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UNIVERSAL COEFFICIENT SEQUENCES FOR COHOMOLOGY THEORIES OF *CW*-SPECTRA

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Kainen [5] showed that there exists a cohomology theory $k^*($ (?) and a natural short exact sequence

$$0 \to \mathrm{Ext}\,(h_{*-1}(X),G) \to k^*(X;G) \to \mathrm{Hom}\,(h_*(X),G) \to 0$$

for any based CW-complex X if h_* is an (additive) homology theory and G is an abelian group. On the other hand, for an (additive) cohomology theory k^* such that k^*(point) has finite type Anderson [3] constructed a homology theory Dk_* and a natural exact sequence

$$0 \to \mathrm{Ext}\,(Dk_{*-1}(\,F),\,Z) \to k^*(\,F) \to \mathrm{Hom}\,(Dk_*(\,F),\,Z) \to 0$$

for any finite CW-complex whose extension to arbitrary CW-complexes is given in a form of a four term exact sequence. He then determined homology theories Dk_* in the special cases $k^*=H^*$, K^* and KO^*. Ordinary cohomology theory and complex K-theory are both self-dual and real K-theory is the dual of sympletic K-theory, i.e., $DH_*=H_*$, $DK_*=K_*$ and $DKSp_*=KO_*$. Moreover he asserted that D^2 is the identity, i.e., $D(Dk)_*=k_*$.

In this note we shall construct a CW-spectrum $\hat{E}(G)$ for every CW-spectrum E and abelian group G by Kainen's method involving an injective resolution of G, and state a relation between E and $\hat{E}(G)$ in a form of a universal coefficient sequence

$$0 \to \mathrm{Ext}\,(E_{*-1}(X),G) \to \hat{E}(G)^*(X) \to \mathrm{Hom}\,(E_*(X),G) \to 0$$

for any CW-spectrum X. And we shall study some properties of $\hat{E}(G)$. For example, under a certain finiteness assumption on $\pi_*(E)$ we show that $\hat{\hat{E}}(R)$ (R) has the same homotopy type of ER where J? is a subring of the rationals Q (Theorem 2). The above universal coefficient sequence combined with Theorem 2 gives us a new criterion for $ER^*(X)$ being Hausdorff (Theorem 3). Also we shall discuss uniqueness of $\hat{E}(G)$ (Theorem 4). Furthermore, using Anderson's technique we investigate the homotopy type of $\hat{E}(G)$ in the special cases $E=H$, K and KSp (Theorem 5). Finally we note that $K^{2n}(K_\wedge \cdots {}_\wedge K)$

and $KO^m(KO\wedge\cdots\wedge KO)$, $m\not\equiv 1$ mod 4, are both Hausdorff (Theorem 6).

1. Duality maps

1.1. Let $u: X'\wedge X\to W$ be a pairing of CW-spectra. Such a pairing defines a homomorphism

$$T=T(u)_E:\ \{Y, E\wedge X'\}\to\{Y\wedge X, E\wedge W\}$$

by the relation $T(f)=(1\wedge u)(f\wedge 1)$ for any CW-spectra Y and E. A pairing $u: X'\wedge X\to W$ is called an *E-duality map* provided $T(u)_E$ is an isomorphism for E fixed and $Y=\Sigma^k$ for all k. If u is an E-duality map, then $T(u)_E$ becomes an isomorphism for any CW-spectrum Y.

Fix CW-spectra X and W and consider the cohomology functor $\{-\wedge X, W\}$ defined on the category of CW-spectra. By the representability theorem, there exists a function spectrum $F(X, W)$ such that T: $\{Y, F(X,W)\}\to\{Y\wedge X, W\}$ is a natural isomorphism for all Y. So we see that the evaluation map

$$e: F(X, W)\wedge X\to W$$

is a S-duality map.

Let $u: X'\wedge X\to W$, $v: Y'\wedge Y\to W$, $f: X\to Y$, $g: Y'\to X'$ be maps such that $v(1\wedge f)$ and $u(g\wedge 1)$ are homotopic. Consider the cofiber sequences

$$X\xrightarrow{f}Y\to Z,\quad Z'\to Y'\xrightarrow{g}X'.$$

We have a CW-spectrum Q and maps $p: X'\wedge X\to Q$, $q: Y'\wedge Y\to Q$, and $r: Z'\wedge Z\to Q$ giving rise to the diagram below homotopy commutative (up to sign)

$$\begin{tikzcd}
 & Y'\wedge X \arrow[dl] \arrow[dr] & & \Sigma^{-1}(X'\wedge Z) \arrow[dl] \arrow[dr] & & Z'\wedge Y \arrow[dl] \arrow[dr] & \\
Y'\wedge Y \arrow[drrr, "q"'] & & X'\wedge X \arrow[dr, "p"] & & Z'\wedge Z \arrow[dl, "r"'] & & Y'\wedge Y \arrow[dlll, "q"] \\
 & & & Q & & &
\end{tikzcd}$$

Since $v(1\wedge f)$ and $u(g\wedge 1)$ are homotopic, an easy diagram chase shows that there exists a map $s: Q\to W$ with $s\,p=u$ and $s\,q=v$ (see [7, Proof of Theorem 13.1]). So we obtain a map

$$w: Z'\wedge Z\to W$$

making the diagram

$$
(1.1)\qquad
\begin{array}{ccccc}
\Sigma^{-1}(X' \wedge Z) & \to & Z' \wedge Z & \leftarrow & Z' \wedge Y \\
\mathrm{I} & & \downarrow w & & \mathrm{I} \\
X' \wedge X & \xrightarrow[U]{} & W & \xleftarrow[V]{} & Y' \wedge Y
\end{array}
$$

homotopy commutative (up to sign).

By use of (1.1) and "five lemma" we have

Lemma 1. *Let* $u\colon X_\wedge X \to W$, $v\colon Y'_\wedge Y \to W$ *be* ***E-duality*** *maps and assume that maps* $f\colon X \to Y$ *and* $g\colon Y' \to X'$ *satisfy the property that* $v(\backslash \wedge f)$ *and* $u(g \wedge 1)$ *are* ***homotopic.*** *Then the above map* $w\colon Z' \wedge Z \to W$ *is an* ***E-duality*** *map.* (Cf., [6, Theorem 6.10]).

Let $C = \{X_n, f_n\}$ and $C' = \{X_n' g_n\}$ be a direct and an inverse sequence of CW-spectra respectively. Pairings $u_n\colon X_n' \wedge X_n \to W$ induce the homomorphism

$$
\begin{aligned}
T\{u_n\}\colon \{Y, E \wedge (\Pi X_n')\} &\to \{Y, \Pi(E \wedge X_n')\} \cong \Pi\{Y, E \wedge X_n'\} \\
&\xrightarrow{\Pi T(u_n)} \Pi\{Y \wedge X_n, E \wedge W\} \cong \{\vee(Y \wedge X_n), E \wedge W\} \\
&\cong \{Y \wedge (\vee X_n), E \wedge W\}\ .
\end{aligned}
$$

Taking $Y = \Pi X_n'$ and $E = S$, there is a map

$$u\colon (\Pi X_n') \wedge (\vee X_n) \to W$$

with the homotopy commutative square

$$
(1.2)\qquad
\begin{array}{ccc}
(\Pi X_n') \wedge X_n & \to & X_n' \wedge X_n \\
\downarrow & & \downarrow u_n \\
(\Pi X_n') \wedge (\vee X_n) & \xrightarrow[u]{} & W\,.
\end{array}
$$

