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THE EFFECT OF DIMENSION ON CERTAIN GEOMETRIC PROBLEMS OF IRREGULARITIES OF DISTRIBUTION

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Suppose that equal numbers of red and blue points, all distinct, lie in the euclidean space $\mathbf{E}^t$, and consider a hyperplane h containing none of the points. If H is one of the open halfspaces determined by h, let $D(h)$ denote $|r(h) - b(h)|$ where $r(h)$ and $b(h)$ are the numbers of red and blue points lying in H. What can be said about the number $\sup D(h)$ as h ranges over all hyperplanes? The present article addresses this and similar problems of discrepancy principally by developing estimates of L^2 integral averages of $D(h)$ with respect to the invariant measure on the planesets of $\mathbf{E}^t$. Special attention is given to the influence of the dimension t.

The aim is to develop inequalities that involve only absolute constants and simple geometric properties of a given pointmass distribution. For example, the following theorem is an immediate corollary to more general results in this article.

THEOREM A. *Let $p_1, p_2, \dots, p_N$ span $\mathbf{E}^t$ and be two-colored as described above. Then there is an absolute positive constant c such that*

$$\sup D(h) \geq c \max\{t, (\delta/\rho)^{1/2} t^{-1/4} [\min(\log N, t)]^{-3/4} \sqrt{N}\}$$

where δ is the minimum distance between distinct points and ρ is the maximum distance, or diameter, of the pointset.

The investigation continues that in [**A1**], but it also draws upon a number of results in [**A2**]. The present work differs markedly from the earlier in that the dimension of the space is taken as a variable. This type of problem can be generalized to convex bodies other than halfspaces, but in this article we shall focus our attention on halfspaces. This seems to be a fundamental setting in which to study the relationship between irregularities of distribution and convexity. Moreover, the methods developed in [**A1**] and [**A2**] may be applied directly to this problem. For an excellent recent report on estimates of discrepancy concerning a wide variety of geometric shapes the reader is referred to the book [**BC**] by J. Beck and W. W. L. Chen.

To formalize these concepts, let ν be a signed Borel measure of total mass 0 supported by a compact subset of $\mathbf{E}^t$. One defines the *separation discrepancy* $D_s(\nu)$ by

$$D_s(\nu) = \sup\{|\nu H| : H \text{ an open halfspace of } \mathbf{E}^t\}. \tag{1}$$

The first question concerning the colored points is equivalent to asking for $D_s(\nu)$ where ν is an atomic measure that assigns weight $+1$ or -1 to each atom p_i. Allowing the plane h to contain atoms of ν will not affect the number $D_s(\nu)$.

Let $\psi = \psi^+ - \psi^-$ be the Hahn decomposition of the signed Borel measure ψ into positive and negative parts, and let the norm $\|\psi\|$ denote the number $|\psi|\mathbf{E}^t = \psi^+\mathbf{E}^t + \psi^-\mathbf{E}^t$. If $\mathbf{B}$ is a subset of $\mathbf{E}^t$, $\Psi^*(\mathbf{B})$ will denote those signed measures ψ of finite norm having compact support in $\mathbf{B}$ and $\Psi(\mathbf{B})$ will denote those ψ in $\Psi^*(\mathbf{B})$ having a total mass 0, i.e., $\psi\mathbf{E}^t = 0$.

The basic questions raised in the first paragraph above could be asked for various classes of Borel measures ν, for example ν in $\Psi(\mathbf{E}^t)$ with $\|\nu\| = c$, a constant. However it turns out that the technicalities of proofs vary along with the structure of ν. In **[A2]** the author treats the more classical family of ν in discrepancy theory, namely those ν in $\Psi(\mathbf{E}^t)$ where ν^+ is Lebesgue t-measure restricted to a fixed set and ν^- is a varying equiweighted atomic measure. The thesis of Allen Rogers **[R]** considers other classes such as those ν possessing a smooth density function. We begin with a result concerning atomic measures from an earlier article **[A1]**, Theorem A.

LEMMA 1. *Let ν be an atomic measure in $\Psi(\mathbf{E}^t)$ supported by the points $p_1, p_2, \ldots, p_N$. Then the separation discrepancy $D_s(\nu)$ satisfies*

$$D_s(\nu) \geq c_t(\delta/\rho)^{1/2}\left\{\sum_i (\nu p_i)^2\right\}^{1/2} \tag{2}$$

where δ is the minimum distance between two distinct p_i, and ρ is the maximum distance. The positive number c_t is an absolute dimensional constant. (*The constant c_2 can be chosen to be* .05, *for example.*)

If ν were to assign measure ± 1 to the $N = s^2$ integral lattice points in $[0, s-1]^2$ so that $\delta = 1$ and $\rho = \sqrt{2N} - \sqrt{2}$, it follows at once from inequality (2) that $D_s(\nu) \geq .042N^{1/4}$ assuming that c_2 is taken as .05. The techniques used in deriving inequality (2) generally lead to very good estimates as demonstrated by the powerful upper bound methods of J. Beck. Often the upper and lower bounds differ only by a factor $\sqrt{\log N}$. However, it must be stated that the method of Beck might have to be modified in application to purely atomic measures and we have not considered this problem. A general result in Beck's recent paper **[B3]** implies that for the plane a factor $(\log N)^4$ will certainly work. Beck's ideas are discussed in Chapter 8 of the book **[BC]**.

