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A UNIQUE CONTINUATION THEOREM FOR AN ELLIPTIC OPERATOR OF TWO INDEPENDENT VARIABLES WITH NON-SMOOTH DOUBLE CHARACTERISTICS

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1. Introduction. Let $P=P(x;\partial/\partial x)$ be an elliptic homogeneous differential operator of order $m(\geqq 2)$ with complex valued C^∞ coefficients defined near the origin in the 2-dimensional real Euclidean space R^2. We say that P is A-ellilpic at x_0 if x_0 has a neighbourhood U such that for any open and connected neighbourhood V $(\subset U)$ of x_0, there is no non-trivial solution $u\in C^m(V)$ of the differential inequality in V

$$(1) \qquad |P(x;\partial/\partial x)u|\leqq C\sum_{|\alpha|<m}|(\partial/\partial x)^\alpha u|$$

such that $u=0$ in some open subset (of V) whose closure contains the point x_0. It is well known that P is A-elliptic at each point where P has simple characteristics, or P has double characteristics and has Lipschitz continuous characteristic roots (see Hörmander [1], Pederson [2]).

In the present paper we shall give a sufficient condition for the operator P to be A-elliptic when P has double characteristics and its symbol $P(x;\xi)$ has a factorization of the form in a neighbourhood of the origin

$$(2) \quad p(x;\xi)=a(x)\prod_{j=1}^{N}(\xi_1^2+2a_j(x)\xi_1\xi_2+b_j(x)\xi_2^2)$$

or

$$(3) \quad p(x;\xi)=a(x)\prod_{j=1}^{N}(\xi_1^2+2a_j(x)\xi_1\xi_2+b_j(x)\xi_2^2)\prod_{j=N+1}^{N+s}(\xi_1+a_j(x)\xi_2)\,.$$

Here a, a_j and b_k are $C^\infty(\omega)$ functions such that $a(0)\neq 0$, $a_j(0)^2=b_j(0)$ $(j=1,\cdots,N)$ and $a_j(0)\neq a_k(0)$ $(1\leq j\neq k\leq N+s)$.

Set $c_j(x)=b_j(x)-a_j(x)^2$ and let R_j be the set of points $y\in\omega$ which has a neighbourhhood where $c_j(x)=k(x)^2$ for some $C^{1+1/2}$ function $k(x)$. Then we have our main result as follows.

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Theorem. *If there is an open neighbourhood* ω_1 $(\subset \omega)$ *of the origin such that the following conditions* (4) *and* (5) *hold for each* $j=1, \cdots, N$, *then* $P(x; \partial/\partial x)$ *is* A-*elliptic at the origin.*

(4) $\operatorname{grad}_x c_j(x)=0$ *when* $c_j(x) = 0$, $x \in \omega_1$.

(5) *The order of zero for* c_j *is finite at each point of* $\omega_1 \backslash R_j$.

In the next section we shall give a proposition on differentiable square roots in order to prove Theorem.

2. Differentiable square roots and the proof of Theorem. Let ω_2 $(\subset \omega_1)$ be an open and connected neighbourhood of the origin such that $\min\{|a_j(x)-a_k(x)|;\ 1\leq j \neq k \leq N+s\} > \max\{2|b_j(x)-a_j(x)^2|^{1/2};\ 1\leq j \leq N\}$. This means that P has at most double characteristics at each point of ω_1. Then, applying the results by Hörmander [1] or Pederson [2], we have P is A-elliptic at each point of $\omega_2 \cap R$ where $R=\bigcap_{j=1}^{N} R_j$. So it is sufficient to prove that $\omega_2 \cap R$ is a dense subdomain of ω_2. From now on, we shall use new notation. Let Ω be domain in R^n $(n\geq 2)$, and let f_j and g_j $(1\leq j<\infty)$ real valued C^∞ functions defined in Ω. We also denote by R_j the set defined §1 for $f_j+\sqrt{-1}g_j$ and Ω instead of c_j and ω. Then we have

Proposition. $R=\bigcap_{j=1}^{N} R_j$ *is a dense subdomain of* Ω *if the following conditions* (6) *and* (7) *are satisfied for each* $j=1, \cdots, N$.

