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INDICES OF UNBOUNDED DERIVATIONS OF C^*-ALGEBRAS

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The paper studies some properties of J-symmetric representations of $*$-algebras on indefinite metric spaces. Making use of this, it defines the index $\operatorname{ind}(\delta, S)$ of a $*$-derivation δ of a C^*-algebra $\mathscr{A}$ relative to a symmetric implementation S of δ. The index consists of six integers which characterize the J-symmetric representation π_S of the domain $D(\delta)$ of δ on the deficiency space $N(S)$ of the operator S. The paper proves the stability of the index under bounded perburbations of the derivation and that, under certain conditions on δ, $\operatorname{ind}(\delta, S)$ has the same value for all maximal symmetric implementations S of δ. It applies the developed methods to the problem of the classification of symmetric operators with deficiency indices $(1, 1)$.

1. Introduction and preliminaries. Let $\mathscr{A}$ be a C^*-subalgebra of the algebra $B(H)$ of all bounded operators on a Hilbert space H. A closed $*$-derivation δ from $\mathscr{A}$ into $B(H)$ is a linear mapping from a subalgebra $D(\delta)$ dense in $\mathscr{A}$ into $B(H)$ such that

(i) $\delta(AB) = \delta(A)B + A\delta(B)$,
(ii) $A \in D(\delta)$ implies $A^* \in D(\delta)$ and $\delta(A^*) = \delta(A)^*$,
(iii) $A_n \in D(\delta)$, $A_n \to A$ and $\delta(A_n) \to B$ implies $A \in D(\delta)$ and $\delta(A) = B$.

An operator S on H *implements* δ if $D(S)$ is dense in H and

$$AD(S) \subseteq D(S) \quad \text{and} \quad \delta(A)|_{D(S)} = i[S, A]|_{D(S)} = i(SA - AS)|_{D(S)}$$

for all $A \in D(\delta)$. If T extends S and also implements δ, then T is called a *δ-extension* of S. If S is symmetric and it does not have symmetric δ-extensions, it is called a *maximal symmetric implementation* of δ.

The case when a symmetric operator S implements the zero derivation on $\mathscr{A}$, i.e., $SA|_{D(S)} = AS|_{D(S)}$, $A \in \mathscr{A}$, was extensively investigated (see, for example, **[6]**, **[21]**, **[22]**). Different sufficient conditions were obtained for S to have a selfadjoint extension T which commutes with $\mathscr{A}$.

The problem of δ-extension of a symmetric operator S which implements a derivation δ on $\mathscr{A}$ has been addressed in a number of papers (see, for example, [7], [9]). In [9] it was proved that any $*$-derivation δ from $\mathscr{A}$ into $B(H)$ implemented by a symmetric operator has a maximal symmetric implementation S. The link between the deficiency indices $n_+(S)$ and $n_-(S)$ of S and finite-dimensional irreducible representations of $\mathscr{A}$ was investigated. This led to introduction in **[10]** of the set $M(\delta, \mathscr{A})$ of all pairs $(n_+(S), n_-(S))$ where S are maximal symmetric implementations of δ.

The investigation of symmetric implementations of derivations δ is deeply related to the investigation of J-symmetric representations of their domains $D(\delta)$ on indefinite metric spaces (see **[8]**, **[9]**, **[10]**). The nature of this relation can be easily seen from the following remarks.

If S is a symmetric operator and S^* is its adjoint, then

$$D(S^*) = D(S) + N_-(S) + N_+(S),$$

where $N_d(S) = \{x \in D(S^*)\colon S^*x = idx\}$, $d = \pm$, are deficiency spaces of S. The numbers $n_\pm(S) = \dim N_\pm(S)$ are called the deficiency indices of S. We define a scalar product on $D(S^*)$ by the formula:

$$\{x, y\} = (x, y) + (S^*x, S^*y).$$

Then $D(S^*)$ becomes a Hilbert space and

$$D(S^*) = D(S) \oplus N_-(S) \oplus N_+(S)$$

is the orthogonal sum of the subspaces $D(S)$, $N_-(S)$ and $N_+(S)$ with respect to $\{\ ,\ \}$. Let $N(S) = N_-(S) \oplus N_+(S)$ and let Q be the projection on $N(S)$ and Q_+ be the projection on $N_+(S)$ in $D(S^*)$. Set $J = 2Q_+ - Q$. Then J is an involution on $N(S)$ and $N(S)$ becomes an indefinite metric space Π_k ($k = \min(n_+(S), n_-(S))$) with the indefinite scalar product

$$[x, y]^S = \{Jx, y\}, \qquad x, y \in N(S).$$

Now if S implements a $*$-derivation δ from $\mathscr{A}$ into $B(H)$ it follows easily that $D(\delta)$ acts on $D(S^*)$ as an algebra of bounded operators. Since $D(S)$ is invariant for $D(\delta)$,

$$\pi_S(A) = QAQ, \qquad A \in D(\delta),$$

is a representation of $D(\delta)$ on $N(S)$. It was proved in **[9]** that π_S is a J-symmetric representation of $D(\delta)$ on $N(S)$ and that there is a one-to-one correspondence between symmetric δ-extensions of S and null subspaces in $N(S)$ invariant for π_S. If S is a maximal implementation of δ, then π_S does not have null invariant subspace in $N(S)$.

Because of the close relation between derivations of C^*-algebras implemented by symmetric operators and J-symmetric representations of $*$-algebras on indefinite metric spaces the study of such representations becomes very important. Section 2 is devoted to this study. For every J-symmetric representation π we introduce a sextuple $\mathrm{ind}(\pi) = (k_+, k_-, d_+(\pi), d_-(\pi), i_+(\pi), i_-(\pi))$ which we call *the index of* π.

Powers **[16]** considered E_0-semigroups α_t of $*$-endomorphisms of $B(H)$ which have strongly continuous semigroups $U(t)$ of intertwining isometries ("spatial" semigroups). If d is the generator of $U(t)$, then $S = id$ is an unbounded maximal symmetric operator, i.e., $n_-(S) = 0$, and it is a maximal symmetric implementation of the generator δ of α_t. Therefore, $N(S) = N_+(S)$ is a Hilbert space, π_S is a $*$-representation of $D(\delta)$ on $N(S)$ and $(0, n_+(S)) \in M(\delta, \mathscr{A})$ where $\mathscr{A}$ is the closure of $D(\delta)$. Powers **[16]** defined the *index* of α_t as the maximal number of non-zero mutually orthogonal projections in the commutant of $\pi_S(D(\delta))$. The examples of CAR-flows **[16]** show that $n_+(S) = \infty$ for all of them and that the index has values $i = 1, 2, \ldots$. In **[17]** Powers and Robinson gave another definition of the index which is independent of the existence of intertwining semigroups of isometries. Arveson **[2]** and **[3]** used another approach to this index theory for E_0-semigroups based on the notion of continuous tensor product systems. He showed that for "spatial" semigroups the Powers-Robinson index can be associated with an integer $i = 1, 2, \ldots$.

Jorgensen and Price **[8]** studied the variety $\mathscr{V}$ of all operators $V\colon H \mapsto N(S)$ such that $VA = \pi_S(A)V$, $A \in D(\delta)$, and showed that $\mathscr{V}$ has a unique scalar form which turns $\mathscr{V}$ into an indefinite metric space. They introduced the V-index as the Krein dimension of $\mathscr{V}$.

In this paper we associate *the index* $\mathrm{ind}(\delta, S)$ with every symmetric implementation S of a derivation δ. In order to do this we consider the J-symmetric representation π_S of $D(\delta)$ and we define $\mathrm{ind}(\delta, S) = \mathrm{ind}(\pi_S)$. If $n_-(S) = 0$, so that S is a maximal symmetric operator, then

$$\mathrm{ind}(\delta, S) = (n_+(S), 0, n_+(S), 0, i_+(\pi_S), 0)$$

where $i_+(\pi_S)$ is the Powers index and $d_+(\pi_S) = n_+(S)$. If

$$\min(n_+(S), n_-(S)) < \infty$$

and if π_S extends to a bounded representation of $\mathscr{A}$ (for example, if $\mathscr{A}$ is commutative), we show that $d_+(\pi_S) = n_+(S)$ and $d_-(\pi_S) = n_-(S)$.

Theorem 3.6 proves that $\operatorname{ind}(\delta, S)$ is stable under perturbations of δ of the form

$$\sigma(A) = \delta(A) + i[B, A],$$

where B is a bounded selfadjoint operator, i.e.,

$$\operatorname{ind}(\delta, S) = \operatorname{ind}(\sigma, S + B).$$

Every derivation implemented by a symmetric operator has an infinite number of maximal symmetric implementations. Therefore the question arises as to whether the index $\operatorname{ind}(\delta, S)$ may be the same for all such implementations. In **[10]** it was shown that if δ has a *minimal* symmetric implementation T (if $\mathscr{A}$ contains the algebra of all compact operators, any closed derivation of $\mathscr{A}$ has such an implementation **[10]**) and if $\min(n_-(T), n_+(T)) < \infty$, then all maximal implementations of δ have the same deficiency indices. In this paper we show that in this case $\operatorname{ind}(\delta, S) = \operatorname{ind}(\delta, S_1)$ for all maximal symmetric implementations S and S_1 of δ.