Even though the inequality (2) appears to give nearly optimal estimates in some circumstances, it is important for the present work to note that if t is taken as a variable, then this inequality says practically nothing about $D_s(\nu)$. It is quite conceivable that if t has a suitably increasing magnitude, then c_t tends to zero so rapidly that the right side of (2) is not comparable to $\{\sum_i (\nu p_i)^2\}^{1/2}$. We shall not be able to give a definitive solution of this problem, but we hope that the inequalities obtained below will serve to inspire further work leading to improved estimates. Somewhat more satisfactory estimates will be given for L^2 averages of $|\nu H|$.

If one considers any atomic measure ν, then the inequality

$$D_s(\nu) \geq \frac{1}{2} \sup \nu(p), \qquad p \in \mathbf{E}^t, \tag{3}$$

gives one "best possible" result for this class of measures because inequality (3) holds for any atomic ν, and it is simple to define atomic measures ν such that equality holds in (3). This type of inequality clearly does not address the problem stated in the first paragraph above. Generally, one must place constraints of both measure and geometry on ν to obtain interesting problems in discrepancy. In inequality (2) the factor $\{\sum_i (\nu p_i)^2\}^{1/2}$ dominates $\sup(\nu p)$, but the ratio δ/ρ must decrease at least as rapidly as $N^{-1/t}$.

Next we define the several classes of measures to be treated in this article. A signed atomic measure ν in $\Psi(\mathbf{E}^t)$ will be termed an *a-measure* if it is supported by a finite set. For a given a-measure the minimum distance between two distinct support points will be denoted by δ. An a-measure ν satisfying $|\nu p| = 1$ for each support point p is termed a *u-measure.* A two-colored (by ± 1) set of lattice points provides a good example of a u-measure with $\delta \geq 1$. For any Borel measure ν on $\mathbf{E}^t$ having compact support the diameter of the support of ν will be denoted by ρ.

A measure $\nu = \nu^+ - \nu^-$ in $\Psi(\mathbf{E}^t)$ for which ν^+ is supported by N points p_i, not necessarily distinct, with $\nu p_i = 1$, and for which ν^-

is Lebesgue t-measure restricted to a set of t-measure N is termed a *U-measure.* The traditional measures occurring in the study of irregularities of distribution are rescaled U-measures with varying assumptions about the geometry of support of ν^-.

2. What structure theorems in $\mathbf{E}^t$ imply about D_S. A famous problem of Sylvester asks for a proof that if N distinct points, not all collinear, lie in $\mathbf{E}^2$, then there is a line containing exactly two of the points. The book by Coxeter [C] contains two beautiful proofs, one by L. M. Kelly and the other by Coxeter. The obvious generalization to planes in $\mathbf{E}^3$ is false.

Let $\mathbf{S}$ be a pointset spanning $\mathbf{E}^t$. Following Motzkin [M], a subset $\mathbf{Q}$ of $\mathbf{S}$ is termed *ordinary* (with respect to $\mathbf{S}$) if $\mathbf{Q}$ spans a $(t-1)$-flat α with $\alpha \cap \mathbf{S} = \mathbf{Q}$, and there is a point p such that $\mathbf{Q}\backslash p$ spans a $(t-2)$-flat. The following fine theorem of Hansen [H] solves Motzkin's generalized Sylvester problem.

THEOREM (*Hansen*). *If a finite pointset* $\mathbf{S}$ *spans* $\mathbf{E}^t$, $t \geq 2$, *then it contains a subset* $\mathbf{Q}$ *that is ordinary with respect to* $\mathbf{S}$.

Given a pointset $\mathbf{S}$ in $\mathbf{E}^t$, a subset of independent points $\mathbf{T}$ is termed *elementary* if the flat β spanned by $\mathbf{T}$ satisfies $\beta \cap \mathbf{S} = \mathbf{T}$. Any nonempty subset of an elementary set is again elementary with respect to $\mathbf{S}$. Edelstein [E] has called attention to an inductive corollary to Hansen's theorem.

