
0 \arrow[r] & \bigoplus_{n<\omega} I_n \arrow[r, "i"] \arrow[u, "\phi_{\oplus_{n<\omega} I_n}"'] & \bigoplus_\omega E(A) \arrow[u, "\phi_{\oplus_\omega E(A)}"', "\wr"]
\end{tikzcd}$$

Hence $\phi_{\oplus_{n<\omega} I_n}$ is a monomorphism.

Choose a free right $E(A)$-module F which admits an epimorphism $\pi\colon F \to \bigoplus_{n<\omega} I_n$. Since $T_A(\bigoplus_{n<\omega} I_n) \subseteq T_A(\bigoplus_\omega E(A))$ yields that the sequence $T_A(F) \to T_A(\bigoplus_{n<\omega} I_n) \to 0$ is A-balanced, because A is faithfully flat, the top-row of the commutative diagram

$$\begin{tikzcd}
H_A T_A(F) \arrow[r, "H_A T_A(\pi)"] & H_A T_A\left(\bigoplus_{n<\omega} I_n\right) \arrow[r] & 0 \\
F \arrow[r, "\pi"] \arrow[u, "\phi_F"', "\wr"] & \bigoplus_{n<\omega} I_n \arrow[r] \arrow[u, "\phi_{\oplus_{n<\omega} I_n}"'] & 0
\end{tikzcd}$$

is exact. Hence $\bigoplus_{n<\omega} I_n \cong H_A T_A(\bigoplus_{n<\omega} I_n)$ and has projective dimension equal to $(\text{gl.}\dim R) - 1$. By Proposition 3.4,

$$A\text{-p.d.}\left(T_A\left(\bigoplus_{n<\omega} I_n\right)\right) = \text{p.d.}\left(\bigoplus_{n<\omega} I_n\right).$$

4. Locally A-projective groups. The combination of the results of Sections 2 and 3 allows to give a more precise estimate of the A-projective dimension of locally A-projective groups than we can obtain from Corollary 3.5:

THEOREM 4.1. *Let A be an abelian group which is flat as an $E(A)$-module and has an endomorphism ring which is discrete in the finite topology. A locally A-projective group G has A-projective dimension at most n if it is an epimorphic image of $\bigoplus_{\omega_n} A$.*

Proof. Since G is an epimorphic image of $\bigoplus_{\omega_n} A$, there exists a family $\{\phi_\nu\}_{\nu<\omega_n} \subseteq H_A(G)$ such that $G = \langle \phi_\nu(A) | \nu < \omega_n \rangle$. Moreover, we can find a finite subset X of A such that $\{\alpha \in E(A) | \alpha(X) = 0\} = 0$. We show that G is an A-balanced epimorphic image of a group isomorphic to a direct summand of $\bigoplus_{\omega_n} A$.

For any indices $\nu_0, \ldots, \nu_m < \omega_n$, choose an A-projective direct summand V of G which contains $\langle \phi_{\nu_0}(X), \ldots, \phi_{\nu_m}(X) \rangle$ and write $G = V \oplus W_1$. Since $H_A(V)$ is projective, Kaplansky's Theorem, [**K**, Proposition 1.1], yields a decomposition $H_A(V) = \bigoplus_{j \in J} P_j$ where P_j is countably generated for all $j \in J$. Hence, we can choose V in such a way that it is isomorphic to a direct summand of $\bigoplus_\omega A$.

We show that $H_A(V)$ contains $\psi(A)$ for every map $\psi \in H_A(G)$ which satisfies $\psi(X) \subseteq V$. Denote the projection of G onto V with kernel W_1 by π. If there is an element a of A such that $(1-\pi)\psi(a) \neq 0$, then we can find a map $\delta \in \operatorname{Hom}(G, A)$ such that $\delta(1-\pi)\psi(a) \neq 0$ because $(1-\pi)\psi(a)$ is contained in an A-projective direct summand of G. The map $\delta(1-\pi)\psi$ is a non-zero element of $E(A)$ with $\delta(1-\pi)\psi(X) = 0$ whose existence contradicts the choice of X. Consequently, $\psi \in H_A(V)$.

