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On Convergence of Galerkin's Approximations for the Regularized 3D Periodic Navier-Stokes Equations

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Abstract

Regularization of the Navier-Stokes equations by adding hyperviscosity term $\mu(-\Delta^2)$, $\mu > 0$ is considered. We proved convergence of Galerkin's approximations to the strong generalized solution of the regularized Navier-Stokes equations; existence and uniqueness of the strong generalized solution.

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1 Introduction

The 3D Navier-Stokes equations describe the motion of a viscous incompressible fluid in $\mathbb{R}^3$. The equations are to be solved for an unknown divergence-free velocity vector-function $u = (u_i)_{1\le i\le 3}$ and scalar function p called pressure [1], [2]. We use dimensionless coordinates and consider the case when the velocity, pressure and the external forces f_i are real periodic functions with the period 2π in all space coordinates x_i, $i = 1,2,3$; that is defined on a 3D torus $\Omega := \mathbb{R}^3/2\pi\mathbb{Z}^3$. The Navier-Stokes equations in the domain $\mathbb{Q}_T = \Omega\times[0,T)$ have the form

$$\frac{\partial u_i}{\partial t} - \nu\Delta u_i = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + f_i; \quad (x,t)\in\mathbb{Q}_T, \ \ \nu > 0,$$

$$\operatorname{div} u = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} = 0, \quad (x,t)\in\mathbb{Q}_T,$$

$$u(x,0) = u^0(x), \ \ \operatorname{div} u^0 = 0.$$

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Notations. Let $\mathbb{Q}_T = \Omega\times[0,T)$, $\mathbb{Q}_\infty = \Omega\times[0,+\infty)$. Norms in the Sobolev spaces $W^{\kappa,2}(\Omega)$ are denoted as

$$\|u\|_{\kappa,2} := \left\{\int_\Omega \left[\left|(-\Delta)^{\kappa/2}u\right|^2 + |u|^2\right] dx\right\}^{1/2}. \tag{1.1}$$

We also use pre-norms

$$\|u\|_{0,\kappa,2} := \left\{\int_\Omega \left|(-\Delta)^{\kappa/2}u\right|^2 dx\right\}^{1/2}.$$

For a mapping $[0,T]\ni t \to f(\cdot,t)\in W^{\kappa,2}(\Omega)$ the norm of the element $f(\cdot,t)\in W^{\kappa,2}(\Omega)$ is denoted as $\|f(\cdot,t)\|_{\kappa,2}$, the $L_2(\Omega)$ norm of a vector-function f as $\|f\|$, a scalar product of vectors f,g in $\mathbb{C}^3$ as $f\cdot g$, magnitude of a $\mathbb{C}^3$ vector f as $|f|$ and a scalar product in the space $L_2(\Omega)$ as $(\cdot,\cdot)$. A scalar product in the Hilbert space $W^{\kappa,2}(\Omega)$ is denoted as $(f,g)_{\kappa,2}$, a norm in the space $L_p(\Omega)$ as $\|\cdot\|_p$, but for the norm in the space $L_2(\Omega)$ we use notation $\|\cdot\|$. A subspace of functions $\{u : u\in L_2(\mathbb{Q}_T),\ u(\cdot,t)\in J_2(\Omega)\}$ is denoted as $L_2^0(\mathbb{Q}_T)$.

A set of *solenoidal* vectors in $C^\infty(\Omega)$ we denote as $J(\Omega)$, and a completion of $J(\Omega)$ in the norm $W^{1,2}(\Omega)$ as $H(\Omega)$. Let $J_2(\Omega)$ be the completion of the set $J(\Omega)$ in $L_2(\Omega)$, and let $\mathbf{P}$ be the orthogonal projection (Leray's projection) of the Hilbert space $L_2(\Omega)$ onto the subspace $J_2(\Omega)$. Direct calculations give for Leray's projection $\mathbf{P}$ an expression

$$(\mathbf{P}f)(x) = \sum_{k\in\mathbb{Z}^3, k\neq 0} \left\{f_k - k(f_k\cdot k)|k|^{-2}\right\}\exp\{i(k\cdot x)\} + f_0. \tag{1.2}$$

through the Fourier coefficients f_k of a function $f\in L_2(\Omega)$ [2]. Evidently on functions $u\in W^{2\kappa,2}(\Omega)\cap H(\Omega)$, $\kappa = 1,2,\ldots$, we have $\mathbf{P}\Delta^\kappa u = \Delta^\kappa u$.