COROLLARY (*Edelstein*). *If a finite pointset* $\mathbf{S}$ *spans* $\mathbf{E}^t$, $t \geq 2$, *it contains an elementary subset* $\mathbf{T}$ *for which* $|\mathbf{T}| \geq [t/2] + 1 \geq t/2$.

THEOREM 1. *Let* ν *be a u-measure supported by a pointset spanning* $\mathbf{E}^t$. *Then*

$$D_S(\nu) \geq ct \tag{4}$$

where $c \geq 1/8$.

Proof. The Edelstein corollary guarantees an elementary subset $\mathbf{T}_1$ of the support points having size at least $t/2$. There is a further elementary subset $\mathbf{T}_2$ of $\mathbf{T}_1$ such that $|\mathbf{T}_2| \geq t/4$ and such that ν assigns the same measure to every point in $\mathbf{T}_2$. There exists a hyperplane through $\mathbf{T}_2$ containing no other support points for ν. The existence of this hyperplane implies that there is a half space H such that $|\nu H| \geq t/8$, and the theorem is proved.

Possibly there are theorems about "monochromatic" flats generated from two-colored pointsets in $\mathbf{E}^t$ which improve the constant 1/8. Is there a proof of Theorem 1 which avoids Hansen's theorem and perhaps gives a better constant?

For the class of U-measures on $\mathbf{E}^t$ there is the easily proved "best possible" estimate

$$D_s(\nu) \geq t/2. \tag{5}$$

If the support of ν^+ does not span $\mathbf{E}^t$, then $D_s(\nu) \geq N/2$.

3. Another measure of discrepancy and its relation to D_s. Following the approach developed in [**A1**] and [**A2**] we introduce the functional $\mathbf{I}^\alpha$ defined on the measures ν in Ψ^* by

$$\mathbf{I}^\alpha(\nu) = \iint |p-q|^\alpha \, d\nu(p)\, d\nu(q). \tag{6}$$

There is the associated bilinear functional $\mathbf{J}$ defined on ν, ψ in Ψ^* by

$$\mathbf{J}^\alpha(\nu, \psi) = \iint |p-q|^\alpha \, d\nu(p)\, d\psi(q). \tag{7}$$

It follows from the work of I. J. Schoenberg [**Sb**] that $\mathbf{J}^\alpha$ is an inner product on the subspace of atomic measures in $\Psi(\mathbf{E}^t)$ for $0 < \alpha < 2$. In fact $\mathbf{J}^\alpha$ is an inner product on $\Psi(\mathbf{E}^t)$ over this range of α since it is true that $\mathbf{I}^\alpha(\nu) = 0$ implies $\nu = 0$. However, this requires further proof.

There is, except for normalization, a natural motion-invariant measure μ on the oriented hyperplanesets of $\mathbf{E}^t$. Although handicapped by the lack of accurate definitions, M. W. Crofton over a century ago developed many of the key ideas and formulas for integration with respect to this measure in $\mathbf{E}^2$. With suitable normalization there is the basic formula for the distance between two points in $\mathbf{E}^t$

$$|p-q| = \frac{1}{2}\mu\{h\colon h \text{ separates } p \text{ and } q\}. \tag{8}$$

The following fundamental lemma relates $\mathbf{I}(\nu)$ to $D_s(\nu)$. It is discussed in §2 of [**A1**].

LEMMA 2. *Let ν belong to $\Psi(\mathbf{E}^t)$. Then*

$$-\mathbf{I}(\nu) = \int (\nu H)^2 \, d\mu(h) \leq D_s(\nu)^2 \mu\mathbf{H}. \tag{9}$$

Here $\mathbf{H}$ *is the set of planes that cut the convex hull of the support of* ν.

Except for $\mathbf{E}^2$ the normalization given by (8) differs from that given in Chapter 14 of the book by Santaló [**Sa**]. Our normalization has the virtue that μ is independent of the dimension t. The measure on planesets in $\mathbf{E}^t$ is induced in a natural manner by the corresponding measure in $\mathbf{E}^{t+1}$. If the measure μ' for planesets in $\mathbf{E}^t$ is scaled in the manner of Santaló, then an investigation of formulas 13.71 and 14.2 in [**Sa**] shows that

$$\mu = (t-1)O_{t-2}^{-1}\mu', \qquad t > 2. \tag{10}$$

The next two lemmas give information on $\mu\mathbf{H}$ for examples that will be useful later.

LEMMA 3. *The* μ*-measure of the set of hyperplanes in* $\mathbf{E}^t$ *that cut a unit* s*-cube* $1 \le s \le t$ *is equal to* $2s$. *The* μ*-measure of the hyperplanes that cut a unit sphere* $\mathbf{S}^{t-1}$ *is equal to* $2(t-1)O_{t-1}/O_{t-2}$ *where* $O_k = 2\pi^{(k+1)/2}(\Gamma((k+1)/2))^{-1}$ *denotes the content of* $\mathbf{S}^k$.

