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ERGODIC PROPERTIES OF RECURRENT SOLUTIONS OF STOCHASTIC EVOLUTION EQUATIONS

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Introduction

It is well known that for solutions of nonsingular finite-dimensional stochastic differential equations (and, more generally, for strongly Feller irreducible Markov processes in locally compact state spaces) there are only two possibilities of asymptotic behavior. If there exists a stationary distribution (that is, an invariant probability measure) then all solutions converge to it in distribution and the strong law of large numbers holds true. On the other hand, if there exists no stationary distribution then all solutions "escape to infinity in distribution" (in the sense specified by the formula (0.4) below). More specifically, consider a stochastic differential equation

$$\text{(0.1)} \qquad \mathrm{d}\xi(t)=b(\xi(t))\mathrm{d}t+\sigma(\xi(t))\mathrm{d}\omega_t, \qquad \xi(0)=x\in \boldsymbol{R}^n,$$

in $\boldsymbol{R}^n$ where the coefficients b: $\boldsymbol{R}^n\to\boldsymbol{R}^n$ and σ: $\boldsymbol{R}^n\to\boldsymbol{R}^{n\times n}$ are, for simplicity, globally Lipschitzian, ω_t is a standard n-dimensional Wiener process and $\sigma(y)\sigma^*(y)>0$, $y\in\boldsymbol{R}^n$. If there exists a stationary distribution μ for the equation (0.1) then

$$\text{(0.2)} \qquad \boldsymbol{P}_x\left[\frac{1}{T}\int_0^T \varphi(\xi(t))\mathrm{d}t\to\int\varphi\mathrm{d}\mu, T\to\infty\right]=1$$

holds for every $x\in\boldsymbol{R}^n$ and every μ-integrable function ϕ: $\boldsymbol{R}^n\to\boldsymbol{R}$, and

$$\text{(0.3)} \qquad P(t,x,A)\to\mu(A), \qquad t\to\infty$$

holds for $x\in\boldsymbol{R}^n$ and $A\in\mathscr{B}(\boldsymbol{R}^n)$ (the Borel sets on $\boldsymbol{R}^n$), where $P=P(t,x,A)$ stands for the transition probability function corresponding to the solutions of (0.1). If there exists no stationary distribution, then

$$\text{(0.4)} \qquad P(t,x,K)\to 0, \qquad t\to\infty$$

holds for any $x \in \boldsymbol{R}^n$ and $K \subset \boldsymbol{R}^n$, K compact ([20], [31], [32]). The aim of the present paper is to check validity of a similar statement for semilinear stochastic evolution equations. In order to demonstrate the principal difference between finite and infinite dimensional stochastic differential equations let us consider the linear stochastic evolution quation of the form

$$\mathrm{d}X = AX\mathrm{d}t + \mathrm{d}W_t, \qquad X(0) = x \tag{0.5}$$

in a real separable Hilbert space H, where A is an infinitesimal generator of a strongly continuous semigroup $S = S(\cdot)$ on H and W_t stands for a cylindrical Wiener process on H with a bounded self-adjoint (not necessarily nuclear) covariance operator $Q > 0$. It is assumed that

$$\int_0^T |S(t)Q^{1/2}|_{\mathrm{HS}}^2 \,\mathrm{d}t < \infty, \qquad T > 0, \tag{0.6}$$

where $|\cdot|_{\mathrm{HS}}$ stands for the Hilbert-Schmidt norm, so there exists a mild solution to (0.5) which is given by the formula

$$X(t) = S(t)x + \int_0^t S(t-r)\mathrm{d}W_r, \qquad t > 0. \tag{0.7}$$

Although the equation (0.5) is a (linear) infinite-dimensional analogue of (0.1) the picture described by (0.2)–(0.4) is no more true. For instance, in [28] (Example 3.8) an equation of the form (0.5) is studied which admits infinitely many stationary distributions, so (0.2) and (0.3) cannot hold. In this example, one has $H = L_2(0, \infty)$, the semigroup generated by A is

$$S(t)x(s) = e^{\lambda t}x(t+s), \qquad t \geq 0, s \geq 0,$$

where $\lambda > 0$ is fixed, and W is a suitable Wiener process in H such that $Q > 0$ and Q is nuclear. (The counterexample in [28] is a modification of Zabczyk's example proposed in [37], Prop. 7. See also [22], Sect. 5, for a discussion of a closely related example with the operator A bounded.) However, even in the case when the semigroup S is exponentially stable and, therefore, there exists a unique stationary distribution, (0.2) and (0.3) need not be true (cf. Remark 3.9 below). Nevertheless, as shown in the present paper, there exists a fairly large class of semilinear stochastic evolution equations, covering stochastic parabolic problems with noise term "enough nondegenerate", for which the dischotomy (0.2)–(0.4) holds true.

The classical proof of (0.2)–(0.4) is based on the well known method of embedded Markov chains or cycles (cf. [16], [20], [21], [32]), and the method used in this paper is basically the same. Some difficulties are caused by lack of local compactness of the state space, and the above mentioned counterexamples suggest that this problem is substantial. It is avoided by an assumption that the solutions live in a smaller space compactly embedded into H which is typically satisfied for parabolic problems. The method of embedded cycles itself can be of some independent interest as it is widely used, for instance, in control theory. Existence and uniqueness of stationary distributions and the related problems of asymptotic and ergodic behavior for stochastic evolution systems were studied by numerous authors, for instance, [2]–[5], [12]–[14], [18], [19], [24]–[29], [36], [37]. The present paper is especially related to the previous results on invariant measures by G. Da Prato and J. Zabczyk ([2], [4], [5], [36], [37]), M. Freidlin ([12]), T. Funaki ([13],[14]) and one of the authors ([26]–[29]). For example, the statement on the uniqueness of the invariant measure (Corollary 3.7) is an extension of some earlier results contained in the above quoted papers.

The paper is divided into four sections including this Introduction. Section 1 contains some basic notation and preliminary results on the equation under consideration. Also, the basic assumptions are formulated there. Since the assumptions are given in a rather general form, some sufficient conditions for the particular assumptions to be fulfilled are also stated. The main result of Section 2 is Theorem 2.9 stating that there exists a σ-finite invariant measure provided the solution is recurrent. For this purpose a Lyapunov type statement for exit from a bounded domain is established (Lemmas 2.2, 2.3) and an embedded Markov chain is constructed. The main result of Section 3 is the strong law of large numbers of Hopf's type (the ratio ergodic theorem, Theorem 3.2) and its consequences, including uniqueness of an invariant measure and the dichotomy of the type (0.2)–(0.4). Two Examples are given at the end of Section 3 (one of them being a system of stochastic reaction-diffusion equations). An abstract theorem on the uniqueness of a σ-finite invariant measure for fairly general Markov processes, that is needed in Section 3, is deferred to Appendix.

Given a Banach space Y we denote by $\mathscr{B}(Y)$ the σ-algebra of Borel sets on Y, $\mathscr{P}(Y)$ stands for the space of probability measures on Y, Var is the total variation of a signed measure over the whole space. A nonzero non-negative σ-finite measure μ on $\mathscr{B}(Y)$ is called an invariant measure for a Markov process with a state space Y and a transition probability function $P=P(t,x,A), t\geq 0, x\in Y, A\in\mathscr{B}(Y)$, if

$$\mu(A)=\int_Y P(t,x,A)\,\mathrm{d}\mu(x)$$

holds for all $A \in \mathscr{B}(Y), t \geq 0$. An invariant measure $\mu \in \mathscr{P}(Y)$ is also called a stationary distribution. We denote by $\boldsymbol{B}(Y)$ and $\boldsymbol{C}(Y)$ the spaces of all bounded measurable and continuous functions on Y, respectively, and $\mathrm{Cl}_Y A$ stands for the closure of the set $A \subset Y$ in the topology of Y. By $\mathscr{L}(X, Y)$ we denote the space of all bounded linear operators defined on a Banach space X with the range in the space Y and we put $\mathscr{L}(X) = \mathscr{L}(X,X)$. The symbols $\mathscr{D}$ and $\mathscr{R}$ stand for the domain and the range of an operator, respectively. By $\boldsymbol{C}_{1,2}(G)$ we denote the space of all functions v: $G \rightarrow \boldsymbol{R}$, $G \subseteq \boldsymbol{R}_+ \times X$ an open set, such that the Fréchet derivatives v_t, v_x, v_{xx} $(t \in \boldsymbol{R}_+, x \in X)$ exist and are continuous on G. The symbol $N(a,\Gamma)$ stands for the Gaussian measure with the mean value a and the covariance operator Γ. Some more notation is introduced in Section 1.

1. Assumptions and preliminary results

In this section the general assumptions are formulated and in some cases sufficient conditions for them to be satisfied are given. Consider a semilinear stochastic evolution equation of the form

$$(1.1) \qquad \mathrm{d}X(t) = (AX(t) + f(X(t)))\mathrm{d}t + \mathrm{d}W_t, \qquad X(0) = x$$

in a real separable Hilbert space H where the linear operator A generates a strongly continuous semigroup S on H, W_t is a cylindrical Wiener process defined on a probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$ with a bounded covariance operator Q which is positive, i.e., $\langle Qy,y \rangle > 0$ holds for all $y \in H, y \neq 0$. The norm in H will be denoted by $|\cdot|_H$ or, if there is no danger of confusion, simply by $|\cdot|$. Our first assumption is

$$(\mathrm{A}1) \qquad \int_0^T |S(t)Q^{1/2}|^2_{\mathrm{HS}}\,\mathrm{d}t < \infty \qquad \textit{for some } T > 0.$$

The assumption (A1) is obviously satisfied if the operator Q is nuclear,which corresponds to the important particular case when the Wiener process is a genuine H-valued Wiener process (see also Examples 3.8 and 3.10). If (A1) is fulfilled then the process

$$Z^x(t) = S(t)x + \int_0^t S(t-r)\,\mathrm{d}W_r, \qquad t \geq 0,\ x \in H$$

is a well defined H-valued Gaussian process with the transition semigroup P_t on $\boldsymbol{B}(H)$ of the form

$$P_t\Phi(x)=\int_H \Phi(y)\,\mathrm{d}N(S(t)x,Q_t)(y), \qquad t\geq 0, x\in H, \Phi\in \boldsymbol{B}(H),$$

where

$$Q_t=\int_0^t S(r)QS^*(r)\,\mathrm{d}r.$$

However, we need the process Z^x to evolve as a Markov process in a smaller space. Therefore we will assume
(A2) *There exists a separable Banach space $E=(E,\|\cdot\|)$ compactly embedded into H such that the restriction $S_{|E}$ is a strongly continuous semigroup on E and the process*

$$\int_0^{\cdot} S(\cdot-r)\,\mathrm{d}W_r$$

has an E-valued, E-continuous modification Z. There exists a family $(\mathscr{F}_t)$ of sub-σ-fields in $\mathscr{F}$ such that the process Z is $(\mathscr{F}_t)$-adapted and for arbitrary $t\geq 0$ the processes

$$\tilde{Z}(r)=Z(t+r)-S(r)Z(t), \qquad r\geq 0,$$

are independent of $\mathscr{F}_t$ and have distributions identical with $Z(r)$.
The condition (A2) has been studied in various aspects, for instance, in [2], [6] and [36] (see also Examples 3.8 and 3.10 below). The following assumption (A3) is in fact a condition on the nonlinear term f. It is formulated in terms of the deterministic integral equation

$$u(t)=S(t)x+\int_0^t S(t-r)f(u(r))\,\mathrm{d}r+\Phi(t), \qquad t\geq 0, \tag{1.2}$$

where $x\in H$, *and*

$$\Phi\in \boldsymbol{C}_0([0,T],E) :=\{\xi\in \boldsymbol{C}([0,T],E);\ \xi(0)=0\}.$$