Applying Leray's projection $\mathbf{P}$ to the Navier-Stokes equations we exclude the presser from the equations and write the Navier-Stokes equations in the equivalent form [2]

$$\frac{\partial u}{\partial t} - \nu\Delta u = -\mathbf{P}(u\cdot\nabla)u + \mathbf{P}f,\quad (x,t)\in\mathbb{Q}_T,\ u(\cdot,t)\in H(\Omega).$$

The Navier-Stokes equations are regularized by adding to the viscosity term $\nu\Delta u$ the hyperviscosity term $-\mu\Delta^2 u$. So the Cauchy problem for the regularized Navier-Stokes equations in $\mathbb{Q}_T$ has the form

$$\frac{\partial u}{\partial t} - \nu\Delta u + \mu\Delta^2 u = -\mathbf{P}(u\cdot\nabla)u + \mathbf{P}f,\quad (x,t)\in\mathbb{Q}_T, \tag{1.3}$$

$$\text{div } u = 0, (x,t)\in\mathbb{Q}_T;\quad u(\cdot,0) = u^0,\ \text{ div } u^0 = 0. \tag{1.4}$$

Generalized solution to problem (1.3), (1.4) can be found in the space $Wr(T)$ obtained as the completion of functions

$$\{u : u\in C^\infty(\mathbb{Q}_T),\ u(\cdot,t)\in H(\Omega)\} \tag{1.5}$$

in the norm

$$\|u\|^2_{Wr(T)} := \sup_{[0,T)}\left\{\|u(\cdot,t)\|^2_{1,2} + \|u(\cdot,t)\|^2_{2,2}\right\} + \int_0^T \left\{\|\partial_t u(\cdot,t)\|^2 + \|u(\cdot,t)\|^2_{4,2}\right\} dt. \tag{1.6}$$

Definition 1.1. 1° *A vector function* $u \in Wr(T),\ T < \infty$ *is called the generalized solution to the regularized Navier-Stokes equations* (1.3), (1.4) *(abbreviation SRNS) in the cylinder* $\mathbb{Q}_T$ *with data*

$$u^0 \in H(\Omega) \cap W^{2,2}(\Omega), \quad f \in L_2(\mathbb{Q}_T) \tag{1.7}$$

if: *a)* $\left\|u(\cdot,t) - u^0(\cdot)\right\|_{2,2} \to 0$ *as* $t \to 0$,

b) $\operatorname{div} u = 0$,

c) the generalized derivatives $u_t,\ u_{x_k},\ u_{x_k x_k},\ u_{x_k x_k x_m x_m}$ *belong to* $L_2(\mathbb{Q}_T)$ *and satisfy equation* (1.3).

2° *A vector function* u *defined in* $\mathbb{Q}_\infty$ *is called the SRNS of problem* (1.3), (1.4) *in the* $\mathbb{Q}_\infty$ *if it is the SRNS in all cylinders* $\mathbb{Q}_T$, $T < \infty$.

The *SRNS* solution is usually referred to as the strong generalized solution. Different regularizations of the Navier-Stokes equations were considered in numerous publications. O. A. Ladyzhenskaya and J. L. Lions in the papers [3], [4] proposed to change the viscosity $\nu\Delta u$ for the hyperviscosity $\nu\Delta u - (-\Delta)^l$, $l > 5/4$ and proved the existence of the global weak solution (in the integral sense) to the regularized Navier-Stokes equations. In the case $l = 2$ we proved the existence of the strong global generalized solution and the convergence of Galerkin's approximations to such solution in the space $Wr(T)$ for all $T < \infty$. There are many publications on the Navier-Stokes equations with hyperviscosity where attractors, a turbulence and computational methods were considered [5], [6], [7], etc.

2 Main Results

Now we deduce a priory estimates for the classical solution to the Navier-Stokes equations.

Lemma 2.1. 1) *The* $C^\infty(\mathbb{Q}_\infty)$ *classical solution to problem* (1.3), (1.4) *satisfies the following inequalities on the interval* $[0,\infty)$*:*

$$\|u(\cdot,t)\| \le \|u_0\| + \int_0^t \|f(\cdot,\tau)\|\,d\tau; \tag{2.1}$$

$$\|u(\cdot,t)\|^2 + 2\int_0^t \left\{\nu\|u(\cdot,\tau)\|_{0,1,2}^2 + \mu\|u(\cdot,\tau)\|_{0,2,2}^2\right\} d\tau$$
$$\le \|u_0\|^2 + 2\left\{\|u_0\| + \int_0^t \|f(\cdot,\tau)\|\,d\tau\right\} \times \int_0^t \|f(\cdot,\tau)\|\,d\tau. \tag{2.2}$$