Theorem 3.2 investigates the link between the deficiency indices of maximal symmetric implementations S of δ and dimensions of irreducible representations of $\mathscr{A}$. It improves the result of **[9]** and, in particular, it shows that if $1 \in \mathscr{A}$ and if $\max(n_+(S), n_-(S)) < \infty$, then there are disjoint sets of irreducible representations $\{\pi_i\}_{i=1}^p$ and $\{\rho_j\}_{j=1}^q$ of $\mathscr{A}$ such that

$$n_+(S) = \sum_{i=1}^{p} \dim \pi_i \quad \text{and} \quad n_-(S) = \sum_{j=1}^{q} \dim \rho_j.$$

If $\max(n_+(S), n_-(S)) = \infty$ and $k = \min(n_+(S), n_-(S)) < \infty$ and if π_S extends to a bounded representation of $\mathscr{A}$ ($1 \in \mathscr{A}$), then there are irreducible representations $\{\pi_i\}_1^p$ of $\mathscr{A}$ such that $k = \sum_{i=1}^p \dim \pi_i$.

Every densely defined symmetric operator S has a $*$-algebra $\mathscr{B}_S$ associated with it: $\mathscr{B}_S = \{A \in B(H)\colon A$ and A^* preserve $D(S)$ and $(SA - AS)|_{D(S)}$ extends to a bounded operator$\}$. The closure $\mathscr{A}_S$ of $\mathscr{B}_S$ is the maximal C^*-subalgebra of $B(H)$ such that S generates a closed $*$-derivation δ_S of $\mathscr{A}_S$ into $B(H)$ and that $D(\delta_S) = \mathscr{B}_S$. In Section 4 we make use of the results of Section 3 and associate a number $\beta(S)$ with every symmetric operator S such that $n_+(S) = n_-(S) = 1$ and such that the representation π_S of the algebra $\mathscr{B}_S$ on $N(S)$ does not have null invariant subspaces. We obtain that $0 \le \beta(S) < 1$ and that $\beta(S) = \beta(T)$ if S and T are isomorphic.

It is well-known (see, for example, [1]) that up to isomorphism there is only one symmetric operator with the deficiency indices $(1, 0)$ and only one with the deficiency indices $(0, 1)$. The variety of symmetric operators with the deficiency indices $(1, 1)$ is much greater. All symmetric differential operators

$$S_a = i\frac{d}{dx}, \quad D(S_a) = \{y(x)\colon y \text{ and } y' \text{ in } L_2(0, a), y(0) = y(a) = 0\},$$

$0 < a < \infty$, have $n_+(S_a) = n_-(S_a) = 1$. Schmudgen [19] showed that S_a and S_b are not isomorphic if $a \neq b$.

Theorem 4.2 investigates the structure of the representations π_{S_a} of the algebras $\mathscr{B}_{S_a}$ on $N(S_a)$ and shows that $\beta(S_a) = e^{-a}$. This provides us with another proof of Schmudgen's result and also shows that $\beta(S)$ takes all values in the interval $[0, 1)$. The question arises as to whether $\beta(S)$ classifies up to isomorphism all the symmetric operators S such that $n_+(S) = n_-(S) = 1$ and such that the representations π_S do not have null invariant subspaces.

2. J-symmetric representations of $*$-algebras. In this section we consider J-symmetric representations of $*$-algebras in indefinite metric spaces. For the benefit of the reader and for the sake of being reasonably self-contained, we call attention to the references [12, 15] and provide some amount of detail about indefinite metric spaces and J-symmetric representations.

Let J be an involution on a Hilbert space H, i.e., $J^* = J$ and $J^2 = 1$. With the indefinite scalar product

$$[x, y] = (Jx, y), \qquad x, y \in H,$$

H becomes an *indefinite metric space.* A subspace L in H is called

(a) *nonnegative* if $[x, x] \geq 0$ for all $x \in L$,

(b) *positive* if $[x, x] > 0$ for all $x \in L$, $x \neq 0$,

(c) *uniformly positive* if there exists $r > 0$ such that $[x, x] \geq r(x, x)$ for all $x \in L$,

(d) *null* if $[x, x] = 0$ for all $x \in L$.

The concepts of *nonpositive, negative, uniformly negative* subspaces are introduced analogously.

Set $Q = (J + 1)/2$. Then $H = H_+ \oplus H_-$, Q is the projection onto H_+, $1 - Q$ is the projection onto H_- and $[x, x] = (x, x)$ if $x \in H_+$ and $[x, x] = -(x, x)$ if $x \in H_-$. Therefore H_+ is uniformly positive and H_- is uniformly negative. Let $k_d = \dim H_d$, $d = \pm$ and let $k = \min(k_-, k_+)$. Then H is called a Π_k-space.

Law of inertia [**12**]. If L is a maximal nonnegative (nonpositive) subspace of H, then

$$\dim L = k_+(k_-).$$

A representation π of a $*$-algebra $\mathscr{A}$ into $B(H)$ is called J-symmetric if for all $A \in \mathscr{A}$ and for all x, y in H

$$J\pi(A^*) = \pi(A)^* J, \quad \text{so that } [\pi(A)x, y] = [x, \pi(A^*)y]. \tag{1}$$

If a subspace L of H is invariant for π, then by π_L we denote the restriction of π to L.

J-symmetric representations π and ρ of a $*$-algebra $\mathscr{A}$ on H and K respectively are called *J-equivalent* if there is a bounded operator U from H onto K such that $U\pi = \rho U$ and such that

$$[Ux, Uy] = [x, y] \quad \text{for all } x, y \in H.$$

For every subspace L in H the subspace

$$L^{[\perp]} = \{y \in H\colon\ [x, y] = 0 \text{ for all } x \in L\}$$

is called *J-orthogonal complement* of L.

It is well-known that there always exists the decomposition

$$H = L \oplus L^{\perp}, \quad L^{\perp} = \{x \in H\colon\ (y, x) = 0 \text{ for all } y \in L\}.$$

In an indefinite metric space the decomposition

$$H = L[+]L^{[\perp]} \tag{2}$$

(the symbol $[+]$ means that the sum is direct and the summands are J-orthogonal) does not always exist.

THEOREM 2.1 ([**12**]). *Let J be an involution on H. Then $H = H_+ \oplus H_-$ where $Q = (J+1)/2$ is the projection onto H_+. Let $k_d = \dim H_d$, $d = \pm$.*

(i) *Let L be a nonnegative* (*nonpositive*) *subspace of H. The decomposition* (2) *holds if and only if L is uniformly positive* (*negative*).

(ii) *If L is an indefinite subspace, then* (2) *holds if and only if L decomposes into a direct sum of two uniformly definite subspaces.*

(iii) (*Iohvidov and Ginzburg, see* [**12**], *page* 118). *Let $k_+ = \infty$. Then all the positive subspaces of H are uniformly positive if and only if $k_- < \infty$.*

For Π_k-spaces ($k < \infty$) Shulman [**20**] obtained the following strong result.

THEOREM 2.2. *If π is a J-symmetric representation of a C^*-algebra $\mathscr{A}$ on a Π_k-space H $(k < \infty)$, then there are maximal negative and maximal positive subspaces N and P respectively such that $H = N[+]P$ and such that N and P are invariant for π. The representation π is similar to a $*$-representation of $\mathscr{A}$.*

Let π be a J-symmetric representation of a $*$-algebra $\mathscr{A}$ on H, let P be a positive invariant subspace of H and let N be a negative invariant subspace of H. Define scalar products on P and N by the formulas:

$$\langle x, y\rangle_P = [x, y], \quad x, y \in P, \quad \text{and } \langle x, y\rangle_N = -[x, y], \quad x, y \in N.$$

Then P and N become pre-Hilbert spaces. Set $\rho = \pi_P$. Since

$$\langle \rho(A)x, y\rangle_P = [\pi(A)x, y] = [x, \pi(A^*)y] = \langle x, \rho(A^*)y\rangle_P,$$

ρ is a $*$-representation of $\mathscr{A}$ on P. Similarly, π_N is a $*$-representation of $\mathscr{A}$ on N.