Proof. Without loss of generality it may be assumed that $s = t$. Let $\mathbf{H}$ be those hyperplanes in $\mathbf{E}^t$ that cut a unit t-cube $\mathbf{C}$. In the notation of Santaló the relation $\mu'(\mathbf{H}) = 2M_{t-2}(\partial\mathbf{C})$ follows from formula 14.2 of [**Sa**] and the fact that the planes are oriented. However formulas 13.48 and 13.45 of [**Sa**] imply that $2M_{t-2}(\partial\mathbf{C}) = 2t(t-1)^{-1}O_{t-2}$. The scaling relation (10) gives the result for the cube. The result for the sphere follows at once from the relation $M_{t-2}(\mathbf{S}^{t-1}) = O_{t-1}$. Here we define the generalized mean curvature $M_k(\partial\mathbf{K})$ to be $tW_{k+1}(\mathbf{K})$ where W_{k+1} denotes the Minkowski Quermassintegrale of order $k+1$.

We state the next lemma without proof. Sudakov [**Su**] gives a partial discussion. The principal difficulty lies in calculating the content of certain spherical simplices, and a proof would take us far afield.

LEMMA 4 (*Sudakov*). *The* μ*-measure of the set of hyperplanes in* $\mathbf{E}^t$ *that cut a regular* s*-simplex* $3 \le s \le t$ *of unit edge length is less than* $6\sqrt{\log s}$.

It is clear from (9) that a lower bound estimate on $-\mathbf{I}(\nu)$ along with an upper bound estimate on $\mu\mathbf{H}$ will yield a lower bound estimate on $D_s(\nu)$. The following lemma points to the approach taken to establish lower bounds on $-\mathbf{I}(\nu)$. Complete discussions of this lemma from two viewpoints are found in [**A1**] and [**A2**].

LEMMA 5. *Let* ϕ *be in* $\Psi^*(\mathbf{E}^t)$ *and let* ν *be in* $\Psi(\mathbf{E}^t)$. *Then*

$$-\mathbf{I}(\phi * \nu) \leq -\|\phi\|^2 \mathbf{I}(\nu) \tag{11}$$

where "$*$" *denotes the usual additive convolution of measures.*

One now searches for measures ϕ, $\|\phi\| = 1$, that allow the determination of a good lower bound on $-\mathbf{I}(\phi * \nu)$. We close this section by stating a corollary to Theorem 9 of **[A1]** about $\mathbf{I}(\nu)$ which with Lemma 2 above forms the basis of the proof of Lemma 1 above. (Also see Theorem 3 of **[A1]**.)

LEMMA 6. *Let* ν *be an a-measure with support in* $\mathbf{E}^t$. *Then*

$$-\mathbf{I}(\nu) \geq \delta c_t \sum_i \nu(p_i)^2 \tag{12}$$

where c_t *is a dimensional constant. One may take* $c_2 = .02$.

4. Technical lemmas based on previous work. The method to be used in the present work is a refinement of that used in **[A1]** and **[A2]** where ϕ is chosen to be an atomic measure on the real line $\mathbb{R}$ considered as the x_{t+1} axis in $\mathbf{E}^{t+1}$. Thus we have $\phi * \nu = \phi \times \nu$, i.e., the ordinary product measure. If ϕ is supported by the points $r_1, r_2, \dots, r_m$ in $\mathbb{R}$, and if p lies in $\mathbf{E}^t$, let ϕ_p denote the measure supported by the points $(r_1, p), (r_2, p), \dots, (r_m, p)$ in $\mathbf{E}^{t+1}$ and satisfying $\phi(r_i, p) = \phi(r_i)$.

For p, q in $\mathbf{E}^t$ define $\mathbf{J}(p, q) = \mathbf{J}(\phi_p, \phi_q)$ where $\mathbf{J}$ is as in (6) above. It is easily seen that the number $\mathbf{J}(p, q)$ depends only upon the measure ϕ and the number $y = |p - q|$, so it is often convenient to use the notation $\mathbf{J}(\phi, y)$ as well.

Lemma 7 below is a direct consequence of standard arguments in measure and integration, and follows from Lemma 10 of **[A2]**.

LEMMA 7. *Let the measure* ϕ *be as above and let* ψ_1, ψ_2 *be measures in* $\Psi^*(\mathbf{E}^t)$. *Then*

$$\mathbf{J}(\phi \times \psi_1, \phi \times \psi_2) = \iint \mathbf{J}(p, q)\, d\psi_1(p)\, d\psi_2(q). \tag{13}$$

However, Lemma 8 below seems to have no simple proof and is proved via an integral representation theorem as is done in Theorem 6 of **[A2]**. A special case is proved by elementary means as Lemma 10 in **[A1]**.