2) *Let*

$$\Phi(t) := \frac{1}{2}\|u_0\|^2 + \left\{\|u_0\| + \int_0^t \|f(\cdot,\tau)\|\,d\tau\right\} \times \int_0^t \|f(\cdot,\tau)\|\,d\tau\,;$$
$$g(t) := c_2\left\{\|u_0\| + \int_0^t \|f(\cdot,\tau)\|\,d\tau\right\} \times \left\{\|u_0\| + \|f(\cdot,t)\| + \int_0^t \|f(\cdot,\tau)\|\,d\tau\right\} + \frac{1}{2}\|f(\cdot,t)\|^2, \tag{2.3}$$
$$c_2 := (\nu^2 + 2\nu\mu + \mu^2) + 2(\nu + \mu).$$

Then the following inequality holds for all $t \in [0,\infty)$, with some constant c_1 no depending on ν and μ,

$$\nu\|u(\cdot,t)\|_{1,2}^2 + \mu\|u(\cdot,t)\|_{2,2}^2 \le \left\{\nu\|u_0\|_{1,2}^2 + \mu\|u_0\|_{2,2}^2\right\} \times \exp\left\{\frac{c_1}{\mu\nu}\Phi(t)\right\}$$
$$+ \int_0^t g(\tau)\exp\left\{\frac{c_1}{\mu\nu}[\Phi(t)-\Phi(\tau)]\right\}d\tau. \tag{2.4}$$

3) *For all $T \in [0,\infty)$, the norm $\|u\|_{Wr(T)}$ of the classical solution satisfies the inequality*

$$\|u\|_{Wr(T)}^2 \le \frac{2}{\nu}\left[\left\{\nu\|u_0\|_{1,2}^2 + \mu\|u_0\|_{2,2}^2\right\} \times \exp\left\{\frac{c_1}{\mu\nu}\Phi(T)\right\}\right.$$
$$\left. + \int_0^T g(\tau)\exp\left\{\frac{c_1}{\mu\nu}[\Phi(T)-\Phi(\tau)]\right\}d\tau\right] \times \Phi(T) + \int_0^T g(\tau)d\tau. \tag{2.5}$$

Proof. Let $u \in C^\infty(\mathbb{Q}_\infty)$ be a real classical solution to problem (1.3). Taking the scalar product in $L_2(\Omega)$ of the left and right hand-sides of equality (1.3) with the solution u, we obtain the inequalities

$$\frac{d}{dt}\|u(\cdot,t)\|^2 \le 2\|u(\cdot,t)\|\,\|f(\cdot,t)\|, \tag{2.6}$$

$$\frac{d}{dt}\|u(\cdot,t)\|^2 + 2\nu\|u(\cdot,t)\|_{0,1,2}^2 + 2\mu\|u(\cdot,t)\|_{0,2,2}^2 \le 2\|u(\cdot,t)\| \times \|f(\cdot,t)\|. \tag{2.7}$$

Inequalities (2.1), (2.2) are a direct consequence of inequalities (2.6), (2.7). Further, taking scalar square in $L_2(\Omega)$ on the left and right hand-side of equality (1.3), and summing up the result with inequality (2.6) multiplied by $(\nu+\mu)$ and with the square of inequality (2.1) multiplied by $(\nu^2+2\nu\mu+\mu^2)$, we obtain the inequality

$$\frac{d}{dt}\left\{\nu\|u(\cdot,t)\|_{1,2}^2 + \mu\|u(\cdot,t)\|_{2,2}^2\right\} + \|\partial_t u(\cdot,t)\|^2 + \nu^2\|u(\cdot,t)\|_{2,2}^2$$
$$+2\,\nu\mu\|u(\cdot,t)\|_{3,2}^2 + \mu^2\|u(\cdot,t)\|_{4,2}^2 \le \|[(u\cdot\nabla)u](\cdot,t)\|^2 + g(t), \tag{2.8}$$

where the function $g(t)$ is defined in (2.3). By the embedding Theorem [8], for the dimension 3, we have $\max_{x\in\Omega}|u(x,t)|^2 \le c\|u(\cdot,t)\|_{2,2}^2$, hence the following inequality holds