Under the assumption that the canonical morphism $E \wedge (\Pi X_n') \to \Pi(\wedge X_n')$ is a homotopy equivalence, we see that

(1.3) *u is an E-duality map if so are all u_n.*

Define maps $f\colon \vee X_n \to \vee X$ and $g\colon \Pi X_n' \to \Pi X_n$ by

$$i_n - i_{n+1} f_n = f \cdot i_n\ , \quad p_n - g_n \cdot p_{n+1} = p_n \cdot g$$

where $i_n\colon X_n \to \vee X_n$, $p_n\colon \Pi X_n' \to X_n$ are the canonical maps. And, construct the telescope TC and the cotelescope T^*C' so that we have the cofiber sequences

$$\vee X_n \xrightarrow{f} \vee X_n \to TC\ ,\ T^*C' \to \Pi X_n' \to \Pi X_n'.$$

Proposition 2. *Let* $C = \{X_n, f_n\}$ *and* $C' = \{X_n', g_n\}$ *be a* ***direct*** *and an inverse sequence of CW-spectra, and* $u_n\colon X_n' \wedge X_n \to W$ *be* ***pairings*** *such that* $u_{n+1}(1 \wedge f_n)$

and $u_n(g_n \wedge 1)$ *are* ***homotopic.*** *Then there exists a map* u: $T^*C' \wedge TC \to W$ *such that the* ***following diagram*** *is homotopy* ***commutative*** *(up to sign):*

$$\begin{array}{ccccc} \Sigma^{-1}(\Pi X_n' \wedge TC) & \to & T^*C' \wedge TC & \leftarrow & T^*C' \wedge (\vee X_n) \\ \downarrow & & \downarrow \bar{u} & & \downarrow \\ (\Pi X_n') \wedge (\vee X_n) & \underset{u}{\longrightarrow} & W & \underset{u}{\longleftarrow} & (\Pi X_n') \wedge (\vee X_n) . \end{array}$$

Moreover, *assuming that the* ***canonical morphism*** $E \wedge (\Pi X_n') \to \Pi(\jmath \wedge X_n')$ *is a homotopy* ***equivalence,*** $\breve{v}$ *is an* ***E-duality*** *map ifso are all* u_n.

Proof. An easy diagram chase shows that $u(l \wedge f)$ and $u(g \wedge 1)$ are homotopic. We apply Lemma 1 and (1.3) to obtain the required map.

1.2. Let G be an abelian group and Γ: $0 \to P_1 \xrightarrow{\phi} P_0 \to G \to 0$ a free resolution. We realize P_i and ϕ by wedges MP_i of sphere spectra and a map $M\phi$: $MP_1 \to MP_0$. The mapping cone $M\Gamma$ of $M\varphi$ forms a Moore spectrum of type G. Then there exists a universal **coefficient** sequence

$$0 \to \mathrm{Ext}(G, \pi_{*+1}(X)) \to \{M\Gamma, X\}_* \xrightarrow{\kappa} \mathrm{Hom}(G, \pi_*(X)) \to 0$$

where κ associates to a map / the induced homomorphism f_* in 0-th homotopy (see [4]). Therefore a Moore spectrum of type G is uniquely determined up to homotopy type. For any CW-spectrum E we define the corresponding spectrum with coefficient group G

$$EG = E \wedge MG$$

where MG is a Moore spectrum of type G.

Let / be a set of primes which may be empty, and denote by I_l the multiplicative set generated by the primes not in /. It is a directed set which is ordered by divisibility. If R is a subring of the rationals Q (with unit), it is just "the integers localized at l'' where / is the set of primes which are not invertible in R. Thus $R = Z_l = I_l^{-1}Z$. Let l^c denote the set of primes $p_k (p_k < p_{k+1})$ not in /, i.e., $l \cap l^c = \{\phi\}$ and $l \cup l^c = \{\text{all primes}\}$. Putting $l_n = p_1^n \cdots p_n^n$, we choose a **cofinal** sequence $J_l = \{l_n\}$ in I_l.

Fix a CW-**spectrum** W. $C_l = \{X_n = W, f_n = l_{n+1}/l_n\}$ and $C_l^* = \{X_n' = W g_n = l_{n+1}/l_n\}$ form respectively a direct and an inverse sequence (indexed by J_l). Denote by W_l, Wf the telescope of C_l and the cotelescope of C_l^*, i.e.,

$$W_l = T\{W f_n = l_{n+1}/l_n\} , \quad W_l^* = T^* \{W, g_n = l_{n+1}/l_n\} .$$

Notice that W_l is homotopy equivalent to $W \wedge S_l$. Since S_l is a Moore spectrum of type Z_l, an easy computation shows that

(1.4) tfZ^S^Z^n/̂'/Z,/ *and* $HZ_{l'}^n(S_l) = 0$ *for* $n \neq 1$

where l' is any set of primes with $l' \cap l^c \neq \{\phi\}$.

Define by ι_n and ρ_n the composite maps $W \xrightarrow{i_n} \vee W \to W_l, W_l^* \to \prod W \xrightarrow{p_n} W$ and consider the cofiber sequences

$$W \xrightarrow{\iota_1} W_l \to \bar{W}_l, \quad \bar{W}_l^* \to Wf \xrightarrow{\rho_1} W.$$

$\bar{S}_l$ is obviously a Moore spectrum of type Z_l/Z, and in addition

(1.5) $\qquad HZ^1_{l'}(\bar{S}_l) \cong \hat{Z}_{l' \cap l}$ *and* ffZ?/(S,) $= 0, \quad n \neq 1$

for any l' with $l' \cap l^c \neq \{\phi\}$.

1.3. Here we construct two useful duality maps.

Proposition 3. *We have maps $\bar{u}$: $W_l^* \wedge S_l \to W$ and $\bar{w}$: $\bar{W}_l^* \wedge \bar{S}_l \to W$ such that the following diagram is commutative (up to sign) for all CW-spectra X and E:*

$$\begin{array}{ccccc} & \nearrow & \{X, E \wedge \bar{W}_l^*\} \to & \{X, E \wedge W_l^*\} & \searrow & \\ \{\sum X, E \wedge W\} & & \downarrow T(\bar{w}) & \downarrow T(\bar{u}) & & \{X, E \wedge W\} . \\ & \searrow & \{X \wedge \bar{S}_l, E \wedge W\} \to & \{X \wedge S_l, E \wedge W\} & \nearrow & \end{array}$$

Proof. Take as u_n: $W \wedge S \to W$ the canonical identification. From (1.2) and Proposition 2 we obtain maps u: $(\prod W) \wedge (\vee S) \to W, \bar{u}$: $Wf \wedge S_l \to W$ with the homotopy commutative squares

$$\begin{array}{ccc} (\prod W) \wedge S & \xrightarrow{p_n \wedge 1} & W \wedge S \\ 1 \wedge i_n \downarrow & & \| \\ (\prod W) \wedge (\vee S) & \xrightarrow[u]{} & W \end{array} \qquad \begin{array}{ccc} W_l^* \wedge (\vee S) & \to & (\prod W) \wedge (\vee S) \\ \downarrow & & \downarrow u \\ W_l^* \wedge S_l & \xrightarrow[\bar{u}]{} & W. \end{array}$$

Putting the above two squares together we see that $\rho_n \wedge 1$ and $u(1 \wedge \iota_n)$ are homotopic. By (1.1) there exists a map W: $\bar{W}f \wedge \bar{S}_l \to W$ making the diagram below homotopy commutative (up to sign)

$$\begin{array}{ccccc} \sum^{-1}(W \wedge \bar{S}_l) & \to & \bar{W}f \wedge \bar{S}_l & \to & \bar{W}_l^* \wedge S_l \\ \downarrow & & \downarrow \bar{w} & & \downarrow \\ W \wedge S & = & W & \xleftarrow[u]{} & W_l^* \wedge S_l . \end{array}$$

Now we need the following result in order to apply Proposition 2.