Denote by $u(.,x,\Phi)$ the solution to (1.2) provided it exists and assume
(A3a) *The function f: $E\rightarrow H$ is H-Lipschitzian on bounded sets in E, that is, $|f(x)-f(y)|\leq L_N|x-y|$ holds for all $x,y\in E, \|x\|+\|y\|\leq N, N>0$, where $L_N<\infty$.*
(A3b) *For every $x\in H$, $\Phi\in \boldsymbol{C}_0([0,T],E)$ there exists a unique solution $u=u(\cdot,x,\Phi)$ of the equation* (1.2), $u\in \boldsymbol{C}([0,T],H)\cap \boldsymbol{C}((0,T],E)$. *Moreover, $u(\cdot,x,\Phi)\in \boldsymbol{C}([0,T],E)$ for $x\in E$.*

The mild solution of the equation (1.1) is defined as an H-continuous

$(\mathscr{F}_t)$-nonanticipative process X satisfying

$$X(t)=S(t)x+\int_0^t S(t-r)f(X(r))\,\mathrm{d}r+Z(t), \qquad t\geq 0. \tag{1.3}$$

The above definition corresponds to the usual concept of a mild solution of a stochastic evolution equation. Taking into account the fact that $X(t)=u(t,x,Z)$ we have by (A3) and [36] (Theorem 3)

Proposition 1.1. *Let* (A1)–(A3) *be satisfied. Then there exists a unique mild solution of the equation* (1.1). *The family of solutions to* (1.1) *for all* $x\in E$ *defines a homogeneous Markov process* $(X(t),\boldsymbol{P}_x)$ *on* E (*and, consequently, on* H).

The following assumption (A4) has more technical nature. It is a kind of "uniform nonexplosion" and "continuous dependence" condition on the solutions of the integral equation (1.2). In the sequel we denote by $\tilde{E}=\tilde{E}(T)$ the space $\boldsymbol{C}_0([0,T],E)$ endowed with the norm $\|g\|_\sim =\sup\{\|g(t)\|;\ t\in[0,T]\}$. Assume
(A4a) *For every* $R>0$

$$\sup\{\|u(t,x,\Phi)\|;\ t\in[0,R], x\in E, \|x\|\leq R, \Phi\in\tilde{E}(R), \|\Phi\|_\sim\leq R\}<\infty,$$

(A4b) $u(\cdot,x_n,\Phi_n)\to u(\cdot,x,\Phi)$ *in* $\tilde{E}(T)$ *for every* $T>0$ *whenever* $x_n\to x$ *in* $E, \Phi_n\to\Phi$ *in* $\tilde{E}(T)$.

In the following proposition we give more explicit sufficient conditions for (A3), (A4) to be satisfied.

Proposition 1.2. *Assume that* f: $E\to H$ *is globally Lipschizian,* $S(t)\in\mathscr{L}(H,E)$ *for* $t>0$ *and the estimate* $\|S(t)\|_{\mathscr{L}(H,E)}\leq ct^{-\alpha}, t\in(0,T)$, *holds for some* $c>0, T>0, 0<\alpha<1$.
Then the assumptions (A3b) *and* (A4) *are fulfilled.*

Proof. (A3b) is shown by a fixed point argument in the Banach space

$$\mathscr{S}=\{y\in\boldsymbol{C}((0,T],E);\ \|y\|_{\mathscr{S}}:=\sup_{t\in(0,T]} t^\alpha\|y(t)\|<\infty\}$$

(cf. [33], Proposition 4.1 for a related proof). Fix $x\in H$, $\Phi\in\tilde{E}$ and for $Y\in\mathscr{S}$ set

$$\Lambda(Y)(t)=S(t)x+\int_0^t S(t-r)f(Y(r))\,\mathrm{d}r+\Phi(t), \qquad t\in(0,T].$$

It can be checked in a standard way that $\Lambda(y)(\cdot)\in \boldsymbol{C}((0,T],E)$. Moreover,

$$\sup_{0\le t\le T} t^\alpha \|\Lambda(Y)(t)\| \le k|x| + \sup_{0\le t\le T} t^\alpha \left\| \int_0^t S(t-r)f(Y(r))\,\mathrm{d}r \right\| + \sup_{0\le t\le T} t^\alpha \|\Phi(t)\|$$

$$\le K_1 + K_2 \sup_t t^\alpha \int_0^t (t-r)^{-\alpha} \|Y(r)\|\,\mathrm{d}r$$

for some constants k, K_1, K_2. Thus

$$\|\Lambda(Y)\|_{\mathscr{S}} \le K_1 + K_2 \|Y\|_{\mathscr{S}} \sup_t t^\alpha \int_0^t (t-r)^{-\alpha} r^{-\alpha}\,\mathrm{d}r < \infty,$$

and it follows that Λ maps $\mathscr{S}$ into $\mathscr{S}$. In order to show contractivity of Λ introduce an equivalent norm $\|\cdot\|_b$,

$$\|y\|_b = \sup_{0\le t\le T} e^{-bt} t^\alpha \|y(t)\|,$$

where $b>0$. For $X, Y\in\mathscr{S}$ we have

$$\|\Lambda(X)-\Lambda(Y)\|_b \le \sup_t e^{-bt} t^\alpha \left\| \int_0^t S(t-r)(f(X(r))-f(Y(r)))\,\mathrm{d}r \right\|$$

$$\le cK_f \|X-Y\|_b \sup_t e^{-bt} t^\alpha \int_0^t (t-r)^{-\alpha} r^{-\alpha} e^{br}\,\mathrm{d}r,$$

where K_f is the Lipschitz constant of f: $E\to H$. Therefore for enough large b the mapping Λ is a contraction and, consequently, the integral equation (1.2) has a unique solution in $\boldsymbol{C}((0,T],E)$ for $x\in H$. For $x\in E$ we have

$$\|u(t,x,\Phi)\| \le \|S(t)\|_{\mathscr{L}(E)} \|x\| + cL \int_0^t (t-r)^{-\alpha}\,\mathrm{d}r + cL \int_0^t (t-r)^{-\alpha} \|u(r,x,\Phi)\|\,\mathrm{d}r + \|\Phi(t)\|$$

for some constant L. For $R>0$ set

$$C_R = R\sup\|S(t)\|_{\mathscr{L}(E)} + cLT^{1-\alpha}(1-\alpha)^{-1} + R.$$

So if $\|x\| < R$, $\|\Phi\|_\sim < R$ then

$$\|u(t,x,\Phi)\| \le C_R + cL\int_0^t (t-r)^{-\alpha}\|u(r,x,\Phi)\|\,\mathrm{d}r, \qquad t\in(0,T].$$

By the generalized Gronwall lemma ([17], Lemma 7.1.1) there exists a constant M_R depending only on C_R, c, L and T such that

$$\sup_{0\le t\le T}\|u(t,x,\Phi)\| \le M_R < \infty,$$

and (A4a) follows. Furthermore, for $x_n\in E, \Phi_n\in\tilde{E}$ we have

$$\|u(t,x_n,\Phi_n) - u(t,x,\Phi)\| \le \|S(t)\|_{\mathscr{L}(E)}\|x_n - x\|$$

$$+\|\Phi_n(t)-\Phi(t)\| + \int_0^t (t-r)^{-\alpha}|f(u(r,x_n,\Phi_n)) - f(u(r,x,\Phi))|\,\mathrm{d}r.$$

By the Lipschitz continuity of f we get

$$\|u(t,x_n,\Phi_n) - u(t,x,\Phi)\| \le K(\|x_n - x\| + \|\Phi_n - \Phi\|_\sim$$

$$+\int_0^t (t-r)^{-\alpha}\|u(r,x_n,\Phi_n) - u(r,x,\Phi)\|\,\mathrm{d}r)$$

and (A4b) follows by the generalized Gronwall lemma. We shall verify (A3b). For $x\in H$ we have

$$\left|\int_0^t S(t-r)f(u(r))\,\mathrm{d}r\right| \le \sup_{0\le t\le T}|S(t)|_{\mathscr{L}(H)}\int_0^t |f(u(r))|\,\mathrm{d}r$$

$$\le KL\int_0^t (1+\|u(r)\|)\,\mathrm{d}r \le const.\ \|u\|_{\mathscr{S}}\int_0^t r^{-\alpha}\,\mathrm{d}r \to 0, \quad t\to 0+,$$

and it follows that $u\in \boldsymbol{C}([0,T],H)$. Furthermore, if $x\in E$ then

$$\left\|\int_0^t S(t-r)f(u(r))\,\mathrm{d}r\right\| \le c\int_0^t (t-r)^{-\alpha}|f(u(r))|\,\mathrm{d}r$$

$$\le cL\sup_{0\le t\le T}(1+\|u(t)\|)\int_0^t (t-r)^{-\alpha}\mathrm{d}r \to 0, \quad t\to 0+$$

by (A4a). Thus $u \in \boldsymbol{C}([0,T],E)$ and (A3b) is verified.

Q.E.D.

The remaining assumptions (A5), (A6) can be viewed as "nondegeneracy conditions". It is well known (cf. [21], Chapter 4) that if the dimension of H is finite then (A5) and (A6) are always satisfied (recall that the covariance operator Q of the Wiener process W is positive). Denote by $P=P(t,x,A), t \geq 0, x \in H, A \in \mathscr{B}(H)$, the transition probability function corresponding to the solution of (1.1). Assume
(A5) (*the strong Feller property in* H)

$$P(t,x_n,A) \to P(t,x,A) \qquad \text{as } x_n \to x \quad \text{in } H$$

for all $t>0, A \in \mathscr{B}(H)$.
The strong Feller property has been studied in the context of stochastic evolution equations in [4], [24], [26]. In the linear case ($f=0$) the assumption (A5) is equivalent to the condition

$$\mathscr{R}(S(t)) \subseteq \mathscr{R}(Q_t^{1/2}) \qquad \text{for all } t>0 \tag{1.4}$$

(cf. [29], Proposition 2.5). For the general case a sufficient condition is given in the statement below.

Proposition 1.3. *Let* (A1)–(A4) *be fulfilled. Let* $\{e_i\}$ *be an orthonormal basis in* H *contained in* $\mathscr{D}(A)$. *Denote by* Π_n *the orthogonal projections onto*

$$H_n = \text{Span}\{e_1, e_2, \cdots, e_n\}$$

and set

$$A_n = \Pi_n A \Pi_n, \qquad S_n(t) = \exp(tA_n),$$

$$B_t = Q_t^{-1/2} S(t), \qquad B_{t,n} = Q_{t,n}^{-1/2} S_n(t) \Pi_n, \qquad Q_{t,n} = \int_0^t S_n(r) \Pi_n Q \pi_n S_n^*(r)\, dr.$$

Assume that $\{e_i\}$ *can be chosen such that*

$$\sup\{|S_n(t)\Pi_n x - S(t)x|;\ t \in [0,T]\} \xrightarrow{n\to\infty} 0, \qquad T>0, \quad x \in \mathscr{D}(A), \tag{1.5}$$

$$\sup_n |S_n(t)|_{\mathscr{L}(H)} \leq C, \qquad t \in [0,T] \tag{1.6}$$

(1.7) $$\sup_n |B_{t,n}|_{\mathscr{L}(H)} + |B_t|_{\mathscr{L}(H)} \leq Ct^{-\beta}, \qquad t \in (0,T]$$

hold for some $C>0$, $\beta \in (0,1)$.