$$\|[(u\cdot\nabla)u](\cdot,t)\|^2 \le c_1\|u(\cdot,t)\|_{0,1,2}^2\|u(\cdot,t)\|_{2,2}^2$$
$$\le \frac{c_1}{\mu}\left\{\nu\|u(\cdot,t)\|_{1,2}^2 + \mu\|u(\cdot,t)\|_{2,2}^2\right\}\|u(\cdot,t)\|_{0,1,2}^2, \tag{2.9}$$

with some constant c_1. From inequalities (2.8), (2.9) we infer the inequality

$$\frac{d}{dt}\left\{\nu\|u(\cdot,t)\|_{1,2}^2 + \mu\|u(\cdot,t)\|_{2,2}^2\right\}$$
$$\le \frac{c_1}{\mu}\left\{\nu\|u(\cdot,t)\|_{1,2}^2 + \mu\|u(\cdot,t)\|_{2,2}^2\right\}\|u(\cdot,t)\|_{0,1,2}^2 + g(t). \tag{2.10}$$

Applying Gromwell's inequality to inequality (2.10) we have

$$\Big\{\nu\|u(\cdot,t)\|_{1,2}^2+\mu\|u(\cdot,t)\|_{2,2}^2\Big\}\le\Big\{\nu\|u(\cdot,0)\|_{1,2}^2+\mu\|u(\cdot,0)\|_{2,2}^2\Big\}\times\exp\left\{\frac{c_1}{\mu\nu}\int_0^t\|u(\cdot,\tau)\|_{0,1,2}^2\,d\tau\right\}$$
$$+\int_0^t g(\tau)\exp\left\{\frac{c_1}{\mu\nu}\int_\tau^t\|u(\cdot,s)\|_{0,1,2}^2\,ds\right\}d\tau \tag{2.11}$$

for all $t\in[0,\infty)$. Note that inequality (2.2) implies the estimate

$$\int_0^t\|u(\cdot,\tau)\|_{0,1,2}^2\,d\tau\le\frac{1}{2\nu}\|u_0\|^2+\frac{1}{\nu}\left\{\|u_0\|+\int_0^t\|f(\cdot,\tau)\|\,d\tau\right\}\times\int_0^t\|f(\cdot,\tau)\|\,d\tau=\Phi(t). \tag{2.12}$$

Thus, substituting estimate (2.12) in inequality (2.11), we obtain inequality (2.4).

Now we replace the term $\|[(u\cdot\nabla)u](\cdot,t)\|^2$ in the right-hand side of inequality (2.8) by its estimate (2.9) and further we replace the term $\big\{\nu\|u(\cdot,t)\|_{1,2}^2+\mu\|u(\cdot,t)\|_{2,2}^2\big\}$ by its estimate (2.11). Then integrating the obtained inequality by t, we obtain estimate (2.5). □

The existence of the SRNS is proved by Galerkin's method. We obtain the convergence of Galerkin's approximations in the space $Wr(T)$ to the SRNS for all $T<+\infty$.

The orthonormal real vector eigenfunctions $f_k\sin(k\cdot x)$, $g_k\cos(k\cdot x)$: $k=(k_1,k_2,k_3)$, $k_i\in\mathbb{Z}$, $f_k\cdot k=0$, $g_k\cdot k=0$ of the operator Δ: $W^{2,2}(\Omega)\cap H(\Omega)\to J_2(\Omega)$ are numerated from 1 to ∞ by the index l and are denoted by a^l. Evidently, the functions $\{a^l\}_{l=1}^\infty$ form the basis in the Hilbert space $J_2(\Omega)$. Galerkin's approximations u^n for the SRNS have the form $u^n(x,t):=\sum_{l=1}^n c_{l,n}(t)a^l(x)$ where the functions $c_{l,n}$ are defined below. The functions $c_{l,n}$ are determined by Galerkin's conditions

$$(\partial_t u^n-f,a^l)+\mu(\Delta u^n,\Delta a^l)+\sum_{i=1}^3\Big\{\nu(\partial_{x_i}u^n,\partial_{x_i}a^l)-(u_i^n u^n,\partial_{x_i}a^l)\Big\}=0,\quad l=1,\dots,n, \tag{2.13}$$

and the initial data $c_{l,n}(0)=c_l$, $l=1,\dots,n$, where $u^0=\sum_{l=1}^\infty c_l a^l$.