Lemma 4. *Let G be a direct product of R-modules G_α and M_α a Moore spectrum of type G_α. Then $\prod M_\alpha$ becomes a Moore spectrum of type G, and the canonical morphism $E \wedge (\prod M_\alpha) \to \prod (i \wedge M_\alpha)$ is a homotopy equivalence if $\pi_*(E)$ has finite type as an R-module.*

Proof. The result of Adams [1, Theorem 15.2] asserts that $HR \wedge \prod M_\alpha \to$

$\prod(HR\wedge M_\alpha)$ is a homotopy equivalence. Thus $\prod M_\alpha$ becomes a Moore spectrum of type G because $\pi_*(\prod M_\alpha)$ is an R-module and hence so is $H_*(\prod M_\alpha)$. In the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi_*(E)\otimes G \arrow[r] \arrow[d, "*"] & \pi_*(E\wedge \prod M_\alpha) \arrow[r] \arrow[d] & \mathrm{Tor}_1^R(\pi_{*-1}(E),\mathrm{G}) \arrow[r] \arrow[d, dash] & 0 \\ 0 \arrow[r] & \prod\pi_*(E)\underset{R}{\otimes} G_\alpha \arrow[r] & \prod\pi_*(E\wedge \mathrm{M.}) \arrow[r] & \prod \mathrm{Tor}_1^R(\pi_{*-1}(E), G_\alpha) \arrow[r] & 0 \end{tikzcd}$$

involving the universal coefficient sequences, the left and right arrows are isomorphisms. The result follows from "five lemma".

Obviously the canonical identification u_n: $W\wedge S\to W$ is an E-duality map for every E. Using Propositions 2, 3 and Lemma 4 we obtain

Theorem 1. *Let G be an **R-module** and M a Moore spectrum of type G. Assume that $\pi_*(E)$ is of finite type as an R-module. Then the maps $\bar{u}$: $M_l^*\wedge S_l\to M$ and $\bar{w}$: $\bar{M}f\wedge \bar{S}_l\to M$ given in Proposition 3 are both **E-duality** maps.*

Remark that $\pi_*((S_l)_\phi^*)$ and $\pi_*((\bar{S}_l)_\phi^*)$ are Z_l-modules. Taking S_l as M and the empty ϕ as / in the above theorem, we compute that

$$H_*((S_l)_\phi^*)\cong HZ_l^*(S_\phi)\,,\quad H_*((\bar{S}_l)_\phi^*)\cong HZ_l^*(\bar{S}_\phi)\,.$$

Thus $\sum(S_l)_\phi^*$ and $\sum(\bar{S}_l)_\phi^*$ are Moore spectra of type $\hat{Z}_l/Z_l$ and of type $\hat{Z}_l$ where $l\neq\{\phi\}$, because of (1.4) and (1.5). So we get

Corollary 5. *Assume that $\pi_*(E)$ is of finite type as an R-module where R is a proper subring of Q. Then there exist natural **isomorphisms** $T(\bar{w})$:$E\hat{R}^*(X)\to E^{*+1}(X\wedge \bar{S}_\phi)$, $T(\bar{u})$:$E\hat{R}/Z^*(X)\to E^{*+1}(X\wedge S_\phi)$ **with** the commutative (up to sign) diagram*

$$\begin{tikzcd} & E\hat{R}^*(X) \arrow[r] \arrow[d, dash, "T(\overline{w})"] & \hat{E}R/Z^*(X) \arrow[dr] \arrow[d, dash, "T(\ddot{v})"] & \\ E^*(X) \arrow[ur] \arrow[dr] & & & E^{*+1}(X)\,. \\ & E^{*+1}(X\wedge \bar{S}_\phi) \arrow[r] & E^{*+1}(X\wedge S_\phi) \arrow[ur] & \end{tikzcd}$$

2. Universal coefficient sequences

2.1. Following Kainen [5] we shall construct a universal coefficient sequence for a generalized cohomology theory. Fix a CW-spectrum E. For every injective abelian group / $\mathrm{Hom}(E_*(-), /)$ forms a cohomology theory defined on the category of CW-spectra. The representability theorem gives us a CW-spectrum $\hat{E}(I)$ and a natural isomorphism

$$T_I\colon\ \{X,\hat{E}(I)\}\to \mathrm{Hom}\,(E_*(X),I)$$

for any CW-spectrum X. Let G be an abelian group and Γ: $0\to G\to I\xrightarrow{\psi} J\to 0$

an injective resolution. Then there exists a unique (up to homotopy) map $\hat{\psi}$: $\hat{E}(I) \to \hat{E}(J)$ whose induced homomorphism coincides with the natural transformation $T_J^{-1} \cdot \psi_* \cdot T_I$. Denote by $\sum \hat{E}(\Gamma)$ the mapping cone of $\hat{\psi}$, i.e.,

$$\hat{E}(\Gamma) \to \hat{E}(I) \to \hat{E}(J)$$

is a cofiber sequence. By homological algebra we obtain a natural exact sequence

$$0 \to \mathrm{Ext}\,(E_{*-1}(X), \mathrm{G}) \to \hat{E}(\Gamma)^*(X) \to \mathrm{Hom}\,(E_*(X), G) \to 0$$

for all X.

Let ϕ: $G \to G'$ be a homomorphism and Γ: $0 \to G \to I \to J \to 0$, Γ': $0 \to G' \to I' \to J' \to 0$ be injective resolutions. For a morphism μ: $\Gamma \to \Gamma'$ which is a lift of ϕ, we may choose a map

$$\mathrm{A}\colon\ \hat{E}(\Gamma) \to \hat{E}(\Gamma')$$

making the diagram with cofiber sequences

$$\begin{array}{ccccccc}
\hat{E}(\Gamma) & \to & \hat{E}(I) & \to & \hat{E}(J) & \to & \sum \hat{E}(\Gamma) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\hat{E}(\Gamma') & \to & \hat{E}(I') & \to & \hat{E}(J') & \to & \sum \hat{E}(\Gamma')
\end{array}$$

homotopy commutative. However μ is not uniquely determined (up to homotopy). The map μ yields the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Ext}\,(E_{*-1}(X), \mathrm{G}) & \to & \hat{E}(\Gamma)^*(X) & \to & \mathrm{Horn}\,(E_*(X), G) & \to & 0 \\
 & & \downarrow \phi_* & & \downarrow \hat{\mu}_* & & \downarrow \phi_* & & \\
0 & \to & \mathrm{Ext}\,(E_{*-1}(X), \mathrm{GO} & \to & \hat{E}(\Gamma')^*(X) & \to & \mathrm{Hom}(E_*(X), G') & \to & 0\,.
\end{array}$$

With an application of "five lemma" we find that $\hat{\mu}$: $\hat{E}(\Gamma) \to \hat{E}(\Gamma')$ is a homotopy equivalence if ϕ: $G \to G'$ is an isomorphism. Thus the homotopy type of $\hat{E}(\Gamma)$ is independent of the choice of an injective resolution Γ of G. So we may put

$$\hat{E}(G) = \hat{E}(\Gamma), \quad \hat{\phi} = \hat{\mu}\,.$$

Consequently we get

Proposition 6. *Let E be a CW-spectrum and G an abelian group. Then there exists a CW-spectrum $\hat{E}(G)$ so that*

$$0 \to \mathrm{Ext}\,(E_{*-1}(X), \mathrm{G}) \xrightarrow{\eta} \hat{E}(G)^*(X) \to \mathrm{Hom}\,(E_*(X), \mathrm{G}) \to 0$$

is a natural exact sequence for any CW-spectrum X. Moreover a homomorphism φ: $G \to G'$ induces a (non-unique) map $\hat{\varphi}$: $\hat{E}(G) \to \hat{E}(G')$ with the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ext}\,(E_{*-1}(X), G) & \to & \hat{E}(G)^*(X) & \to & \mathrm{Hom}\,(E_*(X), G) & \to & 0 \\ & & \downarrow\phi_* & & \downarrow\hat{\phi}_* & & \downarrow\phi_* & & \\ 0 & \to & \mathrm{Ext}\,(E_{*-1}(X), G') & \to & \hat{E}(G')^*(X) & \to & \mathrm{Hom}\,(E_*(X), G') & \to & 0\,. \end{array}$$

(Cf., [5]).