If P and N are uniformly positive and uniformly negative, then they are Hilbert spaces and there are positive r and q such that

$$\begin{aligned} &r\|x\|^2 \le \|x\|_P^2 \le \|x\|^2, \qquad x \in P, \text{ where } \|x\|_P^2 = \langle x, x\rangle_P, \\ &q\|x\|^2 \le \|x\|_N^2 \le \|x\|^2, \qquad x \in N, \text{ where } \|x\|_N^2 = \langle x, x\rangle_N. \end{aligned} \tag{3}$$

We have that

$$\begin{aligned} \|\rho(A)\|_P^2 &= \sup(\langle \rho(A)x, \rho(A)x\rangle_P/\langle x, x\rangle_P) \\ &= \sup((J\pi(A)x, \pi(A)x)/\langle x, x\rangle_P) \\ &\le \sup(\|\pi(A)x\|^2/r\|x\|^2) \\ &= \|\pi(A)\|^2/r. \end{aligned}$$

THEOREM 2.3. *Let L and M be uniformly positive (negative) subspaces of H invariant for π.*

(i) *If $M \cap L^{[\perp]} = \{0\}$, then there is an invariant subspace K in L such that the representations π_M and π_K are equivalent, i.e., there is an isometry U from M onto K with respect to the norms $\|\ \|_M$ and $\|\ \|_K$ such that $U\pi_M(A) = \pi_K(A)U$ for all $A \in \mathscr{A}$. If, in addition, $L \cap M^{[\perp]} = \{0\}$, then the representations π_M and π_L are equivalent.*

(ii) *If L and M are maximal uniformly positive (negative) invariant subspaces, then the representations π_M and π_K are equivalent.*

Proof. Let L and M be uniformly positive. Then, by (3), for x in L and y in M,

$$|[x, y]| = |(Jx, y)| \le \|x\|\,\|y\| \le \|x\|_L\|y\|_M/(r_L r_M)^{1/2}. \tag{4}$$

Since $M \cap L^{[\perp]} = \{0\}$, for every $y \neq 0$ in M there is x in L such that $[x, y] \neq 0$. Therefore y generates a non-zero bounded functional $f_y(x) = [x, y]$ on L. Since L is a Hilbert space, there exists a linear operator S from M into L such that $\operatorname{Ker} S = \{0\}$ and such that for all x in L and y in M,

$$[x, y] = \langle x, Sy\rangle_L.$$

Let K be the closure of the linear manifold $\{Sy\colon y \in M\}$. Then

$$\begin{aligned}\langle x, \pi_L(A)Sy\rangle_L &= \langle \pi_L(A^*)x, Sy\rangle_L = [\pi_L(A^*)x, y] \\ &= [\pi(A^*)x, y] = [x, \pi(A)y] = [x, \pi_M(A)y] \\ &= \langle x, S\pi_M(A)y\rangle_L,\end{aligned}$$

so that $\pi_L(A)S|_M = S\pi_M(A)|_M$ for all A in $\mathscr{A}$. Therefore K is invariant for π and $\pi_K S|_M = S\pi_M|_M$.

Let now y_n converge to 0 in M with respect to $\|\ \|_M$ and let Sy_n converge to x in L with respect to $\|\ \|_L$. Then, by (4),

$$|\langle x, Sy_n\rangle_L| = |[x, y_n]| \leq \|x\|_L \|y_n\|_M / (r_L r_M)^{1/2},$$

so that $\langle x, Sy_n\rangle_L$ converge to 0. Therefore $\langle x, x\rangle_L = 0$, so that $x = 0$. Thus S is a closed operator. Since it is defined on the whole space M, it is bounded. From this and from Gelfand's and Naimark's theorem [**13**, §21] it follows that there is an isometry U from M onto K such that $\pi_K U = U\pi_M$.

Let, in addition, $L \cap M^{[\perp]} = \{0\}$. Then, for every $x \neq 0$ in L, there is y in M such that $[x, y] \neq 0$. Therefore $\operatorname{Im} S$ is dense in L, so that $K = L$. Part (i) is proved.

Let L be maximal uniformly positive. By Theorem 2.1(i), $H = L[+]L^{[\perp]}$. If $R = M \cap L^{[\perp]} \neq \{0\}$, then R is a uniformly positive invariant subspace in $L^{[\perp]}$. Therefore L is not maximal. This contradiction shows that $M \cap L^{[\perp]} = \{0\}$. If M is also maximal uniformly positive, then, similarly, $L \cap M^{[\perp]} = \{0\}$. Therefore part (ii) follows from part (i).

Definition. Let π be a J-symmetric representation of a $*$-algebra $\mathscr{A}$ on a Π_k-space H, where $k = \min(k_-, k_+)$. If P is a uniformly positive subspace in H invariant for π, then we define $i_+(P)$ as the maximal number of non-zero mutually orthogonal projections in the commutant of $\pi_P(\mathscr{A})$ in P and we set $d_+(P) = \dim P$. Set

$$d_+(\pi) = \sup_{P \in \mathscr{P}} d_+(P) \quad \text{and} \quad i_+(\pi) = \sup_{P \in \mathscr{P}} i_+(P)$$

where $\mathscr{P}$ is the set of all uniformly positive invariant subspaces in H. Similarly, we define numbers $d_-(\pi)$ and $i_-(\pi)$ by considering the set $\mathscr{N}$ of all uniformly negative invariant subspaces in H. We shall call the sextuple

$$\operatorname{ind}(\pi) = (k_+, k_-, d_+(\pi), d_-(\pi), i_+(\pi), i_-(\pi))$$

the *index* of π.

By law of inertia, $d_+(\pi) \le k_+$ and $d_-(\pi) \le k_-$. It is clear that if representations π and ρ on spaces H and K respectively are J-equivalent, i.e., there exists a bounded operator T from H onto K such that $[Tx, Ty] = [x, y]$, $x, y \in H$, and such that $\rho T = T\pi$, then $\operatorname{ind}(\pi) = \operatorname{ind}(\rho)$.

THEOREM 2.4. (i) *Let H be a separable Π_k-space and let L be a uniformly positive invariant subspace. Then there exist uniformly positive invariant subspaces $\{L_j\}$ such that $L \subseteq L_j$, that $L_j \subseteq L_{j+1}$ and such that $d_+(\pi) = \lim_{j\to\infty} d_+(L_j)$ and $i_+(\pi) = \lim_{j\to\infty} i_+(L_j)$. The same holds if L is a uniformly negative invariant subspace.*

(ii) *If there is a uniformly positive invariant subspace M such that $d_+(M) = d_+(\pi)$ and that $i_+(M) = i_+(\pi)$, then any uniformly positive invariant subspace L is contained in a uniformly positive invariant subspace P such that $d_+(P) = d_+(\pi)$ and that $i_+(P) = i_+(\pi)$. The same holds if M is uniformly negative.*

(iii) *Let H be a Π_k-space such that $k < \infty$ and let π not have null invariant subspaces. Then there exist maximal uniformly positive and maximal uniformly negative invariant subspaces P and N in H such that $d_-(\pi) = d_-(N)$, $i_-(\pi) = i_-(N)$, $d_+(\pi) = d_+(P)$ and $i_+(\pi) = i_+(P)$.*

Proof. Let L be uniformly positive. If $i_+(L) < i_+(\pi)$, then there exists a uniformly positive invariant subspace M in H such that $i_+(L) < i_+(M)$. Set $R = M \cap L^{[\perp]}$. If $R = \{0\}$, then it follows from Theorem 2.3(i) that π_M is equivalent to a subrepresentation of π_L. Therefore $i_+(M) \le i_+(L)$. This contradiction shows that $R \ne \{0\}$. Set $K = L[+]R$. Then K is a uniformly positive invariant subspace, $L \subset K$ and $M \cap K^{[\perp]} = \{0\}$. By Theorem 2.3(i), $d_+(M) \le d_+(K)$ and $i_+(M) \le i_+(K)$.

If $i_+(\pi) = \infty$, then $d_+(\pi) = \infty$. Since H is separable, there are uniformly positive invariant subspaces $\{M_j\}$ such that

$$i_+(\pi) = \lim_{j\to\infty} i_+(M_j).$$

Using the construction above, we obtain uniformly positive invariant subspaces $\{L_j\}$ such that $L_j \subseteq L_{j+1}$ and that $i_+(M_j) \leq i_+(L_j)$. Therefore

$$i_+(\pi) = \lim_{j\to\infty} i_+(L_j) = \infty.$$

Then obviously

$$\lim_{j\to\infty} d_+(L_j) = d_+(\pi) = \infty.$$

If $i_+(\pi) < \infty$, then, making use of the construction at the beginning of the theorem, we obtain a uniformly positive invariant subspace P such that $L \subset P$ and that $i_+(P) = i_+(\pi)$. If $d_+(P) < d_+(\pi)$, then there is a uniformly positive invariant subspace M such that $d_+(P) < d_+(M)$. Using the construction at the beginning of the theorem, we obtain a uniformly positive invariant subspace K such that $P \subset K$ and that $d_+(M) \leq d_+(K)$. Repeating this process, if necessary, we conclude the proof of part (i).

Part (ii) follows easily from the construction at the beginning of the theorem.