LEMMA 8. *Let ϕ be as above, but in addition lie in $\Psi(\mathbb{R})$, i.e., $\sum \phi r_i = 0$. Then $-\mathbf{J}(\phi, y)$, $y \geq 0$, is a positive strictly decreasing function of y.*

Finally, Lemma 9 follows from Lemma 8 of [A2].

LEMMA 9. *Let the atomic measure ϕ in $\Psi(\mathbb{R})$ have support in the interval $[A, A+h]$ for an h satisfying $0 < h \leq 1/2$. Suppose $\|\phi\| = 1$ and the first n moments of ϕ vanish, i.e., $\sum r_i^k \phi r_i = 0$ for $1 \leq k \leq n$. If $y > 2$, then*

$$0 < -\mathbf{J}(\phi, y) < h^{2n} y^{-(2n+1)}. \tag{14}$$

5. Lower bound inequalities on $-\mathbf{I}(\nu)$ and $\mathbf{D}_s(\nu)$ for a-measures. Suppose that ν is an a-measure with $\delta \geq 2$ having support in $\mathbf{E}^t$. Let ϕ be a measure in $\Psi[0, 1/2]$ for which the first n moments vanish. If ν is supported by the points $p_1, p_2, \ldots, p_N$, one may write the following relations to be justified below.

$$-\mathbf{I}(\phi \times \nu) = -\sum_{ij} \mathbf{J}(p_i, p_j)\nu(p_i)\nu(p_j) \tag{15}$$

$$= -\sum_i \mathbf{I}(\phi)\nu(p_i)^2 - \sum_{i \neq j} \mathbf{J}(p_i, p_j)\nu(p_i)\nu(p_j) \tag{16}$$

$$\geq -\sum_i \mathbf{I}(\phi)\nu(p_i)^2 - \sum_{i \neq j} 2^{-4n}|\nu(p_i)\nu(p_j)| \tag{17}$$

$$\geq -\sum_i \mathbf{I}(\phi)\nu(p_i)^2 - 2^{-4n}(N-1)\sum_i \nu(p_i)^2 \tag{18}$$

$$\geq \{-\mathbf{I}(\phi) - N2^{-4n}\}\sum_i \nu(p_i)^2. \tag{19}$$

Equation (15) follows from Lemma 5 applied to the atomic measures ϕ and ν. Equation (16) comes from (15) via the identities $\mathbf{J}(p_i, p_i) = \mathbf{I}(\phi)$. Inequality (17) is obtained by applying inequalty (14) to $\mathbf{J}(p_i, p_j) = \mathbf{J}(\phi, y)$ where $h = 1/2$ and $y \geq 2$. Inequality (18) follows by applying the Cauchy-Schwarz inequality to the sum $\sum_{i \neq j} |\nu(p_i)||\nu(p_j)|$ while noting that each number $\nu(p_i)$ appears $N-1$ times as a component in each vector. A simple scaling combined with the inequality $-\mathbf{I}(\phi \times \nu) \leq -\mathbf{I}(\nu)$ if $\|\nu\| = 1$ gives the following theorem.

THEOREM 2. *Let ν be an a-measure supported by the points p_1, $p_2, \ldots, p_N$. Let ϕ, $\|\phi\| = 1$, be in $\Psi[0, 1/2]$ with the first n mo-*

ments vanishing. Then

$$-\mathbf{I}(\nu) \geq (\delta/2)\{-\mathbf{I}(\phi) - N2^{-4n}\} \sum_i \nu(p_i)^2 . \tag{20}$$

Key things to observe in inequality (20) are that the dimension of the space supporting ν does not appear and that the number of vanishing moments appears as an exponent while the number of points is linear.

REMARK. In Theorem 2 the number δ may be replaced by δ', defined by

$$\delta' = \min(|p_i - p_j| \colon \nu(p_i)\nu(p_j) < 0) .$$

This simply is because $-\mathbf{J} > 0$, and therefore if $\nu(p_i)\nu(p_j) > 0$, the term $-\mathbf{J}(p_i, p_j)\nu(p_i)\nu(p_j)$ is positive, and if $i \neq j$ it may be omitted in bounding $-\mathbf{I}(\phi \times \nu)$ away from zero.