If Y is a finite CW-spectrum, then the function dual $Y^*=F(Y,S)$ can be taken finite and $E_*(Y)\cong E^{-*}(Y^*), E^*(Y)\cong E_{-*}(Y^*)$. We notice that

(2.1) *there exists a natural exact sequence*

$$0 \to \mathrm{Ext}\,(E^{*+1}(Y),\ G) \to \hat{E}(G)_*(Y) \to \mathrm{Hom}\,(E^*(Y),\ G) \to 0$$

for all finite Y.

Let f: $E\to F$ be a map of CW-spectra. Then f induces a (non-unique) map $\hat{f}$: $\hat{F}(G)\to\hat{E}(G)$ such that the diagram

$$(2.2)\qquad \begin{array}{ccccccccc} 0 & \to & \mathrm{Ext}\,(F_{*-1}(X), G) & \to & \hat{F}(G)^*(X) & \to & \mathrm{Hom}\,(F_*(X), G) & \to & 0 \\ & & \downarrow f^* & & \downarrow \hat{f}_* & & \downarrow f^* & & \\ 0 & \to & \mathrm{Ext}\,(E_{*-1}(X), G) & \to & \hat{E}(G)^*(X) & \to & \mathrm{Hom}\,(E_*(X), G) & \to & 0 \end{array}$$

is commutative. Remark that $\hat{f}$ becomes a homotopy equivalence if so is f. Hence we find that

(2.3) *the homotopy type of $\hat{E}(G)$ depends only on that of E and the isomorphism class of G.*

2.2. For simplicity we write $\hat{E}$ instead of $\hat{E}(Z)$. We shall now show that $\hat{E}(G)$ and $\hat{E}G$ have the same homotopy type under some finiteness assumptions on E and G. First we require the following

Lemma 7. i) *Let G be a direct product of abelian groups G_α, i.e., $G=\prod G_\alpha$. Then $\hat{E}(G)$ is homotopy equivalent to $\prod\hat{E}(G_\alpha)$.*
ii) *Let G be a direct sum of R-modules G_α, i.e., $G=\sum G_\alpha$, and assume that $\pi_*(E)$ is of finiye type as an R-module. Then $\hat{E}(G)$ is homotopy equivalent to $\vee\hat{E}(G_\alpha)$.*

Proof. i) Denote by p_α the canonical projection from G onto G_α. The map $\prod\hat{p}_\alpha$: $\hat{E}(G)\to\prod\hat{E}(G_\alpha)$ induces the composite homomorphism

$$\hat{E}(G)^*(X) \to \prod\hat{E}(G_\alpha)^*(X) \xleftarrow{\cong} (\prod\hat{E}(G_\alpha))^*(X)$$

for any CW-spectrum X. In the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ext}\,(E_{*-1}(X), G) & \to & \hat{E}(G)^*(X) & \to & \mathrm{Hom}\,(E_*(X), G) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \prod\mathrm{Ext}(E_{*-1}(X), G_\alpha) & \to & \prod\hat{E}(G_\alpha)^*(X) & \to & \prod\mathrm{Hom}(E_*(X), G_\alpha) & \to & 0 \end{array}$$

involving the universal coefficient sequences, the left and right arrows are isomorphisms. By "five lemma'' the center becomes an isomorphism, and hence the map $\prod \hat{p}_\alpha$ is a homotopy equivalence.

ii) The canonical injections $i_\alpha\colon G_\alpha \to G$ induce the composite homomorphism

$$(\vee \hat{E}(G_\alpha))^*(Y) \leftarrow \sum E(G_\alpha)^*(Y) \to E(G)^*(Y)$$

for any finite Y. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \sum \mathrm{Ext}^1_R(E_{*-1}(Y), G_\alpha) & \to & \sum \hat{}\,(G_\alpha)^*(Y) & \to & \sum \mathrm{Hom}_R(E_*(Y), G_\alpha) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Ext}^1_R(E_{*-1}(Y), G) & \to & \hat{E}(G)^*(Y) & \to & \mathrm{Hom}_R(E_*(Y), G) & \to & 0\,. \end{array}$$

The vertical arrows on both sides are isomorphisms whenever Y is finite. So the map $\vee \hat{i}_\alpha\colon \vee \hat{E}(G_\alpha) \to \hat{E}(G)$ becomes a homotopy equivalence.

Fix a subring R of Q and assume that $\pi_*(E)$ has finite type as an R-module. For any subrings R', R'', $R' \subset R \subset R''$, the composite maps

(2.4) $\quad e(R')\colon \hat{E}(R)R' \to E(R)R \leftarrow \hat{E}(R), \quad e(R'')\colon E(R)R'' \to E(R'')R'' \leftarrow \hat{E}(R'')$

become homotopy equivalences because all arrows induce isomorphisms in homotopy. So we consider the diagram

$$\begin{array}{ccccccc} E(R)R' & \to & E(R)Q & \to & \hat{E}(R)Q/R' & \to & \sum \hat{E}(R)R' \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ \hat{E}(R) & \to & \hat{E}(Q) & \to & \hat{E}(Q/R) & \to & \sum \hat{E}(R) \end{array}$$

such that the rows are cofiber sequences and the left square is homotopy commutative. Then there exists a homotopy equivalence

(2.5) $$e(Q/R')\colon \hat{E}(R)Q/R' \to \hat{E}(Q/R)$$

(denoted by a dotted arrow in the above diagram) which makes the diagram into a morphism of cofiber sequences. Moreover we get a map

(2.6) $$e(Z_q)\colon \hat{E}(R)Z_q \to E(Z_q)$$

for the R-module Z_q, which becomes also a homotopy equivalence. This gives rise to a homotopy commutative diagram

$$\begin{array}{ccccccc} \hat{E}(R)Z_q & \to & E(R)Q/R & \to & E(R)Q/R & \to & \sum \hat{E}(R)Z_q \\ \mathrm{I} & & \mathrm{I} & & \mathrm{I} & & \mathrm{I} \\ \hat{E}(Z_q) & \to & \hat{E}(Q/R) & \to & \hat{E}(Q/R) & \to & \sum \hat{E}(Z_q) \end{array}$$

where the rows are cofiber sequences associated with the injective resolution

$0 \to Z_q \to Q/R \xrightarrow{q} Q/R \to 0.$

Proposition 8. *Let G be a direct sum or product of finitely generated R-modules G_α. If $\pi_*(E)$ is of finite type as an R-module, then $\hat{E}(G)$ has the same homotopy type of $\hat{E}(R)G$.*

Proof. We may put $G_\alpha = R$ or Z_q. Using (2.4), (2.6) and Lemmas 4, 7 we find that the composite maps

$$E(R)G \leftarrow \vee \hat{E}(R)G_\alpha \to \vee \hat{E}(G_\alpha) \to \hat{E}(G), \quad E(R)G \to \prod \hat{E}(R)G_\alpha \to \prod \hat{E}(G_\alpha) \leftarrow \hat{E}(G)$$

are homotopy equivalences.