Then the strong Feller property (A5) *holds true.*

If f is extendable to a bounded globally Lipschitzian function on H then the proof of Proposition 1.3 follows immediately from [1], Theorem 4.2, where appropriate smoothing properties of the backward Kolmogorov equation corresponding to (1.1) are proved under the conditions (1.5)–(1.7). These results yield

$$\mathrm{Var}\,(P(t,x,\cdot)-P(t,y,\cdot)) \leq C_2 t^{-\beta}|x-y|, \qquad x,y \in H,\, t \in (0,T]$$

for any $T>0$ and some $C_2 = C_2(T) > 0$. For general f we can proceed by the truncation procedure (cf. [29] for details).

The last assumption is

(A6) *For every* $T>0$ *the Gaussian measure* λ *induced by the process* Z *in the space* $\tilde{E} = \boldsymbol{C}([0,T],E)$ *is full in* $\tilde{E}$, *i.e., the closed support* $\mathrm{supp}(\lambda)$ *of* λ *is the whole space* $\tilde{E}$.

Define the operator $\mathscr{K}$: $L_2(0,T,H) \to \boldsymbol{C}_0([0,T],H)$ by

$$\mathscr{K}y(t) = \int_0^t S(t-r)Q^{1/2}y(r)\,\mathrm{d}r, \qquad t \in [0,T].$$

If (A2) is fulfilled and

(1.8) $$\mathscr{K} \in \mathscr{L}(L_2(0,T,H),\tilde{E}),$$

(1.9) $$\mathrm{Cl}_{\tilde{E}}(\mathscr{R}(\mathscr{K})) = \tilde{E}$$

holds for all $T>0$ then (A6) is satisfied. This follows from the fact that the reproducing kernel Hilbert space H_λ of λ on $\tilde{E}$ coincides with $\mathscr{R}(\mathscr{K})$ equipped with the norm

$$|v|_\lambda = \inf\,\{|u|_{L_2(0,T,H)};\ v = \mathscr{K}u\}$$

(cf. [34], the proof of Theorem 3), thus $\mathrm{supp}(\lambda) = \mathrm{Cl}_{\tilde{E}} H_\lambda = \tilde{E}$.

Proposition 1.4. *Assume* (A2) *and let one of the conditions* (D1),(D2) *be fulfilled*:

(D1) $Q^{1/2} \in \mathscr{L}(H,E)$ *and* $\mathscr{R}(Q^{1/2})$ *is dense in* E,

(D2) *$S(t)\in\mathscr{L}(H,E)$ for $t>0$ and $\|S(t)\|_{\mathscr{L}(H,E)}\leq h(t)$, $t\in(0,T)$, where $h\in L_2(0,T)$.*
Then (A6) *is satified.*

Proof. It is straightforward to verify that each of the conditions (D1), (D2) implies (1.8),(1.9) (cf. [29], Proposition 2.7 and Proposition 2.8 for details).

Q.E.D.

The assumption (A6) is used below in the proof of irreducibility of the solution X of (1.1) (Proposition 1.5) and the embedded Markov chain (Lemma 3.1).

Proposition 1.5. *Assume* (A1)–(A4) *and* (A6). *Then the transition probability function $P(t,x,.)$ corresponding to the equation* (1.1) *has the irreducibility property*: (I) *$P(t,x,U)>0$ holds for every $x\in H$, $t>0$, $U\subseteq E$, U open and nonempty.*

Proof. Without loss of generality put $U=\{v\in E;\ \|v-z\|<r\}$ for some $z\in E$, $r>0$. At first assume that $x\in E$ and set

$$\Phi(s)=\frac{s}{t}z+\frac{t-s}{t}x,$$

$$\xi(s)=\Phi(s)-S(s)x-\int_0^s S(s-u)f(\Phi(u))\mathrm{d}u,$$

$s\in[0,T]$. Clearly $\xi\in\tilde{E}$ and by (A4b) there exists an $\varepsilon>0$ such that

$$\|u(\cdot,x,\epsilon)-u(\cdot,x,\xi)\|_{\tilde{E}}<r$$

whenever $\epsilon\in\tilde{E}$, $\|\epsilon-\xi\|_{\sim}<\varepsilon$. Therefore (I) holds true because $X(t)=u(t,x,Z)$ and

$$\boldsymbol{P}[\|Z(\cdot)-\xi\|_{\sim}<\varepsilon]>0$$

by (A6). Now we drop the assumption $x\in E$. For any $\tau>0$, $x\in H$ we have $P(\tau,x,E)=1$ by (A3), so taking $0<\tau<t$ we get

$$P(t,x,U)=\int_H P(t-\tau,y,U)P(\tau,x,\mathrm{d}y)=\int_E P(t-\tau,y,U)P(\tau,x,\mathrm{d}y)>0$$

by the preceding part of the proof.

Q.E.D.

2. Existence of a σ-finite invariant measure

In this section a σ-finite invariant measure for recurrent solutions of (1.1) is constructed. For this purpose some auxiliary results are proved at first. Lemma 2.1 below is a version of the Itô formula. Denote by Tr the trace of an operator and by $\mathscr{L}_1(H)$ the space of nuclear operators on H endowed with the trace norm $|\cdot|_1$. Together with (1.1) let us consider the equation

$$\mathrm{d}\tilde{X}(t)=(A\tilde{X}(t)+g(\tilde{X}(t))\mathrm{d}t+\mathrm{d}W_t, \qquad \tilde{X}(0)=x, \tag{2.1}$$

where $g\colon H\to H$ is globally Lipschitzian.

Lemma 2.1. *Assume* (A1) *and let there exist a function* $v\in \boldsymbol{C}_{1,2}((0,T)\times H)$ *such that* $\langle v_x(t,x),Ax\rangle$,$(t,x)\in(0,T)\times\mathscr{D}(A)$, *is extendable to a continuous function* h *on* $[0,T]\times H$, $v_{xx}(t,x)\in\mathscr{L}_1(H)$ *and*

$$\begin{aligned}&|v_t(t,x)|+|v(t,x)|+|v_x(t,x)|_H+|v_{xx}(t,x)|_1+h(t,x)\\ &\leq k(1+|x|_H^p), \qquad (t,x)\in[0,T]\times H,\end{aligned} \tag{2.2}$$

for some $k\geq 0$, $p\geq 0$. *Then*

$$\begin{aligned}&v(t,\tilde{X}(t))-v(s,\tilde{X}(s))\\ &=\int_s^t [h(u,\tilde{X}(u))+\langle g(\tilde{X}(u)),v_x(u,\tilde{X}(u))\rangle\\ &+\frac{1}{2}Tr(v_{xx}(u,\tilde{X}(u))Q)+v_t(u,\tilde{X}(u))]\mathrm{d}u\\ &+\int_s^t\langle v_x(u,\tilde{X}(u)),\mathrm{d}W_u\rangle\end{aligned} \tag{2.3}$$

holds for $0\leq s\leq t\leq T$.

This statement is basically proved in [18], Theorem 3.2. The difference is that Q need not be a nuclear operator in Lemma 2.1 which is however compensated by nuclearity of v_{xx} (see also [8], Proposition 3.4 for a similar result).

Lemma 2.2. *Let the assumptions of Lemma* 2.1 *be satisfied for every* $T>0$ *with a function* $v\in \boldsymbol{C}_{1,2}(\boldsymbol{R}_+\times H)$ *and let* U *be an open domain in* H, $x\in U$. *Assume that* $v\geq 0$ on $\boldsymbol{R}_+\times U$,

$$(2.4)\qquad \boldsymbol{L}v(s,y)\leq -a(s),\qquad (s,y)\in(0,T)\times(\mathscr{D}(A)\cap U),$$

where $a\geq 0$ *satisfies the condition*

$$(2.5)\qquad b(t)=\int_0^t a(s)\mathrm{d}s\to\infty,\qquad t\to\infty,$$

and

$$(2.6)\qquad \begin{aligned}\boldsymbol{L}v(s,y)&=\langle Ay,v_x(s,y)\rangle+\langle g(y),v_x(s,y)\rangle\\&\quad+\frac{1}{2}Tr(v_{xx}(s,y)Q)+v_t(s,y).\end{aligned}$$

Then

$$(2.7)\qquad \boldsymbol{P}_x[\tau<\infty]=1,$$

where τ *is the exit time of* $\tilde{X}$ *from the domain* U. *Moreover,*

$$(2.8)\qquad \boldsymbol{E}_x b(\tau)\leq v(0,x),\qquad x\in U.$$

Proof. Denote by τ_n the exit time from the ball $\{y\in H;\ |y|<n\}$, $\tau_n(t)=\min(\tau,\tau_n,t)$. Lemma 2.1 and (2.4) yield

$$\begin{aligned}&\boldsymbol{E}_x v(\tau_n(t),\tilde{X}(\tau_n(t)))-v(0,x)\\&=\boldsymbol{E}_x\int_0^{\tau_n(t)}\Big\{h(r,\tilde{X}(r))+v_t(r,\tilde{X}(r))\\&\qquad\qquad+\langle g(\tilde{X}(r)),v_x(r,\tilde{X}(r))\rangle+\frac{1}{2}Tr(v_{xx}(r,\tilde{X}(r))Q)\Big\}\mathrm{d}r\\&\leq -\boldsymbol{E}_x\int_0^{\tau_n(t)}a(r)\mathrm{d}r=-\boldsymbol{E}_x b(\tau_n(t))\end{aligned}$$

for $t\geq 0$. It follows that

$$\boldsymbol{E}_x b(\tau_n(t))\leq v(0,x),$$

thus taking at first $n\to\infty$ we obtain

$$(2.9)\qquad \boldsymbol{E}_x b(\min(\tau,t))\leq v(0,x)$$

and then passing $t\to\infty$ we get (2.8) by the Fatou lemma. Now (2.7) follows directly from (2.5) and (2.8).

Q.E.D.

Denote by τ_U the exit time of the solution of (1.1) from an open set $U\subseteq E$.