Assume that $k = k_-$. Let $\{L_j\}$ be the uniformly positive invariant subspaces as in part (i). Let P be the closure of $\bigcup_j L_j$. Then P is a nonnegative invariant subspace. Since π does not have null invariant subspaces, it follows from Lemma 2.3(iii) **[11]** that P is positive. By Theorem 2.1(iii), P is uniformly positive. Therefore

$$d_+(P) = d_+(\pi) \quad \text{and} \quad i_+(P) = i_+(\pi).$$

The theorem is proved.

Remark 2.5. Even if $0 < k = \min(k_-, k_+) < \infty$, one may find that either one or both of the numbers $d_-(\pi)$ and $d_+(\pi)$ equals 0. If, however, $\mathscr{A}$ is a C^*-algebra, then, by Theorem 2.2, $H = N[+]P$ where N and P are respectively maximal uniformly negative and maximal uniformly positive invariant subspaces. Then, by Theorem 2.4(iii) and by Law of inertia, $d_-(\pi) = \dim N = k_-$ and $d_+(\pi) = \dim P = k_+$. If $H = N_1[+]P_1$ is another decomposition of H, then, by Theorem 2.3, the representations π_N and π_{N_1} are equivalent and the representations π_P and π_{P_1} are equivalent.

Let π be a J-symmetric representation of a $*$-algebra $\mathscr{A}$ on H and assume that $H = N[+]P$ where N and P are respectively uniformly negative and uniformly positive invariant subspaces of H. Let L be a maximal null invariant subspace in H. Then

$$L = \{x + Tx\colon\ x \in L_-\}$$

where L_- is a closed subspace of N invariant for π, T is an isometry from L_- into P $(\langle Tx, Ty\rangle_P = \langle x, y\rangle_N)$ and

$$\pi(A)T|_{L_-} = T\pi(A)|_{L_-} \quad \text{for all } A \text{ in } \mathscr{A}. \tag{5}$$

Set

$$L_+ = \{Tx\colon\ x \in L_-\}, \qquad N_L = N \cap L^{[\perp]} \quad \text{and} \quad P_L = P \cap L^{[\perp]}.$$

From (5) it follows that the representations π_{L_-} and π_{L_+} are equivalent. We also have that

$$N = N_L[+]L_-, \qquad P = P_L[+]L_+ \quad \text{and} \quad L^{[\perp]} = N_L[+]L[+]P_L.$$

The subspaces N_L and P_L are invariant for π.

THEOREM 2.6. *Let π be a J-symmetric representation of a $*$-algebra $\mathscr{A}$ on H and let $H = N[+]P$ where N and P are respectively uniformly negative and positive invariant subspaces. Let L and K be maximal null invariant subspaces, so that $L = \{x + Tx\colon\ x \in L_-\}$ and $K = \{x + Rx\colon\ x \in K_-\}$. Then*

(i) *The representations π_{L_-}, π_{K_-}, π_{L_+} and π_{K_+} are equivalent.*

(ii) *If π_{L_-} is a finite orthogonal direct sum of irreducible representations of $\mathscr{A}$, then the representations π_{N_L} and π_{N_K} are equivalent and the representations π_{P_L} and π_{P_K} are equivalent.*

Proof. Set $M = L \cap K$. Then $M = \{x + Tx\colon\ x \in M_-\}$ where $M_- = \{x \in L_- \cap K_-\colon\ Tx = Rx\}$. Set

$$X = L_-\langle-\rangle M_- \quad \text{and} \quad Y = K_-\langle-\rangle M_-.$$

Then X and Y are closed subspaces in N. Since L and K are invariant for π, M is invariant for π, so that M_- is invariant for π. Since L_- and K_- are invariant for π, X and Y are invariant for π.

The subspace $K \cap L^{[\perp]}$ is a null invariant subspace and $M \subseteq K \cap L^{[\perp]}$. If $K \cap L^{[\perp]} \neq M$, then $L[+](K \cap L^{[\perp]})$ is a null invariant subspace larger than L. Since L is a maximal null invariant subspace, $K \cap L^{[\perp]} = M$. Similarly, $L \cap K^{[\perp]} = M$.

Define a form $Q(x, y)$ on $X \times Y$ by the formula:

$$Q(x, y) = [x + Tx, y + Ry].$$

If for some x in X, $Q(x, y) = 0$ for all y in Y, then $x + Tx \in M$, so that $x \in M_-$. This contradiction shows that $Q(x, y)$ is nondegenerate. Since T and R are isometries, we have that

$$\begin{aligned}|Q(x, y)| &\le |[x, y]| + |[Tx, Ry]| \\ &\le ||x||_N||y||_N + ||Tx||_P||Ry||_P = 2||x||_N||y||_N.\end{aligned}$$

Therefore for every y in Y, $f(x) = Q(x, y)$ is a bounded functional on X. Hence there exists a bounded operator S from Y into X such that

$$Q(x, y) = \langle x, Sy\rangle_N, \qquad x \in X, y \in Y.$$

Since $Q(x, y)$ is nondegenerate, $\mathrm{Ker}(S) = \{0\}$ and $\mathrm{Im}(S)$ is dense in X. Since T and R commute with π,

$$\begin{aligned}\langle x, S\pi(A)y\rangle_N = Q(x, \pi(A)y) &= [x + Tx, \pi(A)y + R\pi(A)y] \\ &= [x + Tx, \pi(A)(y + Ry)] \\ &= [\pi(A^*)(x + Tx), y + Ry] \\ &= [\pi(A^*)x + T\pi(A^*)x, y + Ry] \\ &= Q(\pi(A^*)x, y) = \langle \pi(A^*)x, Sy\rangle_N.\end{aligned}$$

Hence

$$\begin{aligned}\langle \pi(A^*)x, Sy\rangle_N &= -[\pi(A^*)x, Sy] \\ &= -[x, \pi(A)Sy] = \langle x, \pi(A)Sy\rangle_N.\end{aligned}$$

Therefore $S\pi(A)|_Y = \pi(A)S|_Y$. From this and from Gelfand's and Naimark's theorem [**13**, §21] it follows that there is an isometry U from Y onto X such that $U\pi(A)|_Y = \pi(A)U|_Y$. Therefore the representations π_{L_-} and π_{K_-} are equivalent. Similarly, the representations π_{L_+} and π_{K_+} are also equivalent. Since the representations π_{L_-} and π_{L_+} are equivalent, part (i) is proved.

In order to prove part (ii) we shall prove the following lemma.

LEMMA 2.7. *Let π and ρ be equivalent $*$-representations of a $*$-algebra $\mathscr{A}$ on Hilbert spaces H and K respectively. Let H_1 be an invariant subspace of H such that the representation $\pi_1 = \pi_{H_1}$ is irreducible and let K_1 be an invariant subspace of K such that the representation $\rho_1 = \rho_{K_1}$ is irreducible. If π_1 and ρ_1 are equivalent, then the representations $\pi_{H\ominus H_1}$ and $\rho_{K\ominus K_1}$ are equivalent.*

Proof. Let U be an isometry from H onto K such that $U\pi(A) = \rho(A)U$ for all A in $\mathscr{A}$. If $UH_1 = K_1$, the proof is obvious. Let $UH_1 \neq K_1$, let H_2 be the closed span of $H_1 + U^{-1}K_1$ and let K_2 be the closed span of $K_1 + UH_1$. Then H_2 is invariant for π, K_2 is invariant for ρ, $UH_2 = K_2$ and $U\pi|_{H_2} = \rho U|_{H_2}$. Therefore $\pi_{H\ominus H_2}$ is equivalent to $\rho_{K\ominus K_2}$. In order to prove the lemma it is sufficient to show that the representations $\pi_{H_2\ominus H_1}$ and $\rho_{K_2\ominus K_1}$ are equivalent.

Since H_1 and H_2 are invariant for π, $H_2 \ominus H_1$ is invariant for π. Let L and M be subspaces invariant for π. Set $\widetilde{L} =$

$(L \vee M) \ominus M$ and $\widetilde{M} = L \ominus (L \cap M)$. It follows from Proposition 2.1.5 [18] that the representations $\pi_{\widetilde{L}}$ and $\pi_{\widetilde{M}}$ are equivalent. Substituting $U^{-1}K_1$ for L and H_1 for M we obtain that $\widetilde{L} = H_2 \ominus H_1$ and that $\widetilde{M} = U^{-1}K_1 \ominus (U^{-1}K_1 \cap H_1)$. Since π_1 and ρ_1 are irreducible and since $UH_1 \neq K_1$, $U^{-1}K_1 \cap H_1 = \{0\}$. Thus $M = U^{-1}K_1$ and the representations $\pi_{H_2 \ominus H_1}$ and $\pi_{U^{-1}K_1}$ are equivalent. Similarly, we obtain that the representations $\pi_{K_2 \ominus K_1}$ and π_{UH_1} are equivalent. Since π_1 and ρ_1 are equivalent, the representations π_{UH_1} and $\pi_{U^{-1}K_1}$ are equivalent. Therefore $\pi_{H_2 \ominus H_1}$ is equivalent to $\pi_{K_2 \ominus K_1}$. The lemma is proved.