2.3. Let $S \xrightarrow{\iota_\phi} S_\phi \xrightarrow{\tilde{i}} \bar{S}_\phi$ be the cofiber sequence constructed in §1. Assume that for all finite CW-spectra Y we have natural homomorphisms

$$\phi' \colon E^*(Y \wedge \bar{S}_\phi) \to F^*(Y \wedge \bar{S}_\phi), \quad \phi'' \colon E^*(Y \wedge S_\phi) \to F^*(Y \wedge S_\phi)$$

which satisfy the relation that $\phi''(1 \wedge \tilde{\iota})^* = (1 \wedge \tilde{\iota})^* \phi'$. Moreover we assume that $\pi_*(E)$ and $\pi_*(F)$ are R-modules where R is a proper subring of Q. If $\pi_*(F)$ is of finite type, then $F\hat{R}^*(X)$ and $FR/Z^*(X)$ are always Hausdorff for all X [8, III]. Thus $FR^*(X) \cong \varprojlim F\hat{R}^*(X^\lambda)$ and $F\hat{R}/Z^*(X) \cong \varprojlim FR/Z^*(X^\lambda)$ where $\{X^\lambda\}$ runs over the set of all finite subspectra of X. Applying Corollary 5 (and Proposition 3) we obtain natural homomorphisms

$$\psi' \colon E\hat{R}^*(X) \to FR^*(X), \quad \psi'' \colon E\hat{R}/Z^*(X) \to F\hat{R}/Z^*(X)$$

for arbitrary X which gives us the commutative square

$$\begin{array}{ccc} E\hat{R}^*(X) & \to & ER/Z^*(X) \\ \downarrow & & \downarrow \\ F\hat{R}^*(X) & \to & F\hat{R}/Z^*(X). \end{array}$$

Putting $f' = \psi'(1_{E\hat{R}})$ and $f'' = \psi'(1_{E\hat{R}/Z})$ we get the diagram

$$\begin{array}{ccccccc} E & \to & E\hat{R} & \to & E\hat{R}/Z & \to & \sum E \\ & & {\scriptstyle f'} \downarrow & & \downarrow {\scriptstyle f''} & & \\ F & \to & F\hat{R} & \to & F\hat{R}/Z & \to & \sum F \end{array}$$

with cofiber sequences and a homotopy commutative square. Then there exists a map

(2.7) $$/\colon E \to F$$

making the above diagram into a morphism of cofiber sequences. In particular, we obtain the following result which is a useful tool in studying properties of $\hat{E}(G)$.

Lemma 9. *Assume that* $\pi_*(E)$ *and* $\pi_*(F)$ *have finite type as* R*-modules. Iffor any finite* CW*-spectrum* Y *we have natural isomorphisms*

$$\phi' \colon E^*(Y\wedge \bar{S}_\phi)\to F^*(Y\wedge \bar{S}_\phi),\quad \phi'' \colon E^*(Y\wedge S_\phi)\to F^*(Y\wedge S_\phi)$$

such that $\phi''(1\wedge\tilde{\iota})^* - (1\wedge\tilde{\iota})^*\phi'$, *then* E *is homotopy equivalent to* F.

Now we study the homotopy type of $\widehat{\hat{E}(R)}$ by use of Lemma 9.

Theorem 2. *If* $\pi_*(E)$ *ıs of finite type as an* R'*-module, then* $\widehat{\hat{E}(R')}(R)$ *has the same homotopy type of* ER *where* $R'\subset R\subset Q$.

Proof. By [8, (II. 1.10)], (2.1) and Proposition 6 the composite homomorphism

$$E^*(Y)\otimes Q\to \mathrm{Hom}(\mathrm{Hom}(E^*(Y),R'),Q)\xrightarrow{\kappa^*}\mathrm{Hom}(\hat{E}(R')_*(Y),Q)\xleftarrow{\kappa}\widehat{\hat{E}(R')}(Q)^*(Y)$$

is a natural isomorphism for all finite Y. In particular the coefficient $\pi_*(\widehat{\hat{E}(R')}(Q))$ is equal to the Q-module $\pi_*(EQ)$. Therefore $\widehat{\hat{E}(R')}(Q)$becomes homotopy equivalent to EQ.

So we may assume that R is a proper subring of Q. For any finite Y we consider the following diagram

$$\begin{array}{ccccc}
ER^{*+1}(Y\wedge\bar{S}_\phi) & \overset{T(\bar{w})}{\longleftarrow} & (ER)\hat{R}^*(Y) & \leftarrow & E^*(Y)\otimes\mathrm{Ext}(Q/R',R)\\
\downarrow & & \downarrow & & \downarrow\\
ER^{*+1}(Y\wedge S_\phi) & \overset{T(\bar{u})}{\longleftarrow} & (ER)\hat{R}/Z^*(Y) & \leftarrow & E^*(Y)\otimes\mathrm{Ext}(Q,R)
\end{array}$$

$$\begin{array}{ccc}
\to\mathrm{Ext}(\mathrm{Hom}(E^*(Y),Q/R'),R) & \xrightarrow{\kappa^*} & \mathrm{Ext}(\hat{E}(Q/R')_*(Y),R)\\
\downarrow & & \downarrow\\
\to\mathrm{Ext}(\mathrm{Hom}(E^*(Y),Q),R) & \xrightarrow{\kappa^*} & \mathrm{Ext}(\hat{E}(Q)_*(Y),R)
\end{array}$$

$$\begin{array}{ccc}
\xrightarrow{e(Q/Z)^*}\mathrm{Ext}(\hat{E}(R')Q/Z_*(Y),R) & \xrightarrow{\eta} & \widehat{E(R')}(R)^*(Y\wedge S_\phi)\\
\downarrow & & \downarrow\\
\xrightarrow{e(Q)^*}\mathrm{Ext}(\hat{E}(R')Q_*(Y),R) & \xrightarrow{\eta} & \widehat{\hat{E}(R')}(R)^*(Y\wedge S_\phi)
\end{array}$$

(in which we drop the subscript R' on the functors $\underset{R'}{\otimes}$, $\mathrm{Hom}_{R'}$, $\mathrm{Ext}_{R'}$). Note that $\mathrm{Ext}(Q,R)\simeq\hat{R}/R\simeq R\otimes\hat{R}/Z$and $\mathrm{Ext}(Q/R',R)\simeq\hat{R}$. All squares are commutative by Corollary 5, (2.1), (2.5) and Proposition 6. In addition all horizontal arrows are isomorphisms because of Corollary 5, [8, (II. 1.10)], (2.1), (2.4), (2.5) and Proposition 6. Applying Lemma 9 to the above diagram the desirable result is obtaind.

Theorem 2 asserts that we have a natural exact sequence

(2.8) $\quad 0 \to \mathrm{Ext}(\hat{E}(R')_{*-1}(X), R) \to ER^*(X) \to \mathrm{Hom}(\hat{E}(R')_*(X), R) \to 0$

for any X and $R' \subset R \subset Q$ if $\pi_*(E)$ is of finite type as an R'-module.

Using the above universal coefficient sequence we give a new criterion for $ER^*(X)$ being Hausdorff.