Lemma 2.3. *Assume* (A1)–(A3) *and let U be a bounded open set in E. Then*

$$\sup\{\boldsymbol{E}_x\tau_U;\ x\in U\}<\infty. \tag{2.10}$$

Proof. From (A3) it follows that there exists a globally Lipschitzian function $g\colon H\to H$ such that $g(x)=f(x)$ for $x\in U$. Let $W\subset H$ be a bounded open set in H such that $U\subset W$ and denote by τ the exit time of the solution $\tilde{X}$ of the equation (2.1) from W. Clearly $\tau_U\leq\tau$ a.s., hence it suffices to prove

$$\sup\{\boldsymbol{E}_x\tau;\ x\in U\}<\infty. \tag{2.11}$$

We use Lemma 2.2 with the Lyapunov function

$$v(t,x)=e^{\gamma t}[c-(\langle x,z\rangle+d)^{2n}],\qquad (t,x)\in\boldsymbol{R}_+\times H,$$

where $z\in\mathscr{D}(A^*)$, $z\neq0$ is fixed and $\gamma>0$, $c>0$, $d>0$ and $n\in\boldsymbol{N}$ are constants which are specified below. Take $d>0$ such that

$$\langle x,z\rangle+d>1,\qquad x\in W,$$

and $n\in\boldsymbol{N}$ such that

$$-2(\langle x,z\rangle+d)(\langle g(x),z\rangle+\langle x,A^*z\rangle)-(2n-1)\langle Qz,z\rangle<-1,\qquad x\in W,$$

and set

$$c=\sup_{x\in W}(\langle x,z\rangle+d)^{2n}.$$

Thus we get $v(t,x)\geq0$ for $t>0$, $x\in W$ and since

$$\begin{aligned}\boldsymbol{L}v(t,x)=e^{\gamma t}\Big\{&\gamma[c-(\langle x,z\rangle+d)^{2n}]\\&+n(\langle x,z\rangle+d)^{2n-2}[-2(\langle x,z\rangle+d)(\langle g(x),z\rangle\\&+\langle x,A^*z\rangle)-(2n-1)\langle Qz,z\rangle]\Big\},\end{aligned} \tag{2.12}$$

$t\in\boldsymbol{R}_+$, $x\in\mathscr{D}(A)$, we obtain choosing $\gamma>0$ small enough

$$\boldsymbol{L}v(t,x)\leq -k_1e^{\gamma t},\qquad t\in\boldsymbol{R}_+,\ x\in\mathscr{D}(A)\cap W,$$

for some $k_1>0$. Furthermore, the function

$$\langle Ax,v_x(t,x)\rangle=-2ne^{\gamma t}(\langle x,z\rangle+d)^{2n-1}\langle x,A^*z\rangle,\qquad t\in\boldsymbol{R}_+,\ x\in\mathscr{D}(A),$$

is continuously extendable on $\boldsymbol{R}_+\times H$, $v_{xx}(t,x)$ is nuclear and the estimate (2.2) holds with suitably large $k>0$, $p>0$. Thus we can apply Lemma 2.2 which yields

$$\boldsymbol{E}_xe^{\gamma\tau}\leq k_1^{-1}\gamma(1+v(0,x)),\qquad x\in W.$$

Consequently,

$$\sup_{x\in U}\boldsymbol{E}_x\tau_U\leq\sup_{x\in W}\boldsymbol{E}_x\tau<\infty.$$

Q.E.D.

REMARK. The preceding two lemmas are infinite-dimensional analogues of corresponding results for diffusions in $\boldsymbol{R}^n$ (cf. [21], Theorem 3.7.1 and Corollary 3.7.2). Note that the condition $Q>0$ can be relaxed to $Q\neq 0$ (then there exists $z\in\mathscr{D}(A^*)$ such that $\langle Qz,z\rangle>0$).

Lemma 2.4. *Assume* (A1)–(A4). *Then given a bounded set $A\subset E$ there exists a centered open ball W in E containing A and such that*

$$\lim_{t\to 0+}\sup_{x\in A}\boldsymbol{P}_x[\tau_W\leq t]=0, \tag{2.13}$$

where τ_W stands for the exit time of the solution of (1.1) *from W.*

Proof. Take $R>0$ such that $A\subset U_R=\{x\in E;\ \|x\|<R\}$. By (A4) there exists an $M>R$ such that for the solution $u=u(\cdot,x,\Phi)$ of the equation (1.2) we have

$$\sup\{\|u(t,x,\Phi)\|;\ x\in U_R,\ t\in[0,R],\ \Phi\in\tilde{E}(R),\ \|\Phi\|_\sim<R\}<M.$$

The E-continuity of Z and $Z(0)=0$ yield

$$\lim_{t\to 0+}\boldsymbol{P}[\sup_{s\in[0,t]}\|Z(s)\|>R]=0.$$

Since $X(t)=u(t,x,Z(t))$, we can set $W=U_M=\{x\in E;\ \|x\|<M\}$.

Q.E.D.

For $A\subseteq E$ we will denote by τ^A the first hitting time of $\mathrm{Cl}_E A$ by the solution of the equation (1.1).

DEFINITION 2.5. Let U be an open nonempty set in E. Then the process X defined by (1.1) is said to be *U-recurrent* if

$$\boldsymbol{P}_x[\tau^U<\infty]=1 \tag{2.14}$$

holds for all $x\in E$. The process X is said to be *recurrent* if it is U-recurrent for every nonempty open set U in E.

REMARK 2.6. Note that if (A1)–(A3) is assumed and the process X is U-recurrent then (2.14) is satisfied for all $x\in H$, because for any $t>0$, $x\in H$ we have $P(t,x,E)=1$, and therefore

$$\boldsymbol{P}_x[\tau^U<\infty]\geq\int_H \boldsymbol{P}_y[\tau^U<\infty]P(t,x,\mathrm{d}y)=\int_E \boldsymbol{P}_y[\tau^U<\infty]P(t,x,\mathrm{d}y)=1.$$

Now we are ready to construct a Markov chain which plays an essential role in the proofs of the main results of the paper. The Markov chain is obtained by stopping the process X at certain random times similarly as, for instance, in [16], [20], [21], [32]. Assume that the conditions (A1)–(A5) are satisfied and X corresponding to (1.1) is U-recurrent with respect to a bounded open set U in E. Note that since X is a continuous Feller process, it is strongly Markov. Denote by W the open ball in E containing U such that

$$\lim_{t\to 0+}\ \sup_{x\in U} \boldsymbol{P}_x[\tau_W\leq t]=0$$

the existence of which follows from Lemma 2.4. Let τ_0 be the first hitting time of $B=\mathrm{Cl}_E U$ by the process X. Let $\sigma_1=\inf\{t>\tau_0;\ X(t)\notin W\}$ be the first exit time from W after τ_0. By the E-continuity of trajectories of X we have $\sigma_1>\tau_0$ a.s. and by the strong Markov property and Lemma 2.3 it follows that $\sigma_1<\infty$ a.s.. Define further $\tau_1=\inf\{t>\sigma_1;\ X(t)\in B\}$ the first hitting time of B after σ_1 and by induction

$$\sigma_{n+1}=\inf\{t>\tau_n;\ X(t)\notin W\},\quad \tau_{n+1}=\inf\{t>\sigma_{n+1};\ X(t)\in B\},\qquad n\in\boldsymbol{N}.$$

Obviously τ_n, σ_n are stopping times and $\tau_{n-1}<\sigma_n<\tau_n$ for $n\in\boldsymbol{N}$. It follows from [30], Theorem 3, that the random sequence

$$Y_n=X(\tau_n), \qquad n\in\boldsymbol{N},$$

is a homogeneous Markov chain. Denote by $p=p(x,A)$, $x\in H$, $A\in\mathscr{B}(H)$, the transition probability function of the chain Y, i.e., $p(x,A)=\boldsymbol{P}_x[Y_1\in A]$, and set

$$p^n(x,A)=\boldsymbol{P}_x[Y_n\in A].$$

Lemma 2.7. *Assume* (A1)–(A5) *and let the process X be U-recurrent with respect to a bounded open set $U\subset E$. Then the Markov chain (Y_n) is strongly Feller in H, i.e., $p^n(\cdot,A)$ is continuous on H for every $n\in\boldsymbol{N}$, $A\in\mathscr{B}(H)$.*

Proof. Let us choose $n\geq 1$ and a bounded measurable function ψ: $H\to\boldsymbol{R}$ arbitrarily, we aim at proving that the function Φ: $H\to\boldsymbol{R}$, $\Phi(x):=\boldsymbol{E}_x\psi(X(\tau_n))$ is continuous on H. For any $t>0$ we have

$$\begin{aligned}\boldsymbol{E}_x\psi(X(\tau_n))&=\boldsymbol{E}_x\psi(X(\tau_n))\chi_{[\tau_n>t]}+\boldsymbol{E}_x\psi(X(\tau_n))\chi_{[\tau_n\leq t]}\\&=\Phi_t^{(1)}(x)+\Phi_t^{(2)}(x).\end{aligned}$$

First,

$$\begin{aligned}\lim_{t\to 0+}\ \sup_{x\in H}|\Phi_t^{(2)}(x)|&\leq\sup_H|\psi|\ \lim_{t\to 0+}\ \sup_{x\in H}\boldsymbol{P}_x[\tau_n\leq t]\\&=\sup_H|\psi|\ \lim_{t\to 0+}\ \sup_{x\in H}\int_B\boldsymbol{P}_y[\tau_n\leq t]p^0(x,\mathrm{d}y)\\&\leq\sup_H|\psi|\ \lim_{t\to 0+}\ \sup_{x\in H}\int_B\boldsymbol{P}_y[\tau_W\leq t]p^0(x,\mathrm{d}y)=0\end{aligned}$$

by Lemma 2.4, where $p^0(x,.)$ stands for the probability distribution of $X(\tau_0)$. Further, take $u\in(0,t)$ and define a bounded measurable function κ_{t-u} on H by $\kappa_{t-u}(y)=\boldsymbol{E}_y\psi(X(\tau_n))\chi_{[\tau_n>t-u]}$. Then

$$\begin{aligned}\Phi_t^{(1)}(x)&=\boldsymbol{E}_x\boldsymbol{E}_{X(u)}[\psi(X(\tau_n))\chi_{[\tau_n>t-u]}]=\boldsymbol{E}_x\kappa_{t-u}(X(u))\\&=\int_E\kappa_{t-u}(z)P(u,x,dz),\end{aligned}$$

therefore all the functions $\Phi_t^{(1)}$, $t>0$, are continuous by (A5). Thus Φ

is a uniform limit of continuous functions $\Phi_t^{(1)}$ on H and the strong Feller property of (Y_n) follows.

Q.E.D.

In the sequel denote by ∂U the E-boundary of U, that is

$$\partial \mathrm{U} = \mathrm{Cl}_E(U) \backslash U.$$

Proposition 2.8. *Assume* (A1)–(A5) *and let the process X be U-recurrent with respect to a bounded open set $U \subseteq E$. Then there exists a stationary distribution $\tilde{\mu}$ (that is, an invariant measure in $\mathscr{P}(\partial U)$) for the Markov chain Y.*

Proof. The chain Y evolves in the set ∂U and, therefore, also in the bigger set $K = \mathrm{Cl}_H(\partial U)$. The set K is compact in H due to compactness of the embedding $E \hookrightarrow H$. Thus Y is a Markov process which is Feller and takes values in a compact state space. Hence there exists an invariant measure $\tilde{\mu}$ in $\mathscr{P}(K)$, clearly $\tilde{\mu} \in \mathscr{P}(\partial U)$.

Q.E.D.

Now we can formulate the main result of the section.

