

## ON A CONJECTURE OF Y. NAKAI

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Assume that  $A$  is the affine ring of an algebraic variety defined over a characteristic zero field  $k$ . Denote by  $\text{Der}(A/k)$  the algebra of high order derivations of  $A$  to itself, and denote by  $\text{der}(A/k)$  the subalgebra of  $\text{Der}(A/k)$  which is generated by the first order derivations of  $A$  to itself. Recently Y. Nakai in his discussions of high order derivations (see [3]) asked the following question: is the condition  $\text{der}(A/k) = \text{Der}(A/k)$  equivalent to the regularity of  $A$ ? In this note we shall show that if  $A$  is the affine ring of a curve, then  $A$  is regular (i.e. the curve is non-singular) if and only if  $\text{der}(A/k) = \text{Der}(A/k)$ .

1. In all that follows,  $k$  will denote a field of characteristic zero. If  $A$  is a  $k$  algebra, then we shall denote by  $\text{Der}^n(A/k)$  the module of  $n$ -th order derivations over  $k$ , from  $A$  to itself. We shall denote by  $D^n(A/k)$  the  $A$  module of  $n$ -th order differentials, thus  $\text{Der}^n(A/k) = \text{Hom}_A(D^n(A/k), A)$ . We shall denote by  $\text{der}^n(A/k)$  the submodule of  $\text{Der}^n(A/k)$  which consists of linear combinations of derivations of the form  $\delta_1 \cdots \delta_j$ , where  $1 \leq j \leq n$  and  $\delta_i \in \text{Der}^1(A/k)$ . (For a discussion of these concepts see [1] or [3].)

In [1] Theorem 16.11.2 it is shown that if  $A$  is differentially lisse over  $k$  (hence in particular if  $A$  is regular), then each  $n$ -th order derivation is a combination of first order derivations. Thus to prove our assertion we need only prove the following theorem.

**Theorem.** *Suppose that  $k$  is a field of characteristic zero and suppose that  $A$  is the local ring of a point on an irreducible algebraic curve. If  $A$  is not regular, then there exists an integer  $N$  and an element  $w \in \text{Der}^N(A/k)$  such that  $w$  is not an element of  $\text{der}(A/k)$ .*

**Proof.** Denote by  $\bar{A}$  the integral closure of  $A$  in the quotient field  $Q$  of  $A$ . The ring  $\bar{A}$  is semi local, thus  $\bar{A} = B_1 \cap \cdots \cap B_s$ , where each  $B_i$  is a discrete rank one valuation ring with integer valued valuation  $v_i$ . Denote by  $\bar{d}$  the canonical derivation from  $\bar{A}$  to  $D^1(\bar{A}/k)$ . The restriction of  $\bar{d}$  to  $A$  is a  $k$  derivation from

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$A$  to  $D^1(\bar{A}/k)$ , therefore there exists a map  $\sigma$  from  $D^1(A/k)$  to  $D^1(\bar{A}/k)$  which is  $A$  linear such that if  $d$  is the canonical derivation from  $A$  to  $D^1(A/k)$ , then  $\sigma d = \bar{d}$ . Denote by  $K$  the kernel of this map. Because the formation of  $D^1$  commutes with localization (see [1, 16.4.15.])  $Q \otimes_A D^1(A/k) = Q \otimes_{\bar{A}} D^1(\bar{A}/k)$ . Thus  $D^1(A/k)$  has rank one and therefore  $K$  consists entirely of torsion elements. We assert that  $\text{Image}(\sigma) = D^1(A/k)/K$  cannot be free when  $A$  is not regular. To see this first note that  $\text{Hom}_A(D^1(A/k), A) = \text{Hom}_A(D^1(A/k)/K, A)$ . Thus if  $D^1(A/k)/K$  is free, then  $\text{Der}^1(A/k)$  is free. However, Lipman [2] has shown that  $\text{Der}^1(A/k)$  is free if and only if  $A$  is regular.

From now on we shall assume that  $A$  is not regular. The  $\bar{A}$  module  $D^1(\bar{A}/k)$  is projective and hence free because  $\bar{A}$  is regular and semi local. Let  $u$  be a basis for  $D^1(\bar{A}/k)$ . Thus  $\text{Image}(\sigma) = Mu$ , where  $M$  is a finitely generated  $A$  submodule of  $\bar{A}$ . Let  $\varphi$  denote a derivation from  $A$  to  $A$ . The derivation  $\varphi$  extends uniquely to a derivation of  $\bar{A}$  to  $Q$ . Let  $\delta$  denote the derivation from  $\bar{A}$  to  $\bar{A}$  which as an element of  $\text{Hom}_{\bar{A}}(D^1(\bar{A}/k), \bar{A})$  carries  $u$  to 1. Then  $\varphi = \beta\delta$  where  $\beta \in Q$ . The fact that  $\varphi$  carries  $A$  to itself implies that  $\beta M \leq A$ . We know that  $\beta M \neq A$  because  $M$  is not free. Thus  $\beta M$  is contained in the radical  $m$  of  $A$ .

Suppose now that  $x_1, \dots, x_r$  are generators for  $m$ . The module  $M$  is then generated by the elements  $\delta(x_1), \dots, \delta(x_r)$ . We wish to show that  $v_i(\beta) > 0$  for each  $i$  when  $\beta\delta$  carries  $A$  to itself. Let  $t_i$  denote a generator for the maximal ideal of  $B_i$ . Because  $B_i \otimes_{\bar{A}} D^1(\bar{A}/k) = D^1(B_i/k)$ , and  $D^1(B_i/k)$  has basis  $dt_i, u = u_i dt_i$  where  $u_i$  is a unit in  $B_i$ . Thus  $\delta(x_j) = u_i^{-1} \frac{d}{dt_i}(x_j)$  for  $1 \leq j \leq r$  ( $\frac{d}{dt_i}$  is the dual of  $dt_i$ ), therefore  $v_i(\delta x_j) = v_i\left(\frac{d}{dt_i}(x_j)\right)$ . Clearly,  $v_i\left(\frac{d}{dt_i}(x_j)\right) = v_i(x_j) - 1$ . As we have noted before  $\beta\delta x_j$  is in  $m$  for each  $j$ . Hence  $v_i(\beta\delta x_j) \geq \min_j(v_i(x_j))$ , thus  $v_i(\beta) \geq 1$ .

Now denote by  $I$  the  $A$  submodule of  $Q$  which consists of the  $\beta$  such that  $\beta\delta$  is a derivation of  $A$  to itself. The previous discussion has shown that  $I$  is contained in the radical of  $\bar{A}$ , and it is clearly finitely generated. Now if  $\delta_1, \dots, \delta_v$  are first order derivations of  $A$  to itself, then  $\delta_i = \beta_i \delta$  for some  $\beta_i \in I$ . Thus  $\delta_1 \cdots \delta_v = \sum_{j < v} \gamma_j \delta^j + \beta_1 \cdots \beta_v \delta^v$ , and therefore each composition of  $v$  derivations of  $A$  to itself has as coefficient of  $\delta^v$  an element of  $I^v$ . Then, if  $R$  is the radical of  $\bar{A}$ ,  $\beta_1 \cdots \beta_v \in R^v$ . Let  $C$  be the conductor of  $\bar{A}$ , hence there is an  $N$  such that  $R^N \subseteq C$  and  $R^N \neq C$ . So, choose  $y \in C, y \notin R^N$ . Then  $y\delta^N$  is an  $N$ -th order derivation from  $A$  to itself, but  $y\delta^N$  is not in  $\text{der}(A/k)$  because  $y$  is not in  $I^N \subseteq R^N$ .

**References**

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