Next, define the measure Φ_{n-1}, supported on the integer points of the interval $[0, n]$ by $\Phi_{n-1}(k) = (-1)^k C(n, k)$ where C denotes a binomial coefficient. Clearly, $\|\Phi_{n-1}\| = 2^n$. Looking at the various derivatives of $(1-x)^n$ shows Φ_{n-1} to be in $\Psi(\mathbb{R})$ and that the first $n-1$ moments of Φ_{n-1} vanish. By Lemma 1, a binomial coefficient identity and a Sterling's formula estimate one has

$$\begin{aligned} -\mathbf{I}(\Phi_{n-1}) &\geq .02 \sum_k C(n, k)^2 \\ &= .02C(2n, n) \geq .01(\pi n)^{-1/2} 2^{2n} . \end{aligned} \tag{21}$$

Now define the measure ϕ_{n-1} with support in $[0, 1/2]$ by $\phi_{n-1}(0) = 2^{-n}$ and $\phi_{n-1}(k/2n) = 2^{-n}\Phi_{n-1}(k)$ for $1 \leq k \leq n$. Consideration of the scaling properties of $\mathbf{I}$ together with inequality (21) gives

$$-\mathbf{I}(\phi_{n-1}) \geq .002 n^{-3/2} . \tag{22}$$

Choosing n to be approximately $\log N$ in inequality (20) quickly gives the following "dimension-free" result.

LEMMA 10. *Let ν be an a-measure supported by the points p_1, $p_2, \ldots, p_N$, $N \geq 3$. Then there is an absolute positive constant c such that*

$$-\mathbf{I}(\nu) \geq c\delta(\log N)^{-3/2} \sum_i \nu(p_i)^2 . \tag{23}$$

The method used in **[A1]** involved a packing argument that very much depends on the dimension t being fixed. The inequality obtained, (68) of **[A1]**, has a structure similar to (20) above, and the next lemma exploits this to give explicit lower bounds on the constant c_t in Lemma 6 above.

LEMMA 11. *Let* ν *be an a-measure supported by the points* p_1, $p_2, \dots, p_N$ *in* $\mathbf{E}^t$. *Then*

$$-\mathbf{I}(\nu) > c\delta t^{-3/2} \sum_i \nu(p_i)^2 \tag{24}$$

where the constant $c > 10^{-6}$.

Proof. In **[A1]** inequality (43) with $\varepsilon = 1/4$ and $\phi(r)$ chosen as ϕ_t so that $-\mathbf{I}(\phi_t) \geq .002(t+1)^{3/2}$ gives

$$-\mathbf{I}(\nu) \geq (\delta/8)\{.002(t+1)^{-3/2} - 2.14(t+1)/2^{-(t+1)}\} \sum_i \nu(p_i)^2 .$$

For $t \geq 20$ the expression within the braces exceeds $.001t^{-3/2}$. To obtain an inequality for all t, replace t by $21t$ which exceeds $t+20$.

Lemmas 10 and 11 immediately give the following theorem.

THEOREM 3. *Let* ν *be an a-measure with support on* $p_1, p_2, \dots, p_N$ *in* $\mathbf{E}^t$, $N \geq 3$. *Then there is an absolute positive constant* c *such that*

$$-\mathbf{I}(\nu) > c\delta\{\min(\log N, t)\}^{-3/2} \sum_i \nu(p_i)^2 . \tag{25}$$

LEMMA 12. *Let the measure* ν *have support on a pointset in* $\mathbf{E}^t$ *of diameter* ρ. *If* $\mathbf{H}$ *is the set of planes cutting the convex hull of the support points, then for an absolute constant* c

$$\mu\mathbf{H} \leq c\rho\sqrt{t}. \tag{26}$$

Proof. The pointset is contained in a sphere of radius ρ. From Lemma 3 the measure of the planes cutting this sphere is $2\rho(t-1)O_{t-1}/O_{t-2}$. Sterling's formula shows this to be asymptotic to $c\rho\sqrt{t}$.

Often one has more detailed knowledge about $\mu\mathbf{H}$, as is the case for the regular simplex as indicated by Lemma 4.

THEOREM 4. *Let* ν *be an a-measure with support on* $p_1, p_2, \dots, p_N$ *in* $\mathbf{E}^t$, $N \geq 3$. *Then there is an absolute positive constant* c *such that*

$$D_s(\nu) \geq c(\delta/\rho)^{1/2} t^{-1/4}\{\min(\log N, t)\}^{-3/4} \left\{\sum_i \nu p_i^2\right\}^{1/2} . \tag{27}$$

Proof. If $\mathbf{H}$ is the convex hull of the support points, then Lemma 2 and Lemma 12 imply $cpt^{1/2}D_s(\nu)^2 \geq D_s(\nu)^2\mu\mathbf{H} \geq -\mathbf{I}(\nu)$. Applying inequality (25) of Theorem 3 (replacing c by c') leads at once to inequality (27).