Theorem 2.9. *Assume* (A1)–(A6) *and let the process X be U-recurrent for a bounded open set $U \subset E$. Then there exists a σ-finite invariant measure μ for the process X.*

Proof. Let

$$\tau(A) = \int_{\tau_0}^{\tau_1} \chi_A(X(t)) \mathrm{d}t, \qquad A \in \mathscr{B}(H),$$

be the time spent in A "during the first cycle" and set

$$\mu(A) = \int_{\partial U} \boldsymbol{E}_x \tau(A) \mathrm{d}\tilde{\mu}(x), \qquad A \in \mathscr{B}(H). \tag{2.15}$$

We show that μ is a σ-finite measure. Clearly μ is a measure because it is additive and σ-additivity follows from the monotone convergence theorem. Setting

$$A_n = \mathrm{Cl}_H(U_n) = \mathrm{Cl}_H\{x \in E;\ \|x\| < n\}$$

we have

$$\mu\left(H\backslash\left(\bigcup_n A_n\right)\right)\leq\mu(H\backslash E)=0$$

thus for verification of σ-finiteness of μ it suffices to show

$$\mu(A)<\infty \tag{2.16}$$

for any A of the form $A=\mathrm{Cl}_H(\mathscr{U})$, where $\mathscr{U}\subset E$ is bounded in E. By Proposition 1.5 it follows that

$$\boldsymbol{P}_x[\tau_W\leq t]>0, \qquad t>0,\ x\in\partial U, \tag{2.17}$$

and

$$\inf_{y\in\partial W}\boldsymbol{P}_y[\tau^U\leq t]>0, \tag{2.18}$$

since ∂W is relatively compact in H and the function

$$y\mapsto\boldsymbol{P}_y[\tau^U\leq t]$$

is continuous on $H\setminus U$ by (A5). Let ν_x be the probability distribution on $\boldsymbol{R}_+$ of the exit time τ_W of the process X, for which $X(0)=x\in\partial U$. By (2.17) we have $\nu_x((0,t))>0$ for $t>0$, $x\in\partial U$, therefore we obtain by the strong Markov property

$$\begin{aligned}\boldsymbol{P}_x[\tau_1\leq t]&=\int_0^t\int_{\partial W}\boldsymbol{P}_y\,[\tau^U\leq t-s]\boldsymbol{P}_x[\tau_W\in\mathrm{d}s,\ X(\tau_W)\in\mathrm{d}y]\\&\geq\int_0^t\inf\boldsymbol{P}_y[\tau^U\leq t-s]d\nu_x(s)>0, \qquad x\in\partial U.\end{aligned} \tag{2.19}$$

Similarly as in the proof of Lemma 2.7 it can be shown that (A5) implies continuity on H of the function $y\mapsto\boldsymbol{P}_y[\tau_1\leq t]$, therefore we have

$$\inf_{y\in\partial U}\boldsymbol{P}_y[\tau_1\leq t]>0, \tag{2.20}$$

thus for $x\in H$ we obtain

$$\begin{aligned}\boldsymbol{P}_x[\tau(A)\leq t]&=\int_{\partial U}\boldsymbol{P}_y[\tau(A)\leq t]p^0(x,\mathrm{d}y)\\&\geq\int_{\partial U}\boldsymbol{P}_y[\tau_1\leq t]p^0(x,\mathrm{d}y)>0.\end{aligned}$$

Using again (A5) and compactness of A in H we arrive at

$$s=\sup_{x\in A} \boldsymbol{P}_x[\tau(A)>t]<1. \tag{2.21}$$

Set

$$\Omega_t=\{\omega\in\Omega;\ \tau(A)>t\},\qquad \kappa(t)=\inf\{v\geq\tau_0;\ \int_{\tau_0}^{v}\chi_A(X(u))\mathrm{d}u=t\}.$$

It is easy to check that $\kappa(t)$ is a stopping time and the H-continuity of trajectories of X implies $X(\kappa(t))\in A$. Hence by the strong Markov property we get

$$\sup_{x\in A} \boldsymbol{P}_x(\Omega_{2t})=\sup_{x\in A}\int_{\Omega_t}\boldsymbol{P}[\Omega_t|X(\kappa(t))]\mathrm{d}\boldsymbol{P}_x\leq s^2.$$

Proceeding similarly by induction we obtain $\sup\{\boldsymbol{P}_x(\Omega_{nt});\ x\in A\}\leq s^n$, $n\in\boldsymbol{N}$, hence

$$\begin{aligned}\sup_{x\in A} \boldsymbol{E}_x\tau(A)&\leq\sup_{x\in A}\sum_{n\in\boldsymbol{N}}(nt)\boldsymbol{P}_x[(n-1)t\leq\tau(A)\leq nt]\\&\leq t\sum ns^{n-1}<\infty.\end{aligned} \tag{2.22}$$

Without loss of generality we can assume $\mathscr{U}\supset\partial U$ and so (2.22) implies (2.16) which concludes the proof of σ-finiteness of μ. The proof of the invariance of μ is now literally the same as in [20], Theorem 2.1.

Q.E.D.

Corollary 2.10. *Assume* (A1)–(A6) *and let there exist a bounded open set $U\subset E$ such that*

$$\sup\{\boldsymbol{E}_x\tau^U;\ x\in G\}<\infty \tag{2.23}$$

holds for any bounded set $G\subset E$. Then there exists an invariant measure $\mu^*\in\mathscr{P}(E)$.

Proof. From the construction of the invariant measure μ it follows that (2.23) and Lemma 2.3 imply $\mu(E)=\mu(H)<\infty$. Thus we may put $\mu^*(A)=\mu(A)/\mu(H)$, $A\in\mathscr{B}(H)$.

Q.E.D.

3. Ergodicity

The main result of the present section is the strong law of large numbers of the Hopf type presented in Theorem 3.2. As an auxiliary statement we establish at first irreducibility of the Markov chain Y introduced in Section 2. Note that if the process X is U-recurrent for a bounded open set $U \subset E$ then it is also U_R-recurrent for $U_R := \{x \in E; \|x\| < R\}$ with some $R > 0$.

Lemma 3.1. *Assume* (A1)–(A6) *and let the process X be U_R-recurrent for some $R > 0$. Then*

$$p^n(x, V) > 0$$

holds for every $n \in \boldsymbol{N}$, $x \in \partial U$, $\emptyset \neq V \subseteq \partial U$, V open in ∂U in the topology induced from E.

Proof. By the Chapman-Kolmogorov equality it suffices to prove the assertion for $n = 1$. Take $T > 0$, $y \in V$ arbitrary and let $r > 0$ be such that $U_r \supset W$ (for the definition of W see the construction of the Markov chain Y in Section 2). Set $x_1 = (2r/R)x$, $y_1 = (2r/R)y$ and define a continuous curve $\varphi = \{\varphi(t), t \in [0, T]\}$ in E such that $\varphi(0) = x$, $\varphi(T/3) = x_1, \varphi(2T/3) = y_1$, $\varphi(T) = 0$, φ is linear on the intervals $[0, T/3]$ and $[2T/3, T]$ and $\|\varphi(t)\| > r$ for $t \in [T/3, 2T/3]$. Find $\varepsilon > 0$ such that $\|\varphi(t)\| > r + \varepsilon$ for $t \in [T/3, 2T/3]$, $\varepsilon < R$ and

$$\{z \in E;\ \|z - \varphi(t)\| < \varepsilon \qquad \text{for some } t \in [2T/3, T]\} \cap \partial U \subseteq V.$$

Set

$$\xi(t) = \varphi(t) - S(t)x - \int_0^t S(t-u) f(\varphi(u))\mathrm{d}u, \qquad t \in [0, T].$$

Clearly $\xi \in \tilde{E}$ and by (A4b) there exists a $\delta > 0$ such that

$$\|u(t, x, \gamma) - \varphi(t)\| < \varepsilon$$

holds for all $t \in [0, T]$ and $\gamma \in \tilde{E} = \tilde{E}(T)$ satisfying

$$\|\gamma(t) - \xi(t)\| < \delta, \qquad t \in [0, T].$$

It follows that

$$\begin{aligned} p(x, V) = p^1(x, V) = \boldsymbol{P}_x[X(\tau_1) \in V] \geq \boldsymbol{P}_x[\|X - \varphi\|_{\sim} < \varepsilon] \\ \geq \boldsymbol{P}[\|Z - \xi\|_{\sim} < \delta] \end{aligned} \tag{3.1}$$

and the last term in (3.1) is positive by (A6).

Q.E.D.

Theorem 3.2. *Assume* (A1)–(A6) *and let the process defined by the equation* (1.1) *be U-recurrent for a bounded open set* $U \subset E$. *Then*

$$\boldsymbol{P}_x\left[\lim_{T\to\infty}\frac{\int_0^T\varphi(X(t))\mathrm{d}t}{\int_0^T\psi(X(t))\mathrm{d}t}=\frac{\int_E\varphi\mathrm{d}\mu}{\int_E\psi\mathrm{d}\mu}\right]=1 \tag{3.2}$$

holds for every $x\in H$, $\varphi,\psi\colon H\to\boldsymbol{R}$ μ*-integrable such that* $\int_E\psi\mathrm{d}\mu\neq 0$, *where* μ *is the invariant measure constructed in Theorem* 2.9.

Proof. Step 1. At first we establish ergodicity of the chain Y. Due to the irreducibility of Y (Lemma 3.1) and the strong Feller property (Lemma 2.7) the strong law of large numbers in the form

$$\boldsymbol{P}_y\left[\frac{1}{n}\sum_{i=1}^{n}\xi(Y_i)\overset{n\to\infty}{\to}\int_{\partial U}\xi\mathrm{d}\tilde{\mu}\right]=1 \tag{3.3}$$

holds for $\tilde{\mu}$-almost all $y\in\partial U$ and all $\tilde{\mu}$-integrable $\xi\colon\partial U\to\boldsymbol{R}$. Using again Lemmas 2.7 and 3.1 it is easy to see by the Chapman-Kolmogorov equality that the measures $\tilde{\mu}$ and $p^n(x,\cdot)$ are equivalent for $n>1$, $x\in\partial U$. Consequently, (3.3) holds for all $y\in\partial U$ and

$$p^n(x,A)\to\tilde{\mu}(A),\qquad n\to\infty, \tag{3.4}$$

for every $x\in\partial U$, $A\in\mathscr{B}(\partial U)$ (cf. [7], Theorem 5).

Step 2. In this part of the proof we show (3.2) with x replaced by $\tilde{X}$, where $\tilde{X}$ is an $\mathscr{F}_0$-measurable random variable with probability distribution $\tilde{\mu}$. Without loss of generality we can assume that $\varphi\geq 0$. Set

$$\beta_i=\int_{\tau_i}^{\tau_i+1}\varphi(X(t))\mathrm{d}t,\qquad i\in\boldsymbol{N}.$$

The sequence (β_i) is a stationary process and from (3.4) it follows that (β_i) is ergodic, that is,

$$\boldsymbol{P}_{\tilde{X}}\left[\frac{1}{n}\sum_{i=0}^{n-1}\beta_i\overset{n\to\infty}{\to}\boldsymbol{E}_{\tilde{X}}\beta_0=\int_E\varphi\mathrm{d}\mu\right]=1. \tag{3.5}$$

Let

$$k(T)=\begin{cases} n, & \tau_n \leq T < \tau_{n+1}, n \in \boldsymbol{N}, \\ 1, & T < \tau_1, \end{cases}$$

be the number of cycles up to time T. We have

$$\sum_{i=0}^{k(T)-1} \beta_i \leq \int_0^T \varphi(X(t))\mathrm{d}t \leq \sum_{i=0}^{k(T)} \beta_i,$$

thus

$$\boldsymbol{P}_{\tilde{X}}\left[\frac{1}{k(T)} \int_0^T \varphi(X(t))\mathrm{d}t \overset{T\to\infty}{\to} \int_E \varphi \mathrm{d}\mu \right] = 1 \tag{3.6}$$

which together with the same result with φ replaced by ψ implies

$$\boldsymbol{P}_{\tilde{X}}\left[\lim_{T\to\infty} \frac{\int_0^T \varphi(X(t))\mathrm{d}t}{\int_0^T \psi(X(t))\mathrm{d}t} = \frac{\int_E \varphi \mathrm{d}\mu}{\int_E \psi \mathrm{d}\mu} \right] = 1,$$

hence there exists a measurable set $M \subseteq \partial U$, $\tilde{\mu}(M)=1$, such that (3.2) holds for every $x \in M$.