Theorem 3. *Assume that $\pi_*(E)$ has finite type as an R'-module. $(ER)^{n+1}(X)$ is Hausdorff if and only if $\mathrm{Ext}(\hat{E}(R')_n(X)/T\hat{E}(R')_n(X), R)=0$ where $R' \subset R \subset Q$ and TG denotes the torsion subgroup of G.*

Proof. The proof is similar to that of [8, Theorem IV. 4]. Assume that R is a proper subring. Recall that $ER^{n+1}(X)$ is Hausdorff if and only if the boundary homomorphism δ: $E\hat{R}R/Z^n(X) \to ER^{n+1}(X)$ is trivial (cf., [8, Theorem III.1)]. Then Corollary 5 implies that $ER^{n+1}(X)$ is Hausdorff if and only if $(1 \wedge \iota_1)^*: ER^{n+1}(X \wedge S_\phi) \to ER^{n+1}(X)$ is trivial. In the commutative diagram

$$\begin{array}{ccccccccc} & & \mathrm{Ext}(\hat{E}(R')Q_n(X), R) & \to & ER^{n+1}(X \wedge S_\phi) & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \mathrm{Ext}(E(R')_n(X), R) & \to & ER^{n+1}(X) & \to & \mathrm{Hom}(\hat{E}(R')_{n+1}(X), R) & \to & 0, \end{array}$$

the upper arrow is an isomorphism and the lower row is exact by (2.8). On the other hand, the left vertical arrow admits a factorization

$$\mathrm{Ext}(\hat{E}(R')Q_n(X), R) \to \mathrm{Ext}(\hat{E}(R')_n(X)/T\hat{E}(R')_n(X), R) \to \mathrm{Ext}(\hat{E}(R')_n(X), R)$$

such that the former is an epimorphism but the latter is a monomorphism. An easy diagram chase shows that $(1 \wedge \iota_1)^*$ is the zero map if and only if $\mathrm{Ext}(\hat{E}(R')_n(X)/T\hat{E}(R')_n(X), R)=0$. So the result follows immediately.

2.4. Now we discuss uniqueness of $\hat{E}(G)$ under some restrictions on E and G.

Theorem 4. *Let G be a finitely generated R-module with $\mathrm{Tor}(\pi_*(E), G)=0$, and assume that $\pi_*(E)$ is of finite type as an R-module. If F satisfies the property that there exists a natural exact sequence*

$$0 \to \mathrm{Ext}(E_{*-1}(X), G) \to F^*(X) \to \mathrm{Hom}(E_*(X), G) \to 0$$

for any CW-spectrum X, then F has the same homotopy type of $\hat{E}(G)$.

Proof. Assume that R is a proper subring of Q. The torsion subgroup $T=TG$ is a direct summand of G and the quotient $P=G/TG$ is a free R-module. Consider the commutative diagram

$$\begin{array}{ccccccc} \hat{E}(P)^{*+1}(X \wedge \bar{S}_\phi) & \cong & \mathrm{Ext}(EQ/Z_*(X), P) & \to & \mathrm{Ext}(EQ/Z_*(X), G) & \to & F^{*+1}(X \wedge \bar{S}_\phi) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \hat{E}(P)^{*+1}(X \wedge S_\phi) & \cong & \mathrm{Ext}(EQ_*(X), P) & \to & \mathrm{Ext}(EQ_*(X), G) & \to & F^{*+1}(X \wedge S_\phi) \end{array}$$

for any X. (2.7) gives rise to a map f: $\hat{E}(P) \to F$ with the commutative diagram

$$\begin{array}{ccccc} \hat{E}(P)^*(X) & \to & \hat{E}(P)\hat{R}^*(X) & \to & \hat{E}(P)\hat{R}/Z^*(X) \\ \downarrow & & \downarrow & & \downarrow \\ F^*(X) & \to & FR^*(X) & \to & F\hat{R}/Z^*(X). \end{array}$$

Looking at the previous diagram we find that in the above the central arrow is a monomorphism and the right is an isomorphism. So f^*: $\hat{E}(P)^*(X) \to F^*(X)$ becomes a monomorphism whenever $\hat{E}(P)^*(X)$ is Hausdorff, and in addition f induces an isomorphism f_*: $\pi_*(\hat{E}(P)) \otimes Q \to \pi_*(F) \otimes Q$. Denote by F_T the mapping cone of f, thus

$$\hat{E}(P) \xrightarrow{f} F \xrightarrow{g} F_T$$

is a cofiber sequence. $\hat{E}(P)^*(F_T)$ is Hausdorff as $\pi_*(F_T) \otimes Q = 0$ [8, Theorem III. 2]. Therefore we have a short exact sequence

$$0 \to \hat{E}(P)^*(F_T) \to F^*(F_T) \to F_T^*(F_T) \to 0\ .$$

Then we may choose a map h: $F_T \to F$ such that the composite map $g\ h$ is homotopic to the identity. This means that the sequence

$$0 \to \hat{\ }(P)^*(X) \to F^*(X) \to F_T^*(X) \to 0$$

is split exact. F is obviously homotopic to the wedge of $\hat{E}(P)$ and F_T.

We are now left to show that F_T has the same homotopy type of $\hat{E}(T)$ under the assumption that $\mathrm{Tor}(\pi_*(E), G)=0$. Consider the commutative exact diagram

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Ext}(EQ/Z_*(X), P) & \to & \hat{E}(P)^{*+1}(X \wedge \bar{S}_\phi) & \to & \mathrm{Hom}(EQ/Z_{*+1}(X), P) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Ext}(EQ/Z_*(X), G) & \to & F^{*+1}(X \wedge \bar{S}_\phi) & \to & \mathrm{Hom}(EQ/Z_{*+1}(X), G) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & \mathrm{Ext}(EQ/Z_*(X), T) & & F_T^{*+1}(X \wedge \bar{S}_\phi) & & \mathrm{Hom}(EQ/Z_{*+1}(X), T) & \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

in which $\mathrm{Hom}(EQ/Z_{*+1}(X), P)=0$. With an application of "3 x 3 lemma" we get a natural exact sequence

$$0 \to \mathrm{Ext}(EQ/Z_*(X), T) \to F_T^{*+1}(X \wedge \bar{S}_\phi) \to \mathrm{Hom}(EQ/Z_{*+1}(X), T) \to 0$$

for any X. Take a free resolution $0 \to P_1 \xrightarrow{\phi} P_0 \xrightarrow{\psi} T \to 0$ consisting of finitely generated R-modules. The composite homomorphisms

$$\begin{aligned}\mathrm{Ext}(EQ/Z_*(X),P_i) &\leftarrow \mathrm{Ext}(EQ/Z_*(X),R) \underset{R}{\otimes} P_i \\ &\rightarrow \hat{E}(R)^{*+1}(X\wedge \bar{S}_\phi) \underset{R}{\circledR} P_i \rightarrow \hat{E}(R)P_i^{*+1}(X\wedge \bar{S}_\phi)\end{aligned}$$

are isomorphisms. The R-free resolution yields the following commutative exact diagram

$$\begin{tikzcd}
 & & & 0 \\
0 \arrow[r] & \mathrm{Ext}(EQ/Z_{*-1}(X),T) \arrow[r, "\eta'"] & F_T^*(X\wedge\bar{S}_\phi) \arrow[r, "\kappa'"] & \mathrm{Hom}(EQ/Z_*(X),T) \arrow[r] \arrow[u] \arrow[dll, "\partial"'] & 0 \\
 & \mathrm{Ext}(EQ/Z_*(X),P_1) \arrow[r, "\cong", "\eta"'] \arrow[d, "\phi_*"] & \hat{E}(R)P_1^{*+1}(X\wedge\bar{S}_\phi) \arrow[d, "\phi_*"] & & \\
 & \mathrm{Ext}(EQ/Z_*(X),P_0) \arrow[r, "\cong", "\eta"'] \arrow[d, "\psi_*"] & \hat{E}(R)P_0^{*+1}(X\wedge\bar{S}_\phi) & & \\
0 \arrow[r] & \mathrm{Ext}(EQ/Z_*(X),T) \arrow[r, "\iota"'] \arrow[d] & F_T^{*+1}(X\wedge\bar{S}_\phi) \arrow[r, "\kappa"'] & \mathrm{Hom}(EQ/Z_{*+1}(X),T) \arrow[r] & 0 \\
 & 0. & & &
\end{tikzcd}$$