6. I. J. Schoenberg's integral representation of $-\mathbf{I}^\alpha(\nu)$. Let d be a nonnegative number and let α lie in the interval $(0, 2)$. With the notation $\exp x = e^x$ one easily verifies the formula

$$d^\alpha = c_0 \int (1 - \exp(-s^2 d^2)) s^{-1-\alpha}\, ds \tag{28}$$

where the integration is over $(0, \infty)$, and c_0^{-1} equals the value of the integral at $d = 1$.

LEMMA 13 (*Schoenberg*). *Let* ν *be a nonzero atomic measure in* $\Psi(\mathbf{E}^t)$ *supported by the distinct points* $p_1, p_2, \ldots, p_N$. *Then for* $0 < \alpha < 2$

$$-\mathbf{I}^\alpha(\nu) = c_0 \int \left\{ \sum_{i,j} \exp(-s^2 |p_i - p_j|^2) \nu p_i \nu p_j \right\} s^{-1-\alpha}\, ds. \tag{29}$$

Proof. In $\mathbf{I}^\alpha(\nu) = \sum_{i,j} |p_i - p_j|^\alpha \nu p_i \nu p_j$ replace the number $|p_1 - p_j|^\alpha$ by its integral representation (28) for each i, j. Upon grouping the terms as a single integrand it is seen that the expression $(\sum_i \nu(p_i))^2$ appears, but this vanishes because ν is in $\Psi(\mathbf{E}^t)$. Equation (29) follows.

The positive nature of the functional $-\mathbf{I}$ is clear from the first integral representation (9) above. It is well known that the function $\exp(-\beta^2 x^2)$ is of positive type for any β. This implies that the integral in (29) is positive thereby establishing the positivity of $-\mathbf{I}^\alpha$ for α in the interval $(0, 2)$.

7. The unit t-cube as example and counterexample. An interesting example occurs if one takes a u-measure ν supported by the 2^t vertices of the unit t-cube. Here $\delta = 1$, $|\nu p_i| = 1$ for all support points, $\rho = \sqrt{t}$ and $t = \log N / \log 2$. Theorem 3 implies that

$$-\mathbf{I}(\nu) \geq ct^{-3/2} 2^t \tag{30}$$

while Theorem 4 implies that

$$D_s(\nu) \geq ct^{-3/2} 2^{t/2}. \tag{31}$$

The author does not know just how accurate inequality (31) is. $D_s(\nu)$ may always be much larger than $2^{t/2}$, but we think not. Somewhat more can be said about inequality (30).

For the cube there is a clear candidate for the optimal choice of ν, namely, the alternating ± 1 configuration on the vertices. Denote this measure by ν_0. If the cube is placed in the normal manner in the first orthant with $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$ as opposing vertices, then $\nu_0(p_i) = (-1)^k$ where k is the number of times 1 appears as a coordinate for p_i.

LEMMA 14. *Let ν_0 be the measure on the vertices of the t-cube as described above. Then*

$$-\mathbf{I}(\nu_0) = c_0 2^t \int [1 - \exp(-s^2)]^t s^{-2}\, ds \tag{32}$$

where the integration is over $(0, \infty)$.

Proof. Setting $\alpha = 1$, the integrand in Schoenberg's integral formula (29) for $-\mathbf{I}(\nu_0)$ contains 2^{2t} terms. However, it may be observed that the sum of the 2^t terms associated with a given vertex does not depend on which of the 2^t vertices is chosen. The alternating binomial nature of these sums leads to the relation

$$c_0 \int [1 - \exp(-s^2)]^t s^{-2}\, ds = \sum_k (-1)^k C(t, k)\sqrt{k}. \tag{33}$$

Equation (32) follows at once.

COROLLARY. *There are positive constants c_1, c_2 such that*

$$c_1 2^t (\log t)^{-1/2} \leq -\mathbf{I}(\nu_0) \leq c_2 2^t (\log t)^{-1/2}. \tag{34}$$

Proof. If $t > 1$, the function $f(s) = [1 - \exp(-s^2)]^t s^{-2}$ satisfies $f(0) = f(\infty) = 0$ and $f'(s) \geq 0$ for $s \leq \sqrt{\log t}$. It follows easily that the integral of f over each of the intervals $(0, \sqrt{\log t})$ and $(\sqrt{\log t}, \infty)$ is bounded by $c/\sqrt{\log t}$. The right inequality of (34) follows. Consideration of the integral of f over $(\sqrt{\log t}, \infty)$ quickly leads to the left inequality of (34).

A number of examples led to an interesting question, namely, is there an absolute positive constant c such that for any u-measure ν supported by N atoms in Hilbert space one has the inequality

$$-\mathbf{I}(\nu) \geq c\delta N? \tag{35}$$

The corollary shows why all attempts to prove (35) failed. We state the following theorem without any insight as to which, if either, inequality is close to the truth. It gives a negative answer to Problem 5 of **[A1]**.