Conditions (2.13) were obtained formally from system (1.3) by replacing the solution u by Galerkin's approximation u^n, multiplying equations (1.3) by the function a^l and integrating over Ω. Galerkin's conditions (2.13) is a system of ordinary differential equations with respect to the functions c_{jn}:

$$\frac{dc_{jn}}{dt}-\nu\sum_{k=1}^n a_{jk}c_{kn}+\sum_{p,k=1}^n a_{jpk}c_{pn}c_{kn}=f_j,\quad j=1,\dots,n, \tag{2.14}$$

where a_{jk}, a_{jpk} are some constants and $f_j=(f,a^j)$.

Define in $L_2(\Omega)$ a projection $\mathbf{P}_n$: $\mathbf{P}_n f:=\sum_{k=1}^n(f,a^k)a^k$. Evidently Galerkin's approximations u^n satisfy the problem

$$\frac{\partial u^n}{\partial t}-\nu\Delta u^n+\mu\Delta^2u^n=-\mathbf{P}_n(u^n\cdot\nabla)u^n+\mathbf{P}_n f;\quad (x,t)\in\mathbb{Q}_\infty. \tag{2.15}$$

We have $(\mathbf{P}_n(u^n\cdot\nabla)u^n,u^n)=((u^n\cdot\nabla)u^n,u^n)$ and $\|\mathbf{P}_n(u^n\cdot\nabla)u^n\|\le\|(u^n\cdot\nabla)u^n\|$, thus we can apply to equation (2.15) considerations of the Lemma 2.1 and deduce Lemma 2.2.

Lemma 2.2. *Let $u^0 \in H(\Omega)\cap W^{2,2}(\Omega)$ and $f \in L_2(\mathbb{Q}_T)$ for all $T < \infty$. Then:*

1) *Galerkin's approximations u^n satisfy all inequalities of Lemma* 2.1.

2) *Galerkin's approximations u^n, $n = 1,2,\ldots,$ for all $T < \infty$ satisfy the inequality*

$$\left\|u^n\right\|^2_{Wr(T)} \le c\left(T, \nu, \mu, \|u_0\|, \int_0^T \|f(\cdot,s)\|^2 ds\right) \tag{2.16}$$

with some constant depending on $T,\nu,\mu,\|u_0\|,\int_0^T \|f(\cdot,s)\|^2 ds$.

It follows from the orthogonality $(a^j,a^l) = \delta_{j,l}$ that $\|u^n(\cdot,t)\|^2 = \sum_{j=1}^n c^2_{jn}(t)$. Hence inequality (2.1) for the functions u^n implies that Galerkin's approximations $u^n(\cdot,t)$ exist on $[0,\infty)$.

Now we prove the convergence of Galerkin's approximations in the space $W_r(T)$ for all $T < \infty$ and deduce the existence of the SRNS.

Theorem 2.3. *(3D case)*
Let the initial data and the right-hand side f of the Navier-Stokes problem (1.3), (1.4) *satisfy conditions* (1.7)*, then the SRNS to problem* (1.3), (1.4) *exists and is unique in $\mathbb{Q}_\infty$. Galerkin's approximations u^n converge to the SRNS in the norm $\|u\|_{W_r(T)}$ for all $T < \infty$. The SRNS satisfies inequalities* (2.1)-(2.5).

Proof. Fix $T > 0$. By inequality (2.16), the norms $\|u^n\|^2_{Wr(T)}$ of Galerkin's approximations are bounded uniformly in index n. Therefore we can choose from Galerkin's approximations u^n a subsequence $\{u^{n_q}\}$ such that functions u^{n_q}, $u^{n_q}_t$, $u^{n_q}_{x_m}$, $u^{n_q}_{x_i x_j}$ are weakly converging in $L_2(\mathbb{Q}_T)$. Let us study the strong convergence of the sequences $\{u^{n_q}_{x_m}\}$, $\{u^{n_q}\}$ *in* $L_2(\mathbb{Q}_T)$ by using the Friedrich inequality and the argumentation of the book [1], pp 173-178. The Friedrich inequality asserts [1] that for any $\varepsilon > 0$ there exist N_ε functions ω_j, $j = 1,\ldots,N_\varepsilon$, such that an inequality

$$\int_\Omega u^2 dx \le \sum_{j=1}^{j=N_\varepsilon}\left(\int_\Omega u\omega_j dx\right)^2 + \varepsilon\int_\Omega (grad\ u)^2 dx. \tag{2.17}$$

holds for every function from $W^{1,2}(\Omega)$. Evidently for functions $u \in W^{1,2}(\Omega)$ we can chose the set $\{\omega_j\}_{j=1}^{j=N_\varepsilon}$ as 1, $f_k \sin(k\cdot x)$, $g_k\cos(k\cdot x)$, $k = (k_1,\ k_2,\ k_3)$, $k_i \in \mathbb{Z}$; $|k| \le 1/\sqrt{\varepsilon}$. It follows directly from [1] that there exists a subsequence $\{u^{n^1_q}\}$ that converges in $L_2(\mathbb{Q}_T)$.