Step 3. Let $x \in H$ be arbitrary, fix $m>1$ and set

$$c=\frac{\int_E \varphi \mathrm{d}\mu}{\int_E \psi \mathrm{d}\mu}.$$

We have

$$\begin{aligned} &\boldsymbol{P}_x\left[\left(\int_0^T \psi(X(t))\mathrm{d}t \right)^{-1} \int_0^T \varphi(X(t))\mathrm{d}t \to c \right] \\ &= \int_{\partial U} \boldsymbol{P}_y\left[\left(\int_0^T \psi(X(t))\mathrm{d}t \right)^{-1} \int_0^T \varphi(X(t))\mathrm{d}t \to c \right] p^m(x,\mathrm{d}y) \\ &= \int_M \boldsymbol{P}_y\left[\left(\int_0^T \psi(X(t))\mathrm{d}t \right)^{-1} \int_0^T \varphi(X(t))\mathrm{d}t \to c \right] p^m(x,\mathrm{d}y) = 1 \end{aligned}$$

since $p^m(x,M)=1$ due to the equivalence of $p^m(x,\cdot)$ and $\tilde{\mu}$.

Q.E.D.

Corollary 3.3. *Assume* (A1)–(A6) *and let the process defined by the equation* (1.1) *be U-recurrent for a bounded open set* $U \subset E$. *Then it is recurrent.*

Proof. We preserve the notation from Theorem 3.2. Let G be a nonempty bounded open set in E. At first we show

$$(3.7) \qquad \boldsymbol{P}_{\tilde{X}}[\tau^G<\infty]=1.$$

Using (3.5) with $\varphi=\chi_G$ we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\beta_i=\int_E \chi_G \mathrm{d}\mu=\mu(G) \qquad \boldsymbol{P}_{\tilde{X}}\text{-a.s.}$$

since $\mu(G)<\infty$ as shown in the proof of Theorem 2.9. Proposition 1.5 yields

$$\mu(G)=\int_E P(t,x,G)\mathrm{d}\mu(x)>0, \qquad t>0,$$

thus

$$\boldsymbol{P}_{\tilde{X}}\,[\exists n\in\boldsymbol{N};\ \beta_n>0]=1,$$

this proves (3.7). It follows that

$$\boldsymbol{P}_y[\tau^G<\infty]=1$$

holds for any $y\in N\subseteq\partial U$, $\tilde{\mu}(N)=1$. Since $\tilde{\mu}$ and $p^m(x,\cdot)$ are equivalent for $x\in H$, $m>1$, we obtain

$$\boldsymbol{P}_x[\tau^G<\infty]\geq\int_N \boldsymbol{P}_y[\tau^G<\infty]p^m(x,\mathrm{d}y)=1.$$

Q.E.D.

Corollary 3.4. *Assume* (A1)–(A6) *and let the process X defined by the equation* (1.1) *be recurrent. Let μ be the invariant measure provided by Theorem* 2.9. *Then*

(i) *If $\mu(H)<\infty$ then $\mu^*(\cdot)=\mu(\cdot)/\mu(H)$ is a stationary distribution satisfying*

$$(3.8) \qquad \boldsymbol{P}_x\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi(X(t))\mathrm{d}t=\int_H \varphi\mathrm{d}\mu^*\right]=1$$

and

$$(3.9) \qquad \lim_{T\to\infty} P(T,x,A)=\mu^*(A),$$

for all $x\in H$, $A\in\mathscr{B}(H)$ and μ^*-integrable φ: $H\to\boldsymbol{R}$.

(ii) *If* $\mu(H)=\infty$ *then*

$$\lim_{T\to\infty} \boldsymbol{P}_x\left[\frac{1}{T}\int_0^T \varphi(X(t))\mathrm{d}t>\varepsilon\right]=0$$

for any $\varepsilon>0$ *and every* $x\in H$ *and* $\varphi\colon H\to \boldsymbol{R}$ *μ-integrable. In particular,*

$$\frac{1}{T}\int_0^T P(t,x,D)\mathrm{d}t\to 0, \qquad T\to\infty, \tag{3.10}$$

for every $x\in H$, $D\subset E$, *D bounded and measurable.*

Proof. The formula (3.8) is obtained directly from Theorem 3.2 by putting $\psi=1$. Furthermore, by the strong Feller property (A5) and irreducibility (I) (Proposition 1.5) it follows that the transition probabilities $P(t,x,\cdot)$ and the measure μ^* are equivalent for $t>0$, $x\in H$, (see, for instance, [24], Theorem 2.2). Therefore (3.9) is a consequence of (3.8) (cf. [7], Theorem 5). We will prove (ii) by the contradiction. If the assertion is false, then there exists a nonnegative μ-integrable function φ, $x\in H$, a sequence (T_n) of positive numbers growing to infinity and a number $\delta>0$ such that

$$\boldsymbol{P}_x\left[\frac{1}{T_n}\int_0^{T_n} \varphi(X(t))\mathrm{d}t>\delta\right]>\delta, \qquad n\in\boldsymbol{N}. \tag{3.11}$$

Take a sequence (M_k) in $\mathscr{B}(H)$ such that $\mu(M_k)<\infty$, $k\in\boldsymbol{N}$, and $\mu(M_k)\to\infty$, $k\to\infty$. By (3.11) it follows that

$$\boldsymbol{P}_x\left[\frac{\int_0^{T_n}\varphi(X(t))\mathrm{d}t}{\int_0^{T_n}\chi_{M_k}(X(t))\mathrm{d}t}>\delta,\quad n\in\boldsymbol{N}\right]>\delta$$

holds for all $k\in\boldsymbol{N}$. We can use Theorem 3.2 which yields

$$\boldsymbol{P}_x\left[\frac{\int_0^{T_n}\varphi(X(t))\mathrm{d}t}{\int_0^{T_n}\chi_{M_k}(X(t))\mathrm{d}t}\overset{n\to\infty}{\to}\frac{\int_H\varphi\mathrm{d}\mu}{\mu(M_k)}\right]=1 \tag{3.12}$$

and (3.12) contradicts (3.11) for k enough large. Finally, (3.10) follows by the dominated convergence theorem. (Note that $\mu(D)<\infty$ by the proof of Theorem 2.9.)

Q.E.D.

Proposition 3.5. *Let* (A1)–(A6) *be satisfied and assume that the process*

X defined by the equation (1.1) *is not recurrent. Then*

$$\int_0^\infty P(t,x,D)\mathrm{d}t < \infty \tag{3.13}$$

and

$$P(t,x,D) \to 0, \qquad t \to \infty, \tag{3.14}$$

holds for every $x \in H$ *and* $D \subset E$, *D bounded and measurable.*

Proof. Let U be a bounded open set in E such that $U \supset D$. By Corollary 3.3 there exists $z \in H$ such

$$\boldsymbol{P}_z[\tau^U = \infty] > 0.$$

From the strong Feller property (A5) it follows that the function $z \mapsto \boldsymbol{P}_z\,[\tau^U = \infty]$ is continuous on $H \backslash U$ (and, therefore, also on $E \backslash U$), so there exists an open nonempty neighborhood $V \subset E$ such that

$$r = \inf_{y \in V} \boldsymbol{P}_y[\tau^U = \infty] > 0. \tag{3.15}$$

Set

$$\sigma_U = \int_0^\infty \chi_U(X(s))\mathrm{d}s.$$

For $y \in V$ we have by the the strong Markov property

$$1 - \boldsymbol{P}_y[\sigma_U > 1] \geq rP(1,y,V). \tag{3.16}$$

Furthermore, due to relative compactness of U in H Proposition 1.5 together with (A5) imply

$$v = \inf\{P(1,y,V);\ y \in \mathrm{Cl}_E U\} > 0.$$

Thus by (3.16)

$$\boldsymbol{P}_y[\sigma_U > 1] \leq 1 - rv = k < 1, \qquad y \in \mathrm{Cl}_E U,$$

and, consequently,

$$\boldsymbol{P}_x[\sigma_U > 1] \leq k, \qquad x \in H.$$

Using repeatedly the strong Markov property similarly as in the proof of

Theorem 2.9 we obtain by induction

$$\boldsymbol{P}_x[\sigma_U > n] \leq k^n, \qquad x \in H,\ n \in \boldsymbol{N},$$

and hence

$$\int_0^\infty P(t,x,D)\mathrm{d}t \leq \int_0^\infty P(t,x,U)\mathrm{d}t = \boldsymbol{E}_x \int_0^\infty \chi_U(X(t))\mathrm{d}t$$

$$\leq \boldsymbol{E}_x \sum_n \sigma_U \chi_{[n \leq \sigma_U \leq n+1]} \leq \sum_n (n+1)k^n < \infty.$$

The statement (3.14) will be proved by contradiction. Suppose that there exists a sequence (t_n), $t_n \to \infty$, $x \in H$ and positive numbers δ, L, such that

$$P(t_n, x, U_L) > \delta, \qquad n \in \boldsymbol{N}, \tag{3.17}$$

where $U_L = \{y \in E;\ \|y\| < L\}$. By (A4a) there exists a positive M such that $\|u(t,y,\varphi)\| \leq M$ holds for $y \in U$, $t \in [0,1]$ and $\varphi \in \tilde{E}(1)$, $\|\varphi\|_{\sim} < L$. By (A6) we have

$$\eta := \boldsymbol{P}[\sup\{\|Z(t)\|;\ t \in [0,1]\} > L] < 1$$

thus by (A4a) it follows that

$$P(s,y,U_M) \geq 1-\eta, \qquad s \in [0,1],\ y \in U_L. \tag{3.18}$$

From (3.17) and (3.18) we obtain by the Chapman-Kolmogorov equality

$$P(s+t_n, x, U_M) \geq \delta(1-\eta), \qquad s \in [0,1],\ n \in \boldsymbol{N},$$

which contradicts to (3.13).

Q.E.D.

In the following proposition we show that (3.13) in a sense characterizes non-recurrent solutions of (1.1).

Proposition 3.6. *Assume* (A1)–(A6), *let the process* X *defined by the equation* (1.1) *be recurrent. Let* μ *be its* σ*-finite invariant measure provided by Theorem* 2.9. *Then*

$$\int_0^\infty P(t,x,U)\mathrm{d}t = +\infty, \qquad x \in H,$$

for any $U\in\mathscr{B}(H)$ *such that* $\mu(U)>0$.

REMARK. In fact, we will prove more, namely

$$P_x\left[\int_0^\infty \chi_U(X(t))\mathrm{d}t=\infty\right]=1$$

for any $x\in H$, $U\in\mathscr{B}(H)$ such that $\mu(U)>0$. In other words, a recurrent solution to (1.1) defines a Markov process fulfilling the Harris condition, which is known to be sufficient for the existence of a σ-finite invariant measure ([16]). Nevertheless, to establish the Harris recurrence we needed a rather detailed knowledge of the invariant measure μ and its properties.