THEOREM 5. *As ν ranges over all u-measures supported by N atoms in Hilbert space, there are positive numbers c, c' such that*

$$c(\log N)^{-3/2} \le \inf_{\nu}\{-\mathbf{I}(\nu)(\delta N)^{-1}\} \le c'(\log\log N)^{-1/2}. \tag{36}$$

Proof. The first inequality follows from Theorem 3 while the second follows from the corollary to Lemma 14.

Two questions: (i) is $-\mathbf{I}$ minimized by ν_0 among all u-measures supported by the vertices of the t-cube; (ii) what is the true value of $D_s(\nu_0)$?

8. The regular t-simplex. If the u-measure ν is supported by the vertices of a regular t-simplex (t odd), of unit edge length, an easy calculation shows that $-\mathbf{I}(\nu)/2 = [(t+1)/2]^2 - 2C((t+1)/2, 2)$ so that

$$-\mathbf{I}(\nu) = (t+1). \tag{37}$$

Lemmas 2 and 4 imply that $6D_s(\nu)^2\sqrt{\log t} \ge (t+1)$. Therefore

$$D_s(\nu) \ge c(\log t)^{-1/4}t^{1/2}. \tag{38}$$

However, since the positive and negative parts of ν can be separated by a hyperplane, clearly

$$D_s(\nu) = (t+1)/2. \tag{39}$$

In this example, Theorem 1 gives the true order of $D_s(\nu)$ while the mean square averaging method falls far short.

9. Discrepancy of U-measures. While the primary interest of this article is with atomic measures we give several results about U-measures as defined at the end of §1. Since the estimates are somewhat similar to those of a-measures, we include only one proof.

LEMMA 15. *Let ν be a U-measure with ν^+ supported by $p_1, p_2, \ldots, p_N$. There is an absolute constant c' such that for $t \ge t_0$*

$$-\mathbf{I}(\nu) \ge ct^{-3/2}N. \tag{40}$$

If the support of ν is contained in a ball of diameter ρ there is an *absolute constant* c' *such that for* $t \geq t_0$

$$D_s(\nu) \geq c' \rho^{-1/2} t^{-1} \sqrt{N}. \tag{41}$$

Proof. Inequality (41) follows from (40) by choosing $t \geq t_0$ and then applying Lemmas 2 and 12. To obtain (41) we employ results from the paper **[A2]** which is entirely devoted to studying the discrepancy of a class of measures which includes U-measures. If we set $K = N$ in inequality (43) of **[A2]** and insert the values of the constants, which are $c_1 = -\mathbf{I}(\nu)$ and $c_2 = -O_{t-1}(\mathbf{I}(\phi)2^{t+1} - 1)$ are inserted, one obtains

$$-\mathbf{I}(\nu) \geq \{-\mathbf{I}(\phi) + O_{t-1}(\mathbf{I}(\phi)2^{t+1} - 1)\}N. \tag{42}$$

Inequality (43) of **[A2]** was derived under the assumption that ϕ is in $\Psi[-1/4, 1/4]$ with the first t moments vanishing. Choose $\phi = \phi_t$ as constructed above so that $-\mathbf{I}(\phi_t) \geq ct^{-3/2}$. Now O_{t-1} tends to zero roughly as $t^{-\varepsilon t}$, and therefore the right side of (42) is essentially equal to $-\mathbf{I}(\phi_t)N$ for large t. Inequality (40) follows.

LEMMA 16. *The previous lemma remains true if "* t *" is replaced by "* $\log N$.*"*

Lemmas 15 and 16 together with inequality (5) above allow a statement about U-measures that is very similar to Theorem A.

THEOREM 6. *Let* ν *be a* U*-measure on* $\mathbf{E}^t$ *with* ν^+ *supported by the points* $p_1, p_2, \dots, p_n$. *Then there is an absolute positive constant* c *such that*

$$D_s(\nu) \geq \max\{t/2, c\rho^{-1/2} t^{-1/4} [\min(\log N, t)]^{-3/4} \sqrt{N}\}. \tag{43}$$

If the definition of U-measure is extended so that ν^- may be Lebesgue measure restricted to a subset of a convex surface, one still obtains an inequality in the space of (43). In addition to **[A2]** the papers **[B1]**, **[B2]** and **[Sm]** all consider this type of problem. We remark that if the atoms of ν were allowed to assume variable positive values then the " N " of equations (40) and (41) would be replaced by " $\sum_i \nu(p_i)^2$."

In closing it is noted that for U-measures we have no counterexample to the inequality $-\mathbf{I}(\nu) \geq c\sqrt{t}N$. This is the estimate given by the probabilistic upper bound method discussed in §8 of **[A2]**.

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