(6) $\operatorname{grad}_x f_j(x)=\operatorname{grad}_x g_j(x)=0$ *when* $f_j(x)=g_j(x)=0$, $x \in \Omega$.

(7) *At least one of orders of zeros for* f_j *and* g_j *is finite at each point of* $\Omega \backslash R_j$.

Remarks of Theorem. 1) Since $p(x; \xi)$ has factorization (2) or (3), we that the condition:

(4)′ $\operatorname{grad}_x p(x; \xi)=0$ when $p(x; \xi)=\operatorname{grad}_\xi p(x, \xi)=0$, $(x, \xi)\in \omega_1 \times \boldsymbol{C}^2 \backslash 0$

is equivalent to the condition (4).

2) We denote by $D(x, \xi_2)$ the discriminant of polynomial $p(x; \xi)$ in ξ_1. Since $p(x; \xi)$ is homogeneous in $\xi=(\xi_1, \xi_2)$ of order m, we have $D(x, \xi_2)= \delta(x)\xi_2^{m(m-1)}$ and $\delta(x)=\delta_0(x)\prod_{j=1}^{N}(b_j(x)-a_j(x)^2)$. Here $\delta_0 \in C^\infty$ and $\delta_0(0)\neq 0$. So that we have the condition:

(5)′ the order of zero for δ is finite at the origin

implies the condition (5) if we take sufficiently small ω_1.

Remarks of Proposition. 1) In order that x_0 belongs to R_j, it is necessary that the following inequality holds in some neighbourhood of x_0,

(8) $|\mathrm{grad}_x f_j(x)|^2 + |\mathrm{grad}_x g_j(x)|^2 \le C\{|f_j(x)| + |g_j(x)|\}$

for some constant C. Moreover the conditon:

(9) for each $j=1, \cdots, N$, setting $\mathrm{Re}\, c_j = f_j$ and $\mathrm{Im}\, c_j = g_j$, the inequality (8) holds in a neghbourhoof of the origin

is sufficient in order that P is A-elliptic at the origin.

2) There is a pair f and g fo C^∞ funtions satisfying the condition (6), but not satisfying both (7) and (8) such that R is not connected. For example, near $t=0$, $\mathrm{f}(t, x_2) = \exp(-1/t) \sin(1/t)$, $g(t, x_2) = (\log 1/t)^{-1/t}$ if $t<0$, $f=g=0$ if $t<0$.

Proof of Proposition. It is easy to see that R_j and R are open and dense in Ω. So that we have only to show that R_j and R are connected. Now we first prove tha following two Lemmas under the conditions (6) and (7)

Lemma 1. *For any $(n-1)$-dimensional C^∞ manifold Γ in Ω, $\Gamma \cap R_j$ is dense in Γ.*

Lemma 2. *For each point $x \in S_j = \Omega \backslash R_j$, there is a fundamental system $\{U(x)\}$ of open neighbourhoods of x such that $U(x) \backslash S_j$ is connected.*

Proof of Lemma 1. Without loss of generality, we may assume that is Γ defined by the equation $x_1 = \phi(x')$ near $x_0 \in S_j$ and that f_j and g_j vanish at any point x on Γ if x is near x_0. Here $\phi \in C^\infty$ and we use notation $x=(x_1, x')$. In addition, we may assume that the order of zero for f_j is finite at x_0 by the assumption. So, near x_0, f_j and g_j have a factorization of the form

$$f_j(x) = f'_j(x)(x_1 - \phi(x'))^\lambda .$$

If, for some positive integer k, $\partial^k g_j / \partial x_1^k$ does not vanish identically in any neighbourhood of x_0 in Γ, we have

$$g_j(x) = g'_j(x)(x_1 - \phi(x'))^v$$

and if other case occurs, for any positive integer i we have

$$g_j(x) = g_{j,i}(x)(x_1 - \phi(x'))^i .$$

Here λ and v are positive integers which are independent on x and f'_j, g'_j and $g_{j,i}$ are C^∞ functions such that f'_j and g'_j do not vanish identically in any neighbourhood of x_0 in Γ. By the assumption (6), we have $\lambda, v \ge 2$. The above factorizations imply that when for g_j the first case occurs and $\lambda \le v$, or the second case occurs, any point such that f'_j does not vanish is in R_j and when other case occurs, any point such that g'_j dose not vanish is in R_j. This means that $\Gamma \cap R_j$ is dense in Γ.