Proof. We will use the ideas and notation of the proof of Theorem 3.2. Let Y be the solution to (1.1) such that the law of $\tilde{X}:=Y(0)$ is just $\tilde{\mu}$-the invariant measure for the embedded chain. Take $U\in\mathscr{B}(H)$ such that $0<\mu(U)<\infty$. We know that

$$P\left[\frac{1}{n}\sum_{i=1}^{n}\int_{\tau_i}^{\tau_{i+1}} \chi_U(Y(t))\mathrm{d}t \overset{n\to\infty}{\to} \mu(U)\right]=1,$$

hence for almost all ω there exist $i_k(\omega)\in N$, $i_k(\omega)\to\infty$, such that

$$\int_{\tau_{i_k(\omega)}}^{\tau_{i_k+1(\omega)}} \chi_U(Y(t,\omega))\mathrm{d}t\geq\frac{\mu(U)}{2}>0, \qquad k\in N.$$

But this implies

$$\int_0^\infty \chi_U(Y(t,\omega))\mathrm{d}t\geq\sum_{k=1}^{\infty}\int_{\tau_{i_k(\omega)}}^{\tau_{i_k+1(\omega)}} \chi_U(Y(t,\omega))\mathrm{d}t=\infty$$

for almost all ω. As

$$1=P_{\tilde{X}}\left[\int_0^\infty \chi_U(X(t))\mathrm{d}t=\infty\right]=\int_H P_y\left[\int_0^\infty \chi_U(X(t))\mathrm{d}t=\infty\right]\mathrm{d}\tilde{\mu}(y)$$

we have

$$P_y\left[\int_0^\infty \chi_U(X(t))\mathrm{d}t=\infty\right]=1$$

for $\tilde{\mu}$-almost any $y \in H$. Thus we obtain

$$\boldsymbol{P}_x\left[\int_0^\infty \chi_U(X(t))\mathrm{d}t = \infty\right] = \int_H \boldsymbol{P}_y\left[\int_0^\infty \chi_U(X(t))\mathrm{d}t = \infty\right] p^2(x,\mathrm{d}y) = 1$$

for arbitrary $x \in H$ due to the equivalence of $\tilde{\mu}$ and $p^2(x,\cdot)$. (Recall that $p^2(x,\cdot) = \boldsymbol{P}_x[Y_2 \in A]$, (Y_n) being the embedded Markov chain.) Proposition 3.6 follows.

Q.E.D.

Corollary 3.7. *Assume* (A1)–(A6). *Then*:

(i) *If the process X defined by the equation* (1.1) *is recurrent, then there exists a unique–up to a multiplicative constant–σ-finite invariant measure for X.*

(ii) *If the process X is not recurrent, then there exists no finite invariant measure.*

REMARK. Note that the second part of Corollary 3.7 says that the existence of a finite invariant measure for (1.1) implies that the solution to (1.1) is U-recurrent for any $U \subseteq E$ open nonempty. (Cf. [37], Theorem 4, where a closely related result is established for linear equations.)

Proof. The first assertion of Corollary follows immediately from Theorem A.1 (established in the Appendix) if we take into account Proposition 3.6 and the fact that the transition probabilities $P(t,x,\cdot)$, $t>0$, $x \in H$, are equivalent in virtue of irreducibility and the strong Feller property.

The second assertion can be proved as in the finite dimensional case (see [20], Theorem 3.3); we repeat the simple argument for completeness. Let ν be a finite invariant mesure for the non-recurrent solution X, let $D \subset E$ be an arbitrary bounded set. As (3.13) implies

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T P(t,x,D)\mathrm{d}t = 0, \qquad x \in H,$$

we have by the dominated convergence theorem

$$\begin{aligned}\nu(D) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_H P(t,x,D)\mathrm{d}\nu(x)\mathrm{d}t \\ &= \int_H \left[\lim_{T\to\infty} \frac{1}{T}\int_0^T P(t,x,D)\mathrm{d}t\right]\mathrm{d}\nu(x) = 0,\end{aligned}$$

that yields $\nu=0$, which is a contradiction.

Q.E.D.

EXAMPLE 3.8. Suppose that the generator A and the covariance operator Q have the form

$$Ae_i = -\alpha_i e_i, \qquad Qe_i = \lambda_i e_i, \qquad i \in \boldsymbol{N}, \tag{3.19}$$

where $\{e_i\}$ is an orthonormal basis of the Hilbert space H and

$$0 < \alpha_i \overset{i\to\infty}{\to} +\infty; \qquad 0 < \lambda_i \leq \lambda_0 < \infty, \qquad i \in \boldsymbol{N}.$$

In this case we have

$$S(t)x = \sum_i \exp\{-\alpha_i t\}\langle x, e_i\rangle e_i,$$

$$Q_t x = \sum_i \frac{\lambda_i}{2\alpha_i}\Big(1 - \exp\{-2\alpha_i t\}\Big)\langle x, e_i\rangle e_i, \qquad x \in H,$$

which simplifies the discussion of the conditions imposed in this paper. Below we check the particular assumptions (A1)–(A6). The condition (A1) is satisfied if and only if

$$\sum_i \frac{\lambda_i}{\alpha_i} < \infty. \tag{3.20}$$

The remaining conditions depend also on the choice of the Banach space E. Take $E = E_\alpha = \mathscr{D}((-A)^\alpha)$ equipped with the graph norm, $\alpha \in (0,1)$. The embedding $E_\alpha \subsetneq H$ is compact. Assume that (3.20) is strengthened to

$$\sum_i \frac{\lambda_i}{\alpha_i^{2\varepsilon-1}} < \infty \tag{3.21}$$

for some $\varepsilon > 0$. The assumption (A2) is fulfilled for $E = E_\alpha$ with $0 < \alpha < \varepsilon$ due to [6], Theorem 4, and [36], Proposition 1. The condition (A6) is satisfied due to Proposition 1.4 either if $0 < \alpha < 1/2$ (then (D2) holds with $h(t) = ct^{-\alpha}$ for some $c > 0$) or if the sequence $\{(\alpha_i)^\alpha(\lambda_i)^{1/2}\}$ is bounded (then (D1) is fulfilled). Furthermore, we have

$$B_t e_i = Q_t^{-1/2} S(t) e_i = \exp\{-\alpha_i t\} \sqrt{\frac{2\alpha_i}{\lambda_i(1-\exp\{-2\alpha_i t\})}}\, e_i,$$

therefore if there exist positive constants T, c and $\beta \in (0,1)$ such that

$$\sup_i \sqrt{\frac{\alpha_i}{\lambda_i(\exp\{2\alpha_i t\}-1)}} \leq ct^{-\beta}, \qquad t \in [0,T], \tag{3.22}$$

then (1.4)–(1.7) are satisfied. Assume, in addition, that the function f: $E=E_\alpha \to H$, $0<\alpha<\min(1/2,\varepsilon)$, satisfies (A3a) and

(C1) *There exists $C<\infty$ such that $|f(x)-f(y)| \leq C\|x-y\|$ for all $x,y \in E$.*

(For instance, this is always true for f: $H \to H$ globally Lipschitzian.) Then the remaining conditions (A3b), (A4) and (A5) are fulfilled by Propositions 1.2 and 1.3. We can summarize that if A and Q have the form (3.19) with (α_i), (λ_i) satisfying (3.21) and (3.22) and f: $E_\alpha \to H$ satisfies (A3a) and (C1) for some $0<\alpha<\min(1/2,\varepsilon)$ then all assumptions (A1)–(A6) are fulfilled and Theorems 2.9, 3.2, Corollaries 3.4, 3.7 and Proposition 3.5 are applicable. In particular, there exists at most one stationary distribution and if the solutions of (1.1) are recurrent then the strong law of large numbers of Hopf's type holds true.

REMARK 3.9. In the linear case ($f=0$) the strong Feller property can be verified directly and it is easy to see that it is equivalent to (1.4) (cf. [29], Proposition 2.5). We will show–in the situation of Example 3.8 with $f=0$–that if $\mathscr{R}(S(t)) \nsubseteq \mathscr{R}(Q_t^{1/2})$, $t>0$, then the strong law of large numbers does not hold. Indeed, in this case there exists a sequence $i(n) \to \infty$ such that

$$\frac{\alpha_{i(n)}}{\lambda_{i(n)}} \exp\{-2\alpha_{i(n)} n\} \geq n^2.$$

Set $x=\sum_i x_i e_i$, where

$$x_j = \begin{cases} \dfrac{1}{n}, & j=i(n),\ n \in \boldsymbol{N}, \\ 0, & j \neq i(n). \end{cases}$$

For any $t>0$ we have

$$\sum_j \frac{\alpha_j}{\lambda_j} \exp\{-2\alpha_j t\} x_j^2 = \sum_n \frac{\alpha_{i(n)}}{\lambda_{i(n)}} \frac{\exp\{-2\alpha_{i(n)} t\}}{n^2}$$

$$\geq \sum_{n>t} \frac{\alpha_{i(n)}}{\lambda_{i(n)}} \frac{\exp\{-2\alpha_{i(n)}t\}}{n^2}$$

$$= \infty,$$

therefore $S(t)x \notin \mathscr{R}(Q_t^{1/2})$. Since the semigroup $S(\cdot)$ is exponentially stable there exists a unique stationary distribution $\mu^* \in \mathscr{P}(H)$, $\mu^* = N(0,\Gamma)$, where

$$\Gamma = \lim_{t \to \infty} Q_t$$

(cf. [37]). Set $A = \mathscr{R}(\Gamma)$, it is straightforward to verify that $\mathscr{R}(Q_t^{1/2}) = A$ for $t > 0$, hence

$$P(t,x,S(t)x + \mathscr{R}(Q_t^{1/2})) = P(t,x,S(t)x + A) = 1$$

and since $S(t)x \notin A$, we get

$$P(t,x,A) = 0, \qquad t > 0.$$

On the other hand we have $\mu^*(A) = 1$, therefore the measures μ^* and

$$\frac{1}{T} \int_0^T P(t,x,\cdot)\mathrm{d}t$$

are singular for all $T > 0$ and it follows that the strong law of large numbers does not hold in this case. (However, note that $P(t,y,\cdot)$ converges weakly to μ^* as $t \to \infty$ for every $y \in H$, cf. [28], Proposition 3.1.)