We shall now continue the proof of Theorem 2.6. From Lemma 2.7 it follows that if π_1 and ρ_1 are finite orthogonal direct sums of irreducible representations, then the representations $\pi_{H \ominus H_1}$ and $\rho_{K \ominus K_1}$ are equivalent.

Since $N = N_L[+]L_- = N_K[+]K_-$, it follows from (i) that the representations π_{N_L} and π_{N_K} are equivalent. Similarly, the representations π_{P_L} and π_{P_K} are equivalent. The theorem is proved.

3. Indices of derivations of C^*-algebras. In this section we apply the results of Section 2 to bounded and unbounded $*$-derivations of C^*-algebras implemented by symmetric operators.

Let H be a Hilbert space, let δ be a closed $*$-derivation of a C^*-subalgebra $\mathscr{A}$ of $B(H)$ into $B(H)$ and let a symmetric operator S implement δ, i.e.,

$$AD(S) \subseteq D(S) \quad \text{and} \quad \delta(A)|_{D(S)} = i[S, A]|_{D(S)} \quad \text{for all } A \in D(\delta).$$

Recall that $D(S^*)$ becomes a Hilbert space with respect to the scalar product

$$\{x, y\} = (x, y) + (S^*x, S^*y), \qquad x, y \in D(S^*),$$

and that

$$D(S^*) = D(S) \oplus N_+(S) \oplus N_-(S)$$

is the direct orthogonal sum of the subspaces $D(S)$, $N_+(S)$ and $N_-(S)$ with respect to this scalar product. The subspace $N(S) = N_-(S) \oplus N_+(S)$ becomes an indefinite metric space with the indefinite scalar product

$$[x, y]^S = \{Jx, y\}, \quad x, y \in N(S),$$

where J is the involution on $N(S)$ defined in §1. Then $\dim N_d(S) = n_d(S)$, $d = \pm$, are the deficiency indices of S, and we have that

$[x, x]^S = 2(x, x) > 0$ if $x \in N_+(S)$, and $[x, x]^S = -2(x, x) < 0$ if $x \in N_-(S)$. Thus $N(S)$ decomposes into a simultaneously orthogonal and J-orthogonal sum $N(S) = N_+(S) \dotplus N_-(S)$, where $N_+(S)$ and $N_-(S)$ are respectively uniformly positive and negative subspaces in $N(S)$.

It follows easily that for every A in $D(\delta)$

$$AD(S^*) \subseteq D(S^*) \quad \text{and} \quad \delta(A)|_{D(S^*)} = i[S^*, A]|_{D(S^*)}.$$

Set $|||x|||^2 = \{x, x\}$ for $x \in D(S^*)$. Then

$$\begin{aligned}(6)\quad |||Ax|||^2 &= (Ax, Ax) + (S^*Ax, S^*Ax) \\ &= ||Ax||^2 + (AS^*x, AS^*x) \\ &\quad + (\delta(A)x, \delta(A)x) \leq ||A||^2|||x|||^2 + ||\delta(A)||^2||x||^2 \\ &\leq (||A||^2 + ||\delta(A)||^2)|||x|||^2.\end{aligned}$$

Therefore $D(\delta)$ acts as an algebra of bounded operators on $D(S^*)$. Let Q be the projection onto $N(S)$ in $D(S^*)$. Since $D(S)$ is invariant for $D(\delta)$, we have that

$$\pi_S(A) = QAQ, \quad A \in D(\delta),$$

is a representation of $D(\delta)$ on $N(S)$.

THEOREM 3.1 ([**9**]). (i) (*cf.* [**8**]) *π_S is a J-symmetric representation of $D(\delta)$ onto $N(S)$.*

(ii) *There is a one-to-one correspondence between closed symmetric δ-extensions of S and closed null subspaces in $N(S)$ invariant for π_S.*

(iii) *There is a maximal symmetric implementation T of δ which δ-extends S. The representation π_T does not have null invariant subspaces in $N(T)$.*

(iv) *Let S be a maximal symmetric implementation of δ. If*

$$\max(n_-(S), n_+(S)) < \infty$$

or if $\mathscr{A}$ is commutative and $\min(n_-(S), n_+(S)) < \infty$ *then π_S extends to a bounded representation of $\mathscr{A}$ onto $N(S)$.*

Let P and N be respectively uniformly positive and uniformly negative subspaces in $N(S)$ invariant for π_S. Then they become Hilbert spaces with respect to the scalar products $\langle x, y\rangle_P = [x, y]^S$, $x, y \in P$, and $\langle x, y\rangle_N = -[x, y]^S$, $x, y \in N$. Let π_P and π_N be the restrictions of the representation π_S to P and N respectively. Then π_P and π_N are $*$-representations of $D(\delta)$.

From Theorems 2.2 and 3.1 we obtain the following theorem.

THEOREM 3.2. *Let S be a maximal symmetric implementation of δ and let $n = \min(n_-(S), n_+(S)) < \infty$.*

(i) *Let $D(\delta) = \mathscr{A}$ (δ is a bounded derivation) or let π_S extend to a bounded J-symmetric representation of $\mathscr{A}$. Then*

(1) *$N(S) = N[+]P$ where N and P are respectively uniformly negative and uniformly positive subspaces invariant for π_S.*

(2) *Let Z be the maximal subspace in $N(S)$ such that $\pi_S|_Z = 0$ (if, for example, $1 \in \mathscr{A}$, then $Z = \{0\}$.) Then either $Z \subseteq P$ or $Z \subseteq N$.*

(3) *Assume that $n = n_-(S)$. Then there are finite-dimensional irreducible representations $\{\pi_i\}_{i=1}^p$ of $\mathscr{A}$ such that*

$$\pi_S|_N = \begin{cases} \sum_{i=1}^p \oplus \pi_i, & \text{if } Z \subseteq P, \\ \left(\sum_{i=1}^p \oplus \pi_i\right) \oplus \pi_S|_Z, & \text{if } Z \subseteq N. \end{cases}$$

If also $n_+(S) < \infty$, then there are finite-dimensional irreducible representations $\{\rho_j\}_{j=1}^m$ of $\mathscr{A}$ such that

$$\pi_S|_P = \begin{cases} \left(\sum_{j=1}^m \oplus \rho_j\right) \oplus \pi_S|_Z, & \text{if } Z \subseteq P, \\ \sum_{j=1}^m \oplus \rho_j, & \text{if } Z \subseteq N. \end{cases}$$

The sets $\{\pi_i\}$ and $\{\rho_j\}$ are disjoint.

(ii) *Let $D(\delta) \neq \mathscr{A}$ and let π_S be nondegenerate. If $N(S)$ is the closure of $N[+]P$ where N and P are respectively negative and positive closed subspaces invariant for π_S, then π_S extends to a bounded representation of $\mathscr{A}$ and $N(S) = N[+]P$.*

Proof. It follows from (6) that $|||A|||^2 \leq ||A||^2 + ||\delta(A)||^2$, where $|||A|||$ is the norm of an operator A in $D(S^*)$ with respect to the scalar product $\{\ ,\ \}$. If $D(\delta) = \mathscr{A}$, then, since δ is closed, δ is bounded. Therefore

$$|||A|||^2 \leq ||A||^2(1 + ||\delta||^2).$$

Since $||\pi_S(A)|| \leq |||Q|||^2|||A||| = |||A|||$, π_S is a bounded representation of $\mathscr{A}$. Since $\min(n_-(S), n_+(S)) < \infty$, it follows from Theorems 2.1 and 2.2 that $N(S) = N[+]P$, where N and P are respectively uniformly negative and uniformly positive invariant subspaces. Part (i)(1) is proved.

If $x + y \in Z$, $x \in N$, $y \in P$, then $\pi_S(A)x = 0$ and $\pi_S(A)y = 0$. Since Z is maximal, x and y belong to Z. Therefore $Z = Z_N[+]Z_P$ where $Z_N = Z \cap N$ and $Z_P = Z \cap P$. Since S is a maximal symmetric implementation of δ, by Theorem 3.1(iii), π_S does not have null invariant subspaces. Therefore either $Z \subseteq N$ or $Z \subseteq P$. Part (i)(2) is proved.

Let $n = n_-(S)$ and let $Z \subseteq N$. Then the representation π_S is nondegenerate on $N \ominus Z$ and $*$-symmetric with respect to the definite scalar product $\langle x, y\rangle_N = -[x, y]^S$. Since $N \ominus Z$ is finite-dimensional, there are finite-dimensional representations $\{\pi_i\}_{i=1}^p$ of $\mathscr{A}$ such that $\pi_S|_{N\ominus Z} = \sum_{i=1}^p \oplus\pi_i$. If $n_+(S) < \infty$, then similarly there are finite-dimensional representations $\{\rho_j\}_{j=1}^m$ of $\mathscr{A}$ such that $\pi_S|_P = \sum_{j=1}^m \oplus\rho_j$.