For every finite subset Y of $\{\phi_\nu | \nu < \omega_n\}$, we choose a direct summand V_Y of G, which contains $\phi(X)$ for all maps $\phi \in Y$ and is isomorphic to a direct summand of $\bigoplus_\omega A$. Let $\mathfrak{M}$ be the collection of all these V_Y's. The inclusion maps $V_Y \subseteq G$ induce an epimorphism $\nu\colon W \to G$ where $W = \bigoplus\{U | U \in \mathfrak{M}\}$. For every map $\mu \in H_A(G)$, there exist indices $\nu_0, \ldots, \nu_m < \omega$ such that $\mu(X) \subseteq \langle \phi_{\nu_0}(A), \ldots, \phi_{\nu_m}(A) \rangle$. If we denote the set $\{\phi_{\nu_0}, \ldots, \phi_{\nu_m}\}$ by Y, then the result in the previous paragraph yields that V_Y contains $\langle \phi_{\nu_0}(A), \ldots, \phi_{\nu_m}(A) \rangle$ and $\mu(A)$. Thus, $\mu \in H_A(V_Y)$. This shows that the sequence $0 \to \ker \nu \to W \xrightarrow{\nu} G \to 0$ is A-balanced. Furthermore, W is isomorphic to a direct sum of at most $\aleph_n$ abelian groups which are direct summands of $\bigoplus_\omega A$. Therefore, the A-rank of W is at most $\aleph_n$.

Consequently, $H_A(G)$ is a locally projective right $E(A)$-module which is generated by at most $\aleph_n$ elements. Since the induced sequence $H_A(W) \to H_A(G) \to 0$ is exact, we have A-p.d. $G =$ p.d. $H_A(G) \leq n$ by Proposition 3.4 and Theorem 2.3.

The subgroup $U = 2 \cdot \mathbb{Z}^\omega + [\bigoplus_\omega \mathbb{Z}]$ of the locally $\mathbb{Z}$-projective group $\mathbb{Z}^\omega$ is not locally $\mathbb{Z}$-projective, although $S_\mathbb{Z}(U) = U$. On the other hand, the next result yields that the class of locally A-projective groups is closed under A-pure subgroups if $E(A)$ is strongly non-singular and right semi-hereditary.

It also allows to recapture the following property of homogeneous separable abelian groups G [**F1**, Proposition 87.2]: Any pure finite rank subgroup of such a group G is a direct summand of G. To facilitate this, we consider A-purifications which were introduced in [**A8**]: If H

is a subgroup of an A-torsion-free group G with $S_A(H) = H$, then the *A-purification of H in G* is defined to be $H_* = \theta_G(T_A(W))$ where W is the $\mathscr{S}$-closure of $H_A(H)$ in $H_A(G)$. In **[A8]**, we showed that H_* is the smallest A-pure subgroup of G which contains H.

THEOREM 4.3. *Let A be an abelian group which is flat as an $E(A)$-module and has a strongly non-singular, right semi-hereditary endomorphism ring which is discrete in the finite topology*:

(a) *The class of locally A-projective groups is closed under A-pure subgroups.*

(b) *An abelian group G is locally A-projective, if and only if $G = S_A(G)$, and the A-purification U_* is an A-projective direct summand of G for all subgroups U of G which admit an exact sequence $\bigoplus_n A \to U \to 0$ for some $n < \omega$.*

(c) *An A-pure subgroup U of a locally A-projective group is A-projective if it is an epimorphic image of $\bigoplus_\omega A$.*

Proof. (a) Let U be an A-pure subgroup of the locally A-projective group G. Then, $S_A(U) = U$ yields that U is A-solvable by Lemma 3.1. Moreover, $H_A(G)$ is locally projective, and $H_A(U)$ is $\mathscr{S}$-closed in $H_A(G)$. By Proposition 2.4, $H_A(U)$ is locally projective. Consequently, $U \cong T_A H_A(U)$ is a locally A-projective group.

(b) and (c) are deduced in a similar way from the corresponding results of Section 2.

Furthermore, the condition that $E(A)$ is right semi-hereditary may not be omitted from the last result as is shown in Corollaries 2.5 and 3.5.

Let $\mathrm{TL}(A) = \{G | S_A(G) = G$ and $G \underset{\sim}{\subset} A^I$ for some index-set $I\}$ be the class of *A-torsion-less groups*. Similarly, $\mathrm{TL}(E(A))$ denotes the class of torsion-less right $E(A)$-modules.

PROPOSITION 4.4. *Let A be a self-small abelian group which is faithfully flat as an $E(A)$-module and has the property that $S_A(A^I)$ is A-solvable for all index-sets I. The functors H_A and T_A define an equivalence between the categories* $\mathrm{TL}(A)$ *and* $\mathrm{TL}(E(A))$.