EXAMPLE 3.10. Consider the system of reaction-diffusion equations formally described as

$$\frac{\partial}{\partial t} u_i(t,x) = \Delta u_i(t,x) + F_i(u_1(t,x),\cdots,u_n(t,x)) + \eta_i(t,x), \tag{3.24}$$

$$i = 1,2,\cdots,n, \qquad (t,x) \in \boldsymbol{R}_+ \times (0,\pi),$$

$$u_i(0,x) = u_i^0(x), \qquad x \in (0,\pi), \tag{3.25}$$

$$u_i(t,0) = u_i(t,\pi) = 0, \qquad t \in \boldsymbol{R}_+, \qquad i = 1,2,\cdots,n, \tag{3.26}$$

where $F = (F_i)$: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ is globally Lipschitzian and $\eta = (\eta_i)$ stands formally for a space-time white noise in $\boldsymbol{R}^n$. A mathematically rigorous sense to the problem (3.24)–(3.26) is given in a usual manner. An equation of the form

(1.1) is considered, where

$$Ah=\left(\frac{\partial^2}{\partial x^2}h_1,\cdots,\frac{\partial^2}{\partial x^2}h_n\right),\qquad h=(h_i)\in\mathscr{D}(A),$$

$$\mathscr{D}(A)=[H_0^1(0,\pi)\cap H^2(0,\pi)]^n,$$

is an infinitesimal generator of a strongly continuous semigroup on the Hilbert space $H=(L_2(0,\pi))^n$,

$$f(x)(\gamma):=\big(F_1(x(\gamma)),\cdots,F_n(x(\gamma))\big),\qquad x\in H,\quad \gamma\in(0,\pi),$$

and W_t stands for a cylindrical Wiener process on H with the covariance operator $Q=I$ (identity). This example is in a sense a particular case of Example 3.8: Choose the orthonormal basis $\{e_{i,j}\}$ in H, $i\in\boldsymbol{N}$, $j\in J=\{1,\dots,n\}$,

$$e_{i,j}(\gamma)=(0,\cdots,\sqrt{\frac{2}{\pi}}\sin(i\gamma),\cdots,0),\qquad (j\text{-th place}),\qquad \gamma\in(0,\pi),$$

then

$$Ae_{i,j}=i^2e_{i,j},\qquad Qe_{i,j}=e_{i,j},\qquad i\in\boldsymbol{N},\ j\in J.$$

Since

$$\sum_{i=0}^{\infty} i^{2(2\varepsilon-1)}<\infty$$

for any $0<\varepsilon<1/4$, there exist positive constants C, T such that

$$\frac{i}{\sqrt{\exp\{2i^2t\}-1}}\le Ct^{-1/2},\qquad t\in(0,T],\quad i\in\boldsymbol{N}.$$

Moreover, the function f: $H\to H$ is globally Lipschitzian, so similarly as in Example 3.8 we obtain that all assumptions (A1)–(A6) are satisfied for $E=E_\alpha$ with $0<\alpha<1/4$. In the present example we have (see [15], Theorem 1) $E_\alpha=H^{2\alpha}(0,\pi)$, $0<\alpha<1/4$. Another possibility is to choose for E the space $V_\delta=[h_0^\delta[0,\pi]]^n$, $\delta\in(0,1/4)$ fixed, where

$$\begin{aligned}&h_0^\delta[0,\pi]\\&\quad=\{v\in\boldsymbol{C}^{0,\delta}[0,\pi];\ \lim_{t\to0+}\ \sup\{\frac{|v(x)-v(y)|}{|x-y|^\delta};\ x,y\in[0,\pi],\ |x-y|\le t\}=0,\\&\qquad\qquad v(0)=v(\pi)=0\}\end{aligned}$$

is equipped with the norm induced from the space $C^{0,\delta}[0,\pi]$ of δ-Hölder continuous functions. Note that we can take for E neither the space $[C[0,\pi]]^n$(because it is not compactly embedded into $[L_2(0,\pi)]^n$) nor the space $[C^{0,\delta}[0,\pi]]^n$ (since it is not separable). The spaces V_δ are separable (cf. [10], Theorem A) and the restriction $S(\cdot)_{|V_\delta}$ is a strongly continuous semigroup on V_δ (cf. [23], Theorem 2.14). Furthermore, for $\mu>\delta>0$ the space

$$[C_0^{0,\mu}[0,\pi]]^n:=\{v\in[C^{0,\mu}[0,\pi]]^n;\ v(0)=v(\pi)=0\}$$

is continuously embedded into V_δ. Thus (A2) follows from [3], Theorem 2, and (D2) is satisfied because

$$S(t)\in\mathscr{L}([L_2(0,\pi)]^n,\ [C_0^{0,\mu}[0,\pi]]^n),\qquad t>0,$$

and

$$|S(t)|_{\mathscr{L}([L_2(0,\pi)]^n,[C_0^{0,\mu}[0,\pi]]^n)}\leq Ct^{-\mu-1/4},\qquad t\in(0,T],$$

holds for all $\mu\in(0,1/4)$ and some $C=C(\mu)<\infty$. It follows that all assumptions (A1)–(A6) are fulfilled for $E=V_\delta$, $\delta\in(0,1/4)$, and we can again make the same conclusion as in Example 3.8.

Appendix

In this Appendix we aim at establishing a theorem on the uniqueness of a σ-finite invariant measure, which is used in the proof of Corollary 3.7. We will prove the result in a more general setting than we need as it may be of independent interest.

Let $(E,\mathscr{E})$ be a measurable space, $(P_t(x,\cdot))_{t>0}$ a semigroup of Markov transition kernels on $(E,\mathscr{E})$, that is

(i) $\forall A\in\mathscr{E}\quad\forall t>0$, $P_t(\cdot,A)$ *is* $\mathscr{E}$*-measurable,*

(ii) $\forall x\in E\quad\forall t>0$ $P_t(x,\cdot)$ *is a probability measure on* $\mathscr{E}$,

(iii) $\forall A\in\mathscr{E}\quad\forall x\in E\ \forall s,t>0$ $P_{t+s}(x,A)=\displaystyle\int_E P_t(y,A)P_s(x,\mathrm{d}y)$.

We will assume further

(iv) $\forall A\in\mathscr{E}\quad(t,x)\mapsto P_t(x,A)$ *is* $\mathscr{B}((0,\infty))\otimes\mathscr{E}$*-measurable.*

Here we denote by $\mathscr{B}((0,\infty))$ the Borel σ-algebra in $(0,\infty)$. (Note that (iv) is fulfilled e.g. if there exists a measurable E-valued Markov process with the transition probability function $P_t(x,\cdot)$, see [9], Remark following Lemma 5.3.) Let us denote by $\boldsymbol{B}=\boldsymbol{B}(E,\mathscr{E})$ the Banach space of all bounded real $\mathscr{E}$-measurable functions on E equipped with the supremum norm. Set

$$T_t f = \int_E f(y) P_t(\cdot, \mathrm{d}y), \qquad t>0,\ f\in \boldsymbol{B};$$

let T_t^* be the adjoint operator acting on the dual space $\boldsymbol{B}^*$. In particular, any finite measure ψ on $\mathscr{E}$ belongs to the space $\boldsymbol{B}^*$, hence $T_t^*\psi$ is well-defined and one has

$$T_t^*\psi = \int_E P_t(x,\cdot)\mathrm{d}\psi(x), \qquad t>0. \tag{A.1}$$

Obviously, the formula (A.1) makes it possible to define $T_t^*\psi$ for any nonnegative measure ψ on $\mathscr{E}$. Let us define the resolvent kernel by

$$U(x,A) = \int_0^\infty e^{-t} P_t(x,A)\mathrm{d}t, \qquad x\in E,\ A\in\mathscr{E}$$

and for a nonnegative measure ψ set

$$\psi U = \int_E U(x,\cdot)\mathrm{d}\psi(x).$$

A nonnegative measure $\varphi\neq 0$ is called *invariant* provided $T_t^*\varphi=\varphi$ for any $t>0$. Note that if a σ-finite measure q is invariant, then $q=qU$.

For brevity, we will occasionally set $\langle\psi,f\rangle$ for $\int_E f\mathrm{d}\psi$.

Proposition A.1. *Let $(P_t(x,\cdot))_{t\geq 0}$ be a semigroup of transition kernels on $(E,\mathscr{E})$ satisfying* (i)-(iv) *above. Let us assume*:

(A) *All the measures $U(x,\cdot)$, $x\in E$, are equivalent.*

Let μ be a σ-finite invariant measure such that there exists a μ-integrable nonnegative $f\in\boldsymbol{B}$, $f\geq 0$, satisfying

$$\int_0^\infty T_t f(x)\mathrm{d}t = \infty \qquad \textit{for any } x\in E. \tag{A.2}$$

Then any σ-finite invariant measure differs from μ only by a multiplicative constant.

Proof. Fix an arbitrary $x_0\in E$ and set $m=U(x_0,\cdot)$. Obviously, (A) implies that any σ-finite invariant measure is equivalent with m. Let ρ be an arbitrary σ-finite invariant measure. One can find a strictly positive function $h\in\boldsymbol{B}$ satisfying $\langle\mu,h\rangle<\infty$, $\langle\rho,h\rangle<\infty$. Take a constant $c\in\boldsymbol{R}_+$

such that for $\nu = c\rho$ the identity

$$\langle \mu, h\rangle = \langle \nu, h\rangle < \infty \tag{A.3}$$

holds. Our aim is to prove $\mu = \nu$. The measures μ, ν have densities u, v with respect to m, $\mu = um$, $\nu = vm$. By the σ-finiteness of μ, ν, the functions u, v may be taken finite everywhere, hence the measure $(\mu-\nu)^+ \equiv (u-v)^+ m$ is well defined. Using the minimality of the Jordan decomposition it is easy to see that $T_t^*(\mu-\nu)^+(B) \geq (\mu-\nu)^+(B)$ for an arbitrary set $B \in \mathscr{E}$ with $\mu(B)+\nu(B) < \infty$, therefore for any $B \in \mathscr{E}$ by the σ-finiteness.

Taking t, $T > 0$ arbitrary we derive the following estimate:

$$\begin{aligned}
&\langle T_t^*(\mu-\nu)^+ - (\mu-\nu)^+, \int_0^T T_s f \, ds\rangle \\
&= \int_0^T \langle T_t^*(\mu-\nu)^+ - (\mu-\nu)^+, T_s f\rangle ds \\
&= \int_0^T \langle T_{t+s}^*(\mu-\nu)^+ - T_s^*(\mu-\nu)^+, f\rangle ds \\
&= \int_T^{T+t} \langle T_s^*(\mu-\nu)^+, f\rangle ds - \int_0^t \langle T_s^*(\mu-\nu)^+, f\rangle ds \\
&\leq \int_T^{T+t} \langle T_s^*(\mu-\nu)^+, f\rangle ds = \int_T^{T+t} \langle (\mu-\nu)^+, T_s f\rangle ds \\
&\leq \int_T^{T+t} \langle \mu, T_s f\rangle ds = \int_T^{T+t} \langle T_s^*\mu, f\rangle ds = t\langle \mu, f\rangle < \infty.
\end{aligned}$$

Passing $T \to \infty$ and taking (A.2) into account we obtain

$$T_t^*(\mu-\nu)^+ = (\mu-\nu)^+, \qquad t > 0. \tag{A.4}$$

If $(\mu-\nu)^+ = 0$, then $(u-v)^+ = 0$ m-a.e., hence $\mu \leq \nu$ and the equality $\mu = \nu$ follows by (A.3). The proof will be completed if we establish that the case $(\mu-\nu)^+ > 0$ is excluded. Towards this end, set $\Xi = \{x \in E,\ u(x) > v(x)\}$, obviously $(\mu-\nu)^+(E \backslash \Xi) = 0$. As the measure $(\mu-\nu)^+$ is non-zero and invariant by (A.4), it is equivalent to m, so $m(E \backslash \Xi) = 0$. It follows that $u > v$ m-almost everywhere, but this contradicts (A.3).

Q.E.D.

REMARK. The idea of the proof goes back to [11], Th.VI.A, where the case of a single Markov operator in an L^1-space was treated. The result was extended to continuous time Markov processes with a locally compact state space by L. Stettner ([35], Corollary 8).

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