Let $\pi_i = \pi_S|_{L_i}$ be equivalent to $\rho_j = \pi_S|_{K_j}$ where $L_i \subseteq N$ and $K_j \subseteq P$. Let U be the isometry from L_i onto K_j such that $U\pi_i = \rho_j U$. Then the subspace $M = \{x + Ux\colon x \in L_i\}$ is a null subspace in $N(S)$ invariant for π_S, since

$$\begin{aligned}[x + Ux, x + Ux]^S &= [x, x]^S + [Ux, Ux]^S \\ &= -\langle x, x\rangle_N + \langle Ux, Ux\rangle_P = 0\end{aligned}$$

and since

$$\begin{aligned}\pi_S(A)(x + Ux) &= \pi_i(A)x + \rho_j(A)Ux \\ &= \pi_i(A)x + U\pi_i(A)x \in M\end{aligned}$$

for all $x \in L_i$ and all $A \in D(\delta)$. Since S is a maximal symmetric implementation of δ, by Theorem 3.1(iii), π_S does not have null invariant subspaces. Therefore the sets $\{\pi_i\}$ and $\{\rho_i\}$ are disjoint. Part (i) is proved.

Let now $D(\delta) \neq \mathscr{A}$. Since P is positive, by Theorem 2.1, P is uniformly positive and $N(S) = P[+]P^{[\perp]}$. By Law of inertia, $\dim(N) \leq n_-(S) < \infty$. Therefore, since $N \subseteq P^{[\perp]}$, either $N = P^{[\perp]}$ or there is x in $P^{[\perp]}$ which is J-orthogonal to N. If such an x exists, it is J-orthogonal to $N[+]P$ and therefore it is J-orthogonal to H. This contradiction shows that $N = P^{[\perp]}$, so that $H = N[+]P$.

From Lemma 4 [20] it follows that π_S is similar to a $*$-representation of $D(\delta)$. Therefore π_S extends to a bounded representation of $\mathscr{A}$ which completes the proof of the theorem.

If $N(S) = N[+]P$, then, by Law of inertia, $\dim N = n_-(S)$ and $\dim P = n_+(S)$. From this and from Theorem 3.2 we obtain the following corollary.

COROLLARY 3.3. *Let the conditions of Theorem* 3.2(i) *hold and let* $q = \dim Z$. *Then*

$$n_-(S) = \begin{cases} \sum_{i=1}^p \dim \pi_i, & \textit{if } Z \subseteq P, \\ \sum_{i=1}^p \dim \pi_i + q, & \textit{if } Z \subseteq N. \end{cases}$$

If, in addition, $n_+(S) < \infty$, *then*

$$n_+(S) = \begin{cases} \sum_{j=1}^{m} \dim \rho_j + q, & \text{if } Z \subseteq P, \\ \sum_{j=1}^{m} \dim \rho_j, & \text{if } Z \subseteq N. \end{cases}$$

DEFINITION. Let now S be a symmetric implementation of a $*$-derivation δ of a C^*-algebra $\mathscr{A}$ into $B(H)$. Then π_S is a J-symmetric representation of $D(\delta)$ on $N(S)$. We shall call the sextuple

$$\begin{aligned} \operatorname{ind}(\delta, S) &= \operatorname{ind}(\pi_S) \\ &= (n_+(S), n_-(S), d_+(\pi_S), d_-(\pi_S), i_+(\pi_S), i_-(\pi_S)) \end{aligned}$$

the *index of* δ *relative to* S.

From Remark 2.5 and from Theorem 3.2(i) we obtain the following lemma.

LEMMA 3.4. (i) *If* $\max(n_+(S), n_-(S)) < \infty$, *then* $d_+(\pi_S) = n_+(S)$ *and* $d_-(\pi_S) = n_-(S)$.

(ii) *If* $\min(n_+(S), n_-(S)) < \infty$ *and if either* $D(\delta) = \mathscr{A}$ *or the representation* π_S *extends to a bounded representation of* $\mathscr{A}$, *then* $d_+(\pi_S) = n_+(S)$ *and* $d_-(\pi_S) = n_-(S)$.

REMARK 3.5. If $n_-(S) = 0$, so that S is a maximal symmetric operator, then $i_+(\pi_S)$ is the index introduced by Powers [**16**].

Let S be a symmetric implementation of a derivation δ of a C^*-subalgebra $\mathscr{A}$ of $B(H)$ into $B(H)$ and let B be a selfadjoint bounded operator. Then the operator $T = S + B$ is a symmetric implementation of the $*$-derivation $\sigma(A) = \delta(A) + i[B, A]$ of $\mathscr{A}$ into $B(H)$. Then $D(\sigma) = D(\delta)$.

THEOREM 3.6. (i) *The representations* π_S *and* π_T *of* $D(\delta)$ *are* J*-equivalent, i.e., there exists a bounded operator* U *from* $N(S)$ *onto* $N(T)$ *such that* $\pi_T U = U\pi_S$ *and such that* $[Ux, Uy]^T = [x, y]^S$ *for all* $x, y \in N(S)$.

(ii) $\operatorname{ind}(\delta, S) = \operatorname{ind}(\sigma, T)$.

Proof. It is well-known (see [**1**, §100]) that $n_+(S) = n_+(T)$ and that $n_-(S) = n_-(T)$. We shall consider a quadratic form $\langle\langle\ ,\ \rangle\rangle^S$ on $D(S^*)$, given by

$$\langle\langle x, y\rangle\rangle^S = i((x, S^*y) - (S^*x, y)), \qquad x, y \in D(S^*),$$

(see [4], [8]). Given any x and y in $D(S^*)$ and decomposing them

$$x = x_0 + x_+ + x_- \quad \text{and} \quad y = y_0 + y_+ + y_-,$$

where $x_0, y_0 \in D(S)$, $x_+, y_+ \in N_+(S)$ and $x_-, y_- \in N_-(S)$, we obtain that

$$\langle\langle x, y\rangle\rangle^S = 2(x_+, y_+) - 2(x_-, y_-) = [x_+ + x_-, y_+ + y_-]^S. \tag{7}$$

We have that $D(S^*) = D(T^*)$ and that $T^* = S^* + B$. It is clear that

$$\langle\langle x, y\rangle\rangle^S = \langle\langle x, y\rangle\rangle^T, \quad \text{if } x, y \in D(S^*)$$

and that

$$\langle\langle x, y\rangle\rangle^S = 0 \quad \text{if } x, y \in D(S).$$

Therefore the forms $\langle\langle\ ,\ \rangle\rangle^S$ and $\langle\langle\ ,\ \rangle\rangle^T$ generate the same indefinite scalar product on the quotient space $D(S^*)/D(S) = D(T^*)/D(T)$.

Let Q_S and Q_T be the projections onto $N(S)$ and onto $N(T)$ respectively in $D(S^*)$. Then it follows from (7) that for all $x, y \in D(S^*)$,

$$[Q_S x, Q_S y]^S = \langle\langle x, y\rangle\rangle^S = \langle\langle x, y\rangle\rangle^T = [Q_T x, Q_T y]^T. \tag{8}$$

For $x \in N(S)$, set $Ux = Q_T x$. Since $Q_T D(S^*) = N(T)$ and since $Q_T D(S) = \{0\}$, U is a bounded operator which maps $N(S)$ onto $N(T)$. By (8),

$$[x, y]^S = [Ux, Uy]^T.$$

Decomposing any x in $D(S^*)$, $x = y + z$, where $y \in D(S)$ and $z \in N(S)$, we obtain that

$$Q_T Q_S x = Q_T Q_S(y + z) = Q_T z = Q_T(y + z) = Q_T x.$$

Therefore, for any x in $N(S)$ and for any A in $D(\delta)$,

$$U\pi_S(A)x = Q_T Q_S A Q_S x = Q_T Q_S A x = Q_T A x.$$

Since $D(S)$ is invariant for A, $Q_T A = Q_T A Q_T$. Hence

$$U\pi_S(A)x = Q_T A x = Q_T A Q_T x = \pi_T(A)Ux.$$

Thus part (i) is proved. Part (ii) follows from (i).

THEOREM 3.7. *Let S and T be maximal symmetric implementations of δ and let $D = D(S) \cap D(T)$ be dense in H. Set $R = S|_D$. Then R is a symmetric implementation of δ. Let*

(1) $\min(n_+(R), n_-(R)) < \infty$,
(2) $(T - S)|_D$ *extends to a bounded operator* B,

(3) *either* $D(\delta) = \mathscr{A}$ *or* π_R *extends to a bounded representation of* $\mathscr{A}$.

Then the representations π_S *and* π_T *are* J*-equivalent, so that* $\operatorname{ind}(\delta, S) = \operatorname{ind}(\delta, T)$.