Proof of Lemma 2. Take any point x_0 in S_j. Without loss of generality, we may assume that $\alpha(x)$, the order of zero for f_j at x, is finite at x_0. Since $\alpha(x)$ is upper semi-continuous, we can choose an open neighbourhood W of x_0 such that $W \cap S_j = \{x \in W \cap S_j;\ \alpha(x) \leq \alpha(x_0)\}$. Setting $T^{(k)} = \{x \in W \cap S_j;\ \alpha(x) = k\}$, we have $W \cap S_j = \bigcup_{k=2}^{\alpha(x_0)} T^{(k)}$, so that, using the induction on k such that $x \in T^{(k)}$, we prove this Lemma for any $x \in W \cap S_j$. When $x \in T^{(2)}$, $T^{(2)} = W \cap S_j$ and $T^{(2)}$ is contained in a $(n-1)$-dimensional C^∞ manifold near x. Hence, using Lemma 1, the result is clear. When $x \in T^{(k+1)}$, by similar reason, we can take a fundamental system $\{U(x)\}$ of open neighbourhoods of x such that $U(x) \cap S_j = U(x) \cap (\bigcup_{i=2}^{k+1} T^{(i)})$ and that $U(x) \backslash T^{(k+1)}$ is connected. We show that this system $\{U(x)\}$ has that required property. If $U(x) \backslash S_j$ is not connected, we can take two disjoint components C_0 and C_1 and a continuous curve $x(t)$ $(0 \leq t \leq 1)$ in $U(x) \backslash T^{(k+1)}$ such that $x(0) \in C_0$ and $x(1) \in C_1$. Take a small positive number ε such that $B(x(t), \varepsilon) \subset U(x) \backslash T^{(k+1)}$ for any t, $B(x(0), \varepsilon) \subset C_0$ and $B(x(1), \varepsilon) \subset C_1$ where $B(y, \varepsilon)$ is the closed ball with the center y and radius ε. Set $H = \{x = (x_1, x_2, \cdots, x_n) \in R^n;\ x_1 = 0\}$, $H(t) = B(x(t), \varepsilon) \cap [x(t) + H]$ and $t_{\max} = \sup\{t \in [0, 1];\ H(t) \cap C_0 \neq \phi\}$. Since the set $\{t;\ H(t) \cap C_0 \neq \phi\}$ is open in $[0, 1]$ we have $0 < t_{\max} < 1$ and $H(t_{\max}) \cap C_0 = \phi$. On the other hand, by the definition of $t_{\max}$, there are two convergent sequences $\{t_v\}$ and $\{x_v\}$ such that $x_v \in H(t_v) \cap C_0$ with the limit point $t_{\max}$ and y, respectively. This limit point y is in $H(t_{\max}) \cap S_j$ since $H(t_{\max}) \cap C_0 = \phi$. So that $y \in \bigcup_{i=2}^{k} T^{(i)}$. By the induction hypothesis, there is a neighbourhood $U(y)$ of y such that $U(y) \subset U(x)$ and that $U(y) \backslash S_j$ is connceted. Hence, using Lemma 1, we have $U(y) \backslash S_j \subset C_0$ and $H(t_{\max}) \cap C_0 \neq \phi$. This gives a contradiction and then completes the proof of Lemma 2.

Then we can prove easily that R_j and R are connceted if we shall use the similar method by the reduction to absurdity as that in the proof of Lemma 2.

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