Proof. Let M be a submodule of $E(A)^I$ for some index-set I. Because of $H_A(A^I) = H_A(S_A(A^I))$ and the remarks preceding Lemma 3.1, the map $\phi_{H_A(A^I)}$ is an isomorphism. In order to show that ϕ_M is an isomorphism, we consider the following commutative diagram whose rows are exact:

$$
\text{(I)}\qquad
\begin{tikzcd}
0 \arrow[r] & H_A T_A(M) \arrow[r] & H_A T_A H_A(A^I) \\
0 \arrow[r] & M \arrow[u, "\phi_M"] \arrow[r] & H_A(A^I) \arrow[u, "\phi_{H_A(A^I)}", "\wr"']
\end{tikzcd}
$$

Furthermore, an exact sequence $0 \to U \xrightarrow{\alpha} \bigoplus_J E(A) \xrightarrow{\beta} M \to 0$ (1) induces the exact sequence

$$0 \to T_A(U) \xrightarrow{T_A(\alpha)} T_A\left(\bigoplus_J E(A)\right) \xrightarrow{T_A(\beta)} T_A(M) \to 0 \ (2)$$

in which the group $T_A(M)$ is A-solvable as a subgroup of $T_A(E(A)^I) \cong S_A(A^I)$.

Therefore, (2) is A-balanced because A is faithfully flat as an $E(A)$-module; and we obtain the following commutative diagram with exact rows:

$$
\text{(II)}\qquad
\begin{tikzcd}
0 \arrow[r] & H_A T_A(U) \arrow[r, "H_A T_A(\alpha)"] & H_A T_A(\bigoplus_J E(A)) \arrow[r, "H_A T_A(\beta)"] & H_A T_A(M) \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[u, "\phi_U"] \arrow[r, "\alpha"] & \bigoplus_J E(A) \arrow[u, "\phi_{\bigoplus_J^{E(A)}}", "\wr"'] \arrow[r, "\beta"] & M \arrow[u, "\phi_M"] \arrow[r] & 0
\end{tikzcd}
$$

Combining diagrams (I) and (II) yields that ϕ_M is an isomorphism.

On the other hand, if $G \subseteq A^I$ with $S_A(G) = G$, then θ_G is an isomorphism by Lemma 3.1. This shows that H_A and T_A define a category equivalence between $\mathrm{TL}(A)$ and $\mathrm{TL}(E(A))$.

Furthermore, we obtain a converse of the last result under some slight restrictions on A.

THEOREM 4.5. *Let A be a self-small abelian group, which is flat as an $E(A)$-module, and whose endomorphism ring has no infinite set of orthogonal idempotents. The following conditions are equivalent:*

(a) *The functors H_A and T_A are a category equivalence between* $\mathrm{TL}(A)$ *and* $\mathrm{TL}(E(A))$.

(b) (i) *A is a faithful left $E(A)$-module.*

(ii) *$S_A(A^I)$ is A-solvable for all index-sets I.*

Proof. (a) $\Rightarrow$ (b): Let I be a right ideal of $E(A)$ with $IA = A$. Consider the evaluation map $j_I : I \to H_A(IA)$ which is defined by

$[j_I(i)](a) = i(a)$ for all $i \in I$ and $a \in A$. Since A is flat as an $E(A)$-module, there is an isomorphism $\delta\colon T_A(I) \to IA$ with $\delta(i \otimes a) = i(a)$.

Furthermore, the map ϕ_I is an isomorphism since $I \in \mathrm{TL}(E(A))$. Therefore, $I \cong H_A T_A(I) = H_A(IA) = E(A)$, and $I = \phi E(A)$ for some $\phi \in I$. This yields an exact sequence $0 \to U \to A \xrightarrow{\phi} A \to 0$. Since the group U is A-solvable by Lemma 3.1 there exists a non-zero map $\alpha \in H_A(U)$ if $U \neq 0$.

On the other hand, we have a decomposition $E(A) = J_1 \oplus J_2$ where $J_1 = \{\sigma \in E(A) | \phi\sigma = 0\} \neq 0$ and $J_2 \cong E(A)$ because of $I \cong E(A)$ and $\phi\alpha = 0$. Then, $E(A)$ contains an infinite family of orthogonal idempotents which is not possible by the hypotheses on A. Hence, ϕ is an isomorphism, and $I = E(A)$.

Finally, $S(A^I) \in \mathrm{TL}(A)$ yields that $\theta_{S_A(A^I)}$ is an isomorphism because of (a).

(b) $\Rightarrow$ (a) immediately follows from Proposition 4.4.

[**G**, Problem 5.B3] immediately yields that the map $\theta_{S_A(A^I)}$ is an isomorphism if the inclusion map $i\colon B \to A$ factors through a finitely presented module for all finitely generated $E(A)$-submodules B of A. This condition is, for instance, satisfied if A is $\aleph_0$-projective, i.e. every finite subset of A is contained in a finitely generated projective submodule of A. In particular, all groups constructed by [**DG**, Theorem 3.3] belong to this latter class of groups. Another important class of examples is given by

COROLLARY 4.6. *Let A be a self-small abelian group with a left Noetherian endomorphism ring which is flat as an $E(A)$-module. The functors H_A and T_A define a category equivalence between* $\mathrm{TL}(A)$ *and* $\mathrm{TL}(E(A))$ *if and only if A is faithful as an $E(A)$-module.*