Proof. We have that $AD \subseteq D$ for all $A \in D(\delta)$. Therefore R is a symmetric implementation of δ and

$$\delta(A)|_D = i[S, A]|_D = i[T, A]|_D.$$

Hence B belongs to the commutant $\mathscr{A}'$ of $\mathscr{A}$ and

$$R \subseteq S \quad \text{and} \quad R \subseteq T - B.$$

Set $F = T - B$. Then F is a maximal symmetric implementation of δ, $D(T) = D(F)$ and $R = F|_D$. If $D(\delta) = \mathscr{A}$ or if π_R extends to a bounded representation of $\mathscr{A}$, then, by Theorem 2.2, $N(R) = P[+]N$ where P and N are respectively uniformly positive and uniformly negative subspaces invariant for π_R. By Theorem 3.1(ii), there is a maximal null invariant subspace L in $N(R)$ which corresponds to S. Then $L = \{x + Ux\colon x \in L_-\}$ where L_- is a subspace in N invariant for π_R and U is an isometry from L_- into P, i.e., $\langle Ux, Ux\rangle_P = \langle x, x\rangle_N$. Since $\min(n_+(R), n_-(R)) < \infty$, L is finite-dimensional.

In the same way as in Theorem 2.6 set

$$N_L = N \cap L^{[\perp]} \quad \text{and} \quad P_L = P \cap L^{[\perp]}.$$

Then

$$N = N_L[+]L_-, \quad P = P_L[+]L_+ \quad \text{and} \quad L^{[\perp]} = N_L[+]L[+]P_L$$

where $L_+ = \{Ux\colon x \in L_-\}$. It is easy to see that

$$N(S) = N_L[+]P_L \quad \text{and that} \quad \pi_S = \pi_R|_{N(S)}$$

where N_L and P_L are respectively uniformly negative and positive subspaces invariant for π_S.

Similarly, there is a maximal null invariant subspace $K = \{x + Vx\colon x \in K_-\}$ in $N(R)$ which corresponds to F, where K_- is a finite-dimensional subspace in N invariant for π_R and where V is isometry from K_- into P. Then, as above, $N(F) = N_K[+]P_K$, where $N_K = N \cap K^{[\perp]}$ and $P_K = P \cap K^{[\perp]}$ are respectively uniformly negative and uniformly positive subspaces invariant for π_F.

It follows from Theorem 2.6 that the representations $(\pi_S)_{N_L} = (\pi_R)_{N_L}$ and $(\pi_F)_{N_K} = (\pi_R)_{N_K}$ are equivalent and that the representations $(\pi_S)_{P_L} = (\pi_R)_{P_L}$ and $(\pi_F)_{P_K} = (\pi_R)_{P_K}$ are equivalent. Therefore the representations π_S and π_F are J-equivalent, i.e., there exists a bounded operator U from $N(S)$ onto $N(F)$ such that $U\pi_S = \pi_F U$ and $[Ux, Uy]^F = [x, y]^S$ for all $x, y \in N(S)$. By Theorem 3.6, the representations π_T and π_F are J-equivalent, so that π_S and π_T are J-equivalent. The theorem is proved.

DEFINITION. We say that a symmetric implementation T of a $*$-derivation δ from a C^*-subalgebra $\mathscr{A}$ of $B(H)$ into $B(H)$ is *minimal* if for every symmetric implementation S of δ there is a bounded selfadjoint operator B in the commutant of $\mathscr{A}$ such that $T+B \subseteq S$.

In **[10]** it was proved that δ has a minimal implementation if $\mathscr{A}$ contains the algebra $C(H)$ of all compact operators. From this and from Theorem 3.7 we obtain the following theorem.

THEOREM 3.8. *Let δ be a $*$-derivation of a C^*-subalgebra $\mathscr{A}$ of $B(H)$ into $B(H)$. If δ has a minimal implementation T (for example if $C(H) \subseteq \mathscr{A}$), if $\min(n_+(T), n_-(T)) < \infty$ and if either $D(\delta) = \mathscr{A}$ or π_T extends to a bounded representation of $\mathscr{A}$, then the representations π_S and π_{S_1} are J-equivalent for all maximal symmetric implementations S and S_1 of δ, so that* $\operatorname{ind}(\delta, S) = \operatorname{ind}(\delta, S_1)$.

4. Isomorphism of symmetric operators. We shall apply the results about $*$-derivations of C^*-algebras to the investigation of symmetric operators. Every densely defined symmetric operator S has a $*$-algebra associated with it:

$$\mathscr{B}_S = \{A \in B(H) \colon AD(S) \subseteq D(S),\ A^*D(S) \subseteq D(S) \text{ and } (SA - AS)|_{D(S)} \text{ extends to a bounded operator}\}.$$

By $\mathscr{A}_S$ we denote the norm closure of $\mathscr{B}_S$. Then $\mathscr{A}_S$ is a C^*-algebra, $\delta_S(A)|_{D(S)} = i[S, A]|_{D(S)}$ is a closed $*$-derivation from $\mathscr{A}_S$ into $B(H)$ and $D(\delta_S) = \mathscr{B}_S$. If S implements a $*$-derivation δ of a C^*-subalgebra $\mathscr{A}$ of $B(H)$ into $B(H)$, then $D(\delta) \subseteq \mathscr{B}_S$ and $\mathscr{A} \subseteq \mathscr{A}_S$. Thus $\mathscr{A}_S$ is the largest C^*-subalgebra of $B(H)$ on which S generates a closed $*$-derivation and π_S is a J-symmetric representation of $\mathscr{B}_S$ on $N(S)$.

Problems. (i) Is S always a maximal symmetric implementation of δ_S? In other words, does $\pi_S(\mathscr{B}_S)$ have null invariant subspaces in $N(S)$ or not? If $\pi_S(\mathscr{B}_S)$ has such subspaces, there exists a maximal

δ_S-extension T of S such that $\mathscr{B}_S \subseteq \mathscr{B}_T$ and that $\pi_T(\mathscr{B}_S)$ does not have null invariant subspaces in $N(T)$.

(ii) Let $\pi_S(\mathscr{B}_S)$ have no null invariant subspaces in $N(S)$. Assume also that π_S extends to a bounded J-symmetric representation $\widetilde{\pi}_S$ of $\mathscr{A}_S$ and that $N(S) = N[+]P$ where N and P are respectively uniformly negative and positive invariant subspaces for $\widetilde{\pi}_S$. Are the restrictions of $\widetilde{\pi}_S$ to N and P always irreducible?

Symmetric operators S and T on H and H_1 respectively are isomorphic if there exists an isometry V from H onto H_1 such that

$$VD(S) = D(T) \quad \text{and} \quad VS|_{D(S)} = TV|_{D(S)}. \tag{9}$$

Ginzburg **[5]** and Phillips **[14]** showed that in any Π_k-space H there is a one-to-one correspondence between maximal nonpositive subspaces N in H and operators K from H_- into H_+ such that $||K|| \leq 1$: $N = \{x + Kx\colon\ x \in H_-\}$. If, in addition, N is uniformly negative, then $||K|| < 1$.

For every symmetric operator S we denote by $\mathscr{K}(S)$ the set of all operators K from the Hilbert space $N_-(S)$ into the Hilbert space $N_+(S)$ (with respect to the scalar product $\{\ ,\ \}$) such that $|||K||| < 1$ ($|||K|||$ is the norm of an operator K in $N(S)$ with respect to the scalar product $\{\ ,\ \}$) and such that the subspaces $\{x + Kx\colon\ x \in N_-(S)\}$ are invariant for the representation π_S of the algebra $\mathscr{B}_S$.

The following lemma gives necessary conditions for two symmetric operators to be isomorphic in terms of the representations π_S of the algebras $\mathscr{B}_S$ and in terms of the sets $\mathscr{K}(S)$.