Define homomorphisms

$$\bar{\psi}\colon \hat{E}(R)P_0^*(X\wedge\bar{S}_\phi)\rightarrow F_T^*(X\wedge\bar{S}_\phi),\quad \delta\colon F_T^*(X\wedge\bar{S}_\phi)\rightarrow \hat{\ } \qquad \wedge\bar{S}_\phi)$$

by the composite maps $\bar{\psi}=\eta'\ \psi_*\ \eta^{-1}$, $\delta=\eta\ 3\ \kappa'$. By an easy diagram chase we show that the long sequence

$$\rightarrow \hat{E}(R)P_1^*(X\wedge S_\phi) \rightarrow E(R)P_0^*(X\wedge S_\phi) \xrightarrow{\bar{\psi}} F_T^*(X\wedge S_\phi) \xrightarrow{\delta} E(R)P_1^{*+1}(X\wedge S_\phi) \rightarrow$$

is exact for all X.

Next we consider the commutative exact diagram

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \\
 & E(R)P_1^*(X) \arrow[r, "\phi_*"] \arrow[d, "\iota"] & \hat{E}(R)P_0^*(X) \arrow[d, "\iota"] & F_T^*(X) \arrow[d, "\iota'", "\simeq"'] \\
 \arrow[r] & \hat{E}(R)P_1\hat{R}^*(X) \arrow[r, "\psi_*"] \arrow[d] & E(R)P_0R^*(X) \arrow[r, "\psi^*"] \arrow[d] & F_TR^*(X) \arrow[r] & \\
 & \hat{E}(R)P_1\hat{R}/Z^*(X) \arrow[r, "\cong", "\phi_*"'] \arrow[d] & \hat{E}(R)P_0\hat{R}/Z^*(X) \arrow[d] & \\
 & & &
\end{tikzcd}$$

in which the middle row is rewritten the previous long exact sequence by the aid of Corollary 5. As is easily seen, we get an exact sequence

$$E(R)P_1^*(X) \xrightarrow{\phi_*} E(R)P_0^*(X) \xrightarrow{\rho} F_T^*(X)$$

for any X. Taking $e'=\rho(1_{\hat{E}(R)P_0})$ the composite map $e'\ \phi$ becomes homotopic to the zero map. Therefore e' admits a factorization (up to homotopy)

$$\hat{E}(R)P_0 \to \hat{E}(R)T \xrightarrow{e} F_T .$$

This yields the commutative triangle

$$\begin{tikzcd} & \hat{E}(R)T^*(X) \arrow[d, "e_*"] \\ \hat{E}(R)^*(X) \underset{R}{\otimes} T \arrow[ur] \arrow[r, "\rho'"'] & F_T^*(X) \end{tikzcd} .$$

If $\mathrm{Tor}(\pi_*(E), T)=0$, then $\mathrm{Tor}(\pi_*(\hat{E}(R)),T)=0$ and hence the above e_* is an isomorphism. So F_T becomes homotopy equivalent to $\hat{E}(T)$ because of Proposition 8. Putting this and the previous result together, the required result is obtained from Lemma 7.

3. Complex and real K-theories

3.1. First we shall construct an injective resolution

$$\Gamma(G)\colon 0 \to G \to I_G \to J_G \to 0$$

for every abelian group G which is functorial in G (see [5]). Let $A(G)$ denote the direct sum of copies A_g of A which runs over the set of all elements g of G, where $A=Z$, Q or Z/Q. G admits the canonical free resolution $0 \to P \to Z(G) \to G \to 0$. Consider the commutative exact diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ & P \arrow[r, equal] \arrow[d] & P \arrow[d] & & \\ 0 \arrow[r] & Z(G) \arrow[r] \arrow[d] & Q(G) \arrow[r] \arrow[d] & Q/Z(G) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G \arrow[r] \arrow[d] & I_G \arrow[r] \arrow[d] & J_G \arrow[r] & 0 \\ & 0 & 0 & & \end{tikzcd}$$

and take the lower row in the above diagram as $\Gamma(G)$. Note that J_G is isomorphic to $Q/Z(G)$.

Let $\mu\colon E \wedge F \to W$ be a pairing of CW-spectra with $\pi_0(W) \cong Z$ and $\pi_{-1}(W)$ torsion free. This gives rise to the natural homomorphism

$$\bar{\mu}\colon FG^*(X) \to \mathrm{Hom}(E_*(X), \pi_0(WG))$$

for any G.

We shall require the following result in studying the duals of K-theories.

Lemma 10. *If μ induces isomorphisms $\bar{\mu}$: $\pi_*(FG')\to\mathrm{Hom}(\pi_{-*}(E),G')$ in the cases $G'=I_G, J_G$, then we have a homotopy equivalence f: $FG\to\hat{E}(G)$ with $\bar{\mu}=\kappa\ f_*$.*

Proof. In the commutative diagram

$$\begin{array}{ccccc}
FG^*(X) & \to & \mathrm{Hom}(E_*(X), G) & \leftarrow & \hat{E}(G)^*(X) \\
\downarrow & & \downarrow & & \downarrow \\
FI_G{}^*(X) & \to & \mathrm{Hom}(E_*(X), I_G) & \leftarrow & \hat{E}(I_G)^*(X) \\
\downarrow & & \downarrow & & \downarrow \\
FJ_G{}^*(X) & \to & \mathrm{Hom}(E_*(X), J_G) & \leftarrow & \hat{E}(J_G)^*(X),
\end{array}$$

the last two left-hand arrows are isomorphisms. So the above diagram yields the homotopy commutative diagram

$$\begin{array}{ccccccc}
FG & \to & FI_G & \to & FJ_G & \to & \sum FG \\
 & & & & \downarrow & & \\
\hat{E}(G) & \to & \hat{E}(I_G) & \to & \hat{E}(J_G) & \to & \sum\hat{E}(G)
\end{array}$$

with cofiber sequences. Choose a map

$$f\colon FG\to\hat{E}(G)$$

making the above diagram homotopy commutative. Then it becomes a homotopy equivalence from our hypothesis. The composite map $\kappa\ f_*$: $FG^*(X)\to \hat{E}(G)^*(X)\to\mathrm{Hom}(E_*(X),G)$ coincides with the homomorphism $\bar{\mu}$ induced by the pairing μ, because $\mathrm{Hom}(E_*(X),G)\to\mathrm{Hom}(E_*(X),I_G)$ is a monomorphism.

3.2. Let us denote by H, K, KO and KSp the Eilenberg-MacLane spectrum, the BU-, BO- and BSp-spectrum respectively. We now investigate the homotopy types of $\hat{H}(G)$, $\hat{K}(G)$ and $\widehat{KSp}(G)$.