5. Examples. We give an example of a strongly non-singular, semi-hereditary ring which is neither a valuation domain, finite dimensional, nor hereditary:

PROPOSITION 5.1. *Let $R = \prod_{n<\omega} R_n$ where addition and multiplication are defined coordinatewise, and each R_i is a principal ideal domain. The ring R is semi-hereditary and strongly non-singular. An ideal I of R is essential in R if and only if $\pi_i(I) \neq 0$ for all $i < \omega$ where $\pi_i\colon R \to R_i$ is the projection onto the ith-coordinate.*

Proof. Denote the ring $\prod_{n<\omega} Q_n$ by $S^\circ R$ where Q_n is the field of quotients of R_n for all $n < \omega$. It is self-injective and regular; and R is an essential R-submodule of $S^\circ R$. Thus, $S^\circ R$ is the maximal ring of quotients of R by [**G**, Theorem 2.10]. Suppose $x = (r_n^{-1}s_n)_{n<\omega} \in S^\circ R$ where $r_n = 1$ if $s_n = 0$ and $(r_n, s_n) = 1$ otherwise. If $r = (m_n)_{n<\omega} \in R$ with $rx \in R$, then $r_n|m_n$ for all $n < \omega$. Thus, $\{r \in R | rx \in R\} = (r_n)_{n<\omega}R$ is a finitely generated ideal of R. By [**G**, Theorem 3.10], R is a strongly non-singular ring.

Moreover, let $x_1, \dots, x_m$ be in R, say $x_i = (x_{i,n})_{n<\omega}$ for $i = 1, \dots, m$. We set $I_n = \langle x_{1,n}; \cdots ; x_{m,n}\rangle$, which is an ideal of R_n. Then,

$$I = \langle x_1, \dots, x_m\rangle = \left\{ \left(\sum_{j=1}^{m} x_{j,n} r_{j,n} \right)_{n<\omega} \middle| \, r_{j,n} \in R_n \right\} = \prod_{n<\omega} I_n.$$

If $Y \subseteq \omega$ is the set of all $n < \omega$ with $I_n \neq 0$, then $I_n \cong R_n$ for $n \in Y$ yields that $I = \prod_{n<\omega} I_n \cong \prod_{n\in Y} I_n \cong \prod_{n\in Y} R_n$ is a projective R-module. Thus, R is semi-hereditary.

Suppose, that E is an essential ideal of R. We denote the embedding into the ith coordinate of R by δ_i. If $\pi_n(E) = 0$ for some $n < \omega$, then $E \cap \delta_n(R_n) = 0$ yields a contradiction.

On the other hand, assume $\pi_i(I) \neq 0$ for all $i < \omega$ where I is an ideal of R. Let $r = (r_n)_{n<\omega}$ be a non-zero element of R. If $r_m \neq 0$, then there is $e \in I$ such that $s_m = \pi_m(e) \neq 0$. Then,

$$0 \neq r\delta(s_m) = \delta_m(r_m s_m) = e\delta_m(r_m) \in rR \cap I.$$

Hence, I is essential in R.

COROLLARY 5.2. *The ring $R = \mathbb{Z}^\omega$ is strongly non-singular and semi-hereditary, but not hereditary.*

Proof. Let I be the ideal $(2, \dots) \cdot R + [\bigoplus_\omega \mathbb{Z}]$ of R. If I were projective, then it would be finitely generated by Sandomierski's Theorem, [**CH**, Proposition 8.24] since it contains the cyclic essential submodule $(2, \dots)R$. Hence, we can find an index $n_0 < \omega$ such that m_n is even for all elements $(m_i)_{i<\omega} \in I$ and all indices $n_0 \leq n < \omega$. However, because of $\bigoplus_\omega \mathbb{Z} \subseteq I$, this is not possible. Therefore, I is not projective; and R is not hereditary.

By [**DG**, Theorem 3.3] in conjunction with [**A7**, Theorem 2.8], there exists a proper class of abelian groups A with $E(A) \cong \mathbb{Z}^\omega$ which are faithfully flat and have the additional property that $E(A)$ is discrete in the finite topology.

However, the groups considered in this paper need not be torsion-free:

EXAMPLE 5.3. Let $A = \prod\{\mathbb{Z}/p\mathbb{Z}$ is prime$\}$. Then, $E(A)$ is strongly non-singular, semi-hereditary and discrete in the finite topology; and A is flat as an $E(A)$-module. However, $E(A)$ is not hereditary.

Proof. As a product of fields, $E(A)$ is self-injective. By **[O2]**, a hereditary, self-injective ring is semi-simple Artinian. But this is not the case for $E(A)$.

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