LEMMA 4.1. *Let symmetric operators S on H and T on L be isomorphic and let V be the isometry from H onto L such that $VS = TV$. then $V\mathscr{B}_S V^* = \mathscr{B}_T$ and there exists an isometry U from $N(S)$ onto $N(T)$ ($|||Ux||| = |||x|||$, $x \in N(S)$) such that $UN_d(S) = N_d(T)$, $d = \pm$, and such that*

$$\pi_T(VAV^*) = U\pi_S(A)U^*, \qquad A \in \mathscr{B}_S,$$

and

$$\mathscr{K}(T) = U\mathscr{K}(S)U^* = \{UKU^*\colon\ K \in \mathscr{K}(S)\}.$$

Proof. We have that $V^*V = 1_H$ and $VV^* = 1_L$. From this and from (9) we obtain that

$$V^*D(T^*) = D(S^*), \qquad V^*D(T) = D(S), \qquad S^*V^*|_{D(T)} = V^*T^*|_{D(T)},$$
$$VD(S^*) = D(T^*), \quad SV^*|_{D(T)} = V^*T|_{D(T)}, \quad VS^*|_{D(S)} = T^*V|_{D(S)}.$$

Therefore it follows immediately that

$$VN_d(S) = N_d(T) \quad \text{and} \quad V^*N_d(T) = N_d(S), \quad d = \pm,$$

and that

$$V\mathcal{B}_S V^* = \mathcal{B}_T \quad \text{and} \quad V\mathcal{A}_S V^* = \mathcal{A}_T.$$

We also have that for $x, y \in D(S^*)$,

$$\begin{aligned}\{Vx, Vy\} &= (Vx, Vy) + (T^*Vx, T^*Vy) \\ &= (x, y) + (VS^*x, VS^*y) \\ &= (x, y) + (S^*x, S^*y) = \{x, y\}.\end{aligned}$$

Therefore V generates an isometry $U = Q_T V Q_S$ from $N(S)$ onto $N(T)$, where Q_S is the projection onto $N(S)$ in $D(S^*)$ and where Q_T is the projection onto $N(T)$ in $D(T^*)$. Since $VQ_S = Q_T V$,

$$\begin{aligned}\pi_T(VAV^*) &= Q_T VAV^* Q_T \\ &= Q_T V Q_S A Q_S V^* Q_T = U\pi_S(A)U^* \quad \text{for all } A \in \mathcal{B}_S.\end{aligned}$$

Let $K \in \mathcal{K}(S)$. Then $|||K||| < 1$ and the subspace $N = \{x + Kx\colon x \in N_-(S)\}$ is invariant for the representation π_S of the algebra $\mathcal{B}_S$. Set $K^1 = UKU^*$. Then $|||K^1||| < 1$ and the subspace $M = UN = \{y + K^1y\colon y \in N_-(T)\}$ is invariant for the representation π_T of the algebra $\mathcal{B}_T$, since

$$\pi_T(VAV^*)M = U\pi_S(A)U^*UN = U\pi_S(A)N \subseteq UN = M$$

for all $A \in \mathcal{B}_S$. Therefore $K^1 \in \mathcal{K}(T)$.

If $K^1 \in \mathcal{K}(T)$, similarly we obtain that $U^*K^1U = K$ belongs to $\mathcal{K}(S)$ which concludes the proof of the lemma.

It follows from Lemma 4.1 that in order to prove that two symmetric operators S and T are not isomorphic it is sufficient to show that there does not exist an isometry U from $N(S)$ onto $N(T)$ such that $UN_d(S) = N_d(T)$, $d = \pm$, and such that $\mathcal{K}(T) = U\mathcal{K}(S)U^*$.

We shall now consider symmetric operators S such that $n_+(S) = n_-(S) = 1$. We shall also assume that the representations π_S of $\mathcal{B}_S$ on $N(S)$ do not have null invariant subspaces. By Theorem 3.1(iv), π_S extend to bounded representations of C^*-algebras $\mathcal{A}_S$. It follows from Theorem 3.2 that $N(S) = N[+]P$ where N and P are respectively negative and positive subspaces invariant for π_S and that the representations $\pi_S|_N$ and $\pi_S|_P$ are not equivalent. Then N and P are the only subspaces in $N(S)$ invarinat for π_S, $\dim N = \dim P = 1$

and $N = \{x + Kx \colon x \in N_-(S)\}$, where K are operators from $N_-(S)$ into $N_+(S)$ such that $|||K||| < 1$. Set

$$\beta(S) = |||K||.$$

Then $0 \leq \beta(S) < 1$ and from Lemma 4.1 it follows that $\beta(S) = \beta(T)$ if S and T are isomorphic.

For every $\lambda \in [0, 1)$, we shall construct a symmetric operator S such that $n_-(S) = n_+(S) = 1$ and such that $\beta(S) = \lambda$. The question arises as to whether $\beta(S)$ classifies up to isomorphism all the symmetric operators S such that $n_+(S) = n_-(S) = 1$ and such that π_S do not have null invariant subspaces.

It is easy to construct a symmetric operator S such that $\beta(S) = 0$. Let

$$S_+ = i\frac{d}{dx},$$
$$D(S_+) = \{y(x) \colon y \text{ and } y' \text{ in } L_2(-\infty, 0),\ y(-\infty) = y(0) = 0\},$$
$$S_- = i\frac{d}{dx},$$
$$D(S_-) = \{y(x) \colon y \text{ and } y' \text{ in } L_2(0, \infty),\ y(0) = y(\infty) = 0\}.$$

Set $S = S_+ \oplus S_-$ on $H = L_2(-\infty, 0) \oplus L_2(0, \infty)$. Then $n_+(S) = n_-(S) = 1$ and it can be shown that $N_+(S)$ and $N_-(S)$ are invariant for π_S. Therefore $K = 0$, so that $\beta(S) = 0$.

Let us consider the following symmetric differential operators

$$S_a = i\frac{d}{dx},$$
$$D(S_a) = \{y(x) \colon y \text{ and } y' \text{ in } L_2(0, a),\ y(0) = y(a) = 0\},$$

$0 < a < \infty$. It is well-known that $n_-(S_a) = n_+(S_a) = 1$ for all $0 < a < \infty$. Schmudgen **[19]** showed that S_a and S_b are not isomorphic if $a \neq b$. Using Lemma 4.1 we shall give another proof of this result and show that $0 < \beta(S_a) = e^{-a} < 1$, so that $\beta(S)$ takes all values in $[0, 1)$.

THEOREM 4.2. *For every $a \neq 0$, the representation π_{S_a} of $\mathscr{B}_{S_a}$ does not have null invariant subspaces and $\beta(S_a) = e^{-a}$. The symmetric operators S_a and S_b are only isomorphic if $a = b$.*

Proof. We have that

$$(S_a)^* = i\frac{d}{dx} \quad \text{and} \quad D((S_a)^*) = \{y(x) \colon y \text{ and } y' \text{ in } L_2(0, a)\}.$$

Set $h = h(x) = e^x$ and $g = g(x) = e^{a-x}$. Then

$$h(x),\ g(x) \in D((S_a)^*),\qquad (S_a)^* h(x) = ih(x) \text{ and } (S_a)^* g(x) = -ig(x),$$

so that $N_-(S_a) = \{g(x)\}$ and $N_+(S_a) = \{h(x)\}$. We also have that

$$\begin{aligned} |||h|||^2 &= ||h(x)||^2 + ||S_a^* h(x)||^2 \\ &= 2||h(x)||^2 = |||g|||^2 = e^{2a} - 1. \end{aligned} \tag{10}$$

Let A be the bounded operator of multiplication by x, i.e., $Ay(x) = xy(x)$. Then

$$AD(S_a) \subseteq D(S_a) \quad \text{and} \quad i[S_a, A]|_{D(S_a)} = -1|_{D(S_a)}.$$

Therefore $A \in \mathscr{B}_{S_a}$. Set

$$\begin{aligned} y(x) &= h(x) - e^{-a} g(x) = e^x - e^{-x}, \\ z(x) &= g(x) - e^{-a} h(x) = e^{a-x} - e^{x-a}. \end{aligned} \tag{11}$$

Then $y(x)$ and $z(x)$ form a basis in $N(S_a)$ and

$$\begin{aligned} Ay(x) &= x(e^x - e^{-x}) = a(e^x - e^{-x}) + f(x) = ay(x) + f(x), \\ Az(x) &= x(e^{a-x} - e^{x-a}) = q(x), \end{aligned}$$

where the functions $f(x)$ and $q(x)$ belong to $D(S_a)$. Therefore

$$\pi_{S_a}(A) y(x) = y(x) \quad \text{and} \quad \pi_{S_a}(A) z(x) = 0.$$

Since g and h are J-orthogonal, we have that

$$\begin{aligned} [y, y]^{S_a} &= [h, h]^{S_a} + e^{-2a}[g, g]^{S_a} = |||h|||^2 - e^{-2a}|||g|||^2 \\ &= (e^{2a} - 1)(1 - e^{-2a}) > 0 \end{aligned}$$

and

$$[z, z]^{S_a} = [g, g]^{S_a} + e^{-2a}[h, h]^{S_a} = (e^{2a} - 1)(e^{-2a} - 1) < 0.$$

Therefore the subspaces $P = \{y(x)\}$ and $N = \{z(x)\}$ are respectively positive and negative subspaces in $N(S_a)$ invariant for $\pi_{S_a}(A)$. Moreover, they are the only subspaces in $N(S_a)$ invariant for $\pi_{S_a}(A)$. Therefore $\pi_{S_a}(\mathscr{B}_{S_a})$ does not have null invariant subspaces and it follows from Theorem 3.2 that the subspaces N and P are invariant for the representation π_{S_a} of the algebra $\mathscr{B}_{S_a}$. Thus $\mathscr{K}(S_a)$ consists of only one operator K and, by (11),

$$Kg(x) = -e^{-a}h(x).$$

It follows from (10) that $|||K||| = e^{-a}$. Thus $0 < \beta(S_a) < 1$.

If $a \neq b$, $\beta(S_a) \neq \beta(S_b)$, so that S_a and S_b are not isomorphic. The theorem is proved.

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