Theorem 5. *For any abelian group G $\hat{H}(G)$, $\hat{K}(G)$ and $\widehat{KSp}(G)$ have the same homotopy types of HG, KG and KOG respectively* (cf., [3]).

Proof. The proof is essentially due to Anderson [3].

The $\hat{H}$ and K cases: Let E denote either H or K, and μ_E: $E\wedge E\to E$ the usual pairing. As is well known, $\bar{\mu}_E$: $\pi_*(E)\to\mathrm{Hom}(\pi_{-*}(E),Z)$ is an isomorphism. This implies that $\bar{\mu}_E$: $\pi_*(EG)\to\mathrm{Hom}(\pi_{-*}(E),G)$ is an isomorphism for all G. The result follows immediately from Lemma 10.

The $\widehat{KSp}$ case: There is a well known pairing μ_{KSp}: $KSp\wedge KO\to KSp$. We see easily that $\bar{\mu}_{KSp}$ $\pi_n(KO)\to\mathrm{Hom}(\pi_{-n}(KSp),Z)$ is an isomorphism except $n\equiv 1, 2 \bmod 8$, and hence $\bar{\mu}_{KSp}$ $\pi_n(KOA)\to\mathrm{Hom}(\pi_{-n}(KSp),A)$ is so for all n and Q-modules A. Fix a Q-module A. For any subgroup B of A we define homomorphisms λ_n by the composite maps

$$\pi_n(KO)\otimes A/B \xrightarrow{\cong} \pi_n(KOA/B) \to \mathrm{Hom}(\pi_{-n}(KSp), A/B) \quad \text{when } n \not\equiv 2,\ 3 \bmod 8$$

$$\pi_{n-1}(KO)\otimes B \xrightarrow{\cong} \pi_{n-1}(KOB) \xleftarrow{\cong} \pi_n(KOA/B) \to \mathrm{Hom}(\pi_{-n}(KSp),\ A/B) \quad \text{when } n \equiv 2,\ 3 \bmod 8$$

which are natural with respect to B. Let η_1 be the generator of $\pi_1(KO)\cong Z_2$ and define as the multiplications by η_1 ϕ: $\pi_1(KO)\to\pi_2(KO)$ and ϕ: $\pi_{-3}(KSp)\to \pi_{-2}(KSp)$. Then we remark that ϕ's are isomorphisms and $\phi^*\ \lambda_2=\lambda_3(\phi\otimes 1)$ The simpletification ε_{Sp}: $K\to KSp$ induces a natural transformation $K^*(Y)\to KSp^*(Y)$ of $KO^*(\ \ )$-modules for all finite Y. So we get a weak homotopy commutative diagram

$$\begin{tikzcd} K\wedge KO \arrow[r] \arrow[d] & K\wedge K \arrow[r] & K \arrow[d] \\ KSp\wedge KO \arrow[rr] & & KSp. \end{tikzcd}$$

(In fact this diagram is homotopy commutative by use of Corollary 13 below). This yields the commutative diagram

$$\begin{tikzcd} \pi_2(KOA/B) \arrow[rr] \arrow[d] & & \mathrm{Hom}(\pi_{-2}(KSp),\pi_0(KSpA/B)) \arrow[d] \\ \pi_2(KA/B) \arrow[r] & \mathrm{Hom}(\pi_{-2}(K),\ \pi_0(KA/B)) \arrow[r] & \mathrm{Hom}(\pi_{-2}(K),\ \pi_0(KSpA/B)). \end{tikzcd}$$

The left vertical arrow is a monomorphism because $\pi_1(KOA/B)=0$, and the lower horizontal ones are isomorphisms. Therefore the upper becomes a monomorphism, and hence so are both λ_2 and λ_3. Let $\{B_\lambda\}$ be the set of all finitely generated subgroups of B. As is easily checked, λ_n are isomorphisms for all n and B_λ because B_λ is free. On the other hand, A/B is isomorphic to the direct limit of A/B_λ and $\mathrm{Hom}(\pi_{-n}(KSp),A/B)\cong \varinjlim \mathrm{Hom}(\pi_{-n}(KSp),A/B_\lambda)$. So we see immediately that λ_n are isomorphisms for any subgroup B. Thus

$$\bar{\mu}_{KSp}\colon\ \pi_n(KOG') \to \mathrm{Hom}(\pi_{-n}(KSp),G')$$

is an isomorphism for any quotient group G' of a Q-module. Taking I_G and J_G as the above G' and applying Lemma 10 we get the desirable result.

In other words, Theorem 5 says that there exist universal coefficient sequences

$$\begin{array}{ll} & 0\to \mathrm{Ext}(H_{n-1}(X),\ G) \to HG^n(X) \to \mathrm{Hom}(H_n(X),\ G)\to 0 \\ (3.1) & 0\to \mathrm{Ext}(K_{n-1}(X),\ G) \to KG^n(X) \to \mathrm{Hom}(K_n(X),G)\to 0 \\ & 0\to \mathrm{Ext}(KO_{n+3}(X),\ G)\to KOG^n(X)\to \mathrm{Hom}(KO_{n+4}(X),G) \to 0 \end{array}$$

for any CW-spectrum X.

Theorem 3 combined with (3.1) implies the following

Corollary 11. i) *$HR^{n+1}(X)$ is Hausdorfff and only if* $\mathrm{Ext}(H_n(X)/TH_n(X), R)=0$.
ii) *$KR^{n+1}(X)$ is Hausdorff if and only if* $\mathrm{Ext}(K_n(X)/TK_n(XR)=0$.
iii) *$KOR^{n+1}(X)$ is Hausdorff if and only if* $\mathrm{Ext}(KO_{n+4}(X)/TKO_{n+4}(XR)=0$.

3.3. Finally we shall make a comment for Hausdorff-ness of K-theories.

Proposition 12. *Let E be a CW-spectrum such that $\pi_*(E)$ is of finite type as an R-module and fix a degree n. If $\pi_k(X)\otimes\pi_{k-n}(E)\otimes Q=0$ for all k, then $E^{n+1}(X)$ is Hausdorff.* (Cf., [8, Theorem III, 2]).

Proof. Under our assumtions we compute

$$E\hat{Z}/Z^n(X)\simeq\prod H^k(X;\pi_{k-n}(E)\otimes\hat{Z}/Z)\simeq\prod\mathrm{Hom}(H_k(X),\pi_{k-n}(E)\otimes\hat{Z}/Z)=0 .$$

Then the result is immediate from [8, Theorem III. 1].

For CW-spectra E and X whose rational homotopy groups are sparse we have

Corollary 13. *Let E be a CW-spectrum such that $\pi_*(E)$ is of finite type as an R-module. Assume that $\pi_m(E)\otimes Q=\pi_m(X)\otimes Q=0$ unless $m=0$ mod n. Then $E^{m+1}(X)$ is Hausdorff whenever $m\not\equiv 0$ mod n.*

As is well known, $\pi_{2n+1}(K)=0$ and $\pi_m(KO)\otimes Q=0$ if $m\not\equiv 0$ mod 4. Therefore Corollary 13 implies

Theorem 6. i) *$K^{2n}(K\wedge\cdots\wedge K)$ is Hausdorff.*
ii) *$KO^m(KO\wedge\cdots\wedge KO)$ is Hausdorff whenever $m\not\equiv 1$ mod 4.*

REMARK. Informations on $K_*(K)$ and $KO_*(KO)$ have been obtained by Adams, Harris and Switzer [2].

As an immediate corollary we have

Corollary 14. *Complex and real K-theories K^*, KO^* (defined on the category of CW-spectra) possess an associative and commutative multiplication.*

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