Applying the Friedrich inequality to the function $u := \partial_{x_k}(u^{n^1_i} - u^{n^1_j})$ and integrating it with respect to the variable t from 0 to T, we have

$$\begin{aligned}\int_0^T\int_\Omega\left|\partial_{x_k}(u_l^{n^1_i} - u_l^{n^1_j})\right|^2 dxdt \le \sum_{j=1}^{N_\varepsilon}\int_0^T\left[\int_\Omega\left\{\partial_{x_k}\left(u_l^{n^1_i} - u_l^{n^1_j}\right)\right\}\omega_j dx\right]^2 dt \\ + \varepsilon\int_0^T\int_\Omega\sum_{m=1}^3\left|\partial^2_{x_k x_m}\left(u_l^{n^1_i} - u_l^{n^1_j}\right)\right|^2 dxdt.\end{aligned} \tag{2.18}$$

Note that Galerkin's approximations $u^{n^1_i}$ satisfy inequality (2.16). Therefore, the last integral in the right-hand side of inequality (2.18) does not exceed a fixed constant multiplied by ε. The first integral in the right-hand side of inequality (2.18) can be considered arbitrarily small for the large values n_i^1, n_j^1 because the sequence $\{u^{n_i^1}\}$ converges in $L_2(\mathbb{Q}_T)$, and hence the sequence

$$\int_\Omega \omega_m(x)\partial_{x_k} u_l^{n_i^1}(x,t)dx = -\int_\Omega \{\partial_{x_k}\omega_m(x)\}\, u_l^{n_i^1}(x,t)dx$$

converges for almost all $t \in [0,T]$. Therefore we obtain

$$\int_0^T \left[\int_\Omega \left\{\partial_{x_k}(u_l^{n_i^1} - u_l^{n_j^1})\right\}\omega_m dx\right]^2 dt \to 0$$

as $n_j^1,\ n_i^1 \to \infty$. Thus, the right-hand side of (2.18) can be considered arbitrarily small for sufficiently large indices $n_i^1,\ n_j^1$. This proves that the sequence $\{u_{x_k}^{n_i^1}\}$ converges strongly in $L_2(\mathbb{Q}_T)$. Passing to subsequences we get the sequences $\{u_{x_k}^{\widetilde{n_i}}\}$, $k = 1,2,3$, converging in $L_2(\mathbb{Q}_T)$. To simplify the notation in what follows for these converging sequences we use the notation $\{u_{x_k}^{n_i}\}$, $k = 1,2,3$.

a) *Now let us prove that the sequence* $\{(u^{n_i}\cdot\nabla)u^{n_i}\}$ *strongly converges in* $L_2(\mathbb{Q}_T)$. With this goal in mind we deduce from the multiplicative inequalities [1], [9] the following inequality

$$\int_\Omega w^2(\partial_{x_i}u)^2 dx \le c\,\|w\|_{1,2}^2\,\left\|\partial_{x_i}u\right\|\,\|u\|_{2,2}\,. \tag{2.19}$$

Then we put $w = u_l^{n_i} - u_l^{n_j}$, $v = u_k^{n_i}$ in the above inequality (2.19) and obtain

$$\begin{aligned}\int_0^T dt\int_\Omega \left|(u_l^{n_i} - u_l^{n_j})\partial_{x_l}u_k^{n_i}\right|^2 dx \le c\left\{\max_{[0,T]}\left\|u_l^{n_i}(\cdot,t)\right\|_{1,2} + \max_{[0,T]}\left\|u_l^{n_j}(\cdot,t)\right\|_{1,2}\right\}^2 \\ \times\int_0^T \left\|(u_l^{n_i} - u_l^{n_j})(\cdot,t)\right\|_{1,2}\times\left\|u_k^{n_i}(\cdot,t)\right\|_{2,2}dt.\end{aligned} \tag{2.20}$$

Due to inequality (2.16) the numbers

$$\sup_{[0,T]}\left\|u_l^{n_i}(\cdot,t)\right\|_{1,2};\ \ \sup_{[0,T]}\left\|u_l^{n_j}(\cdot,t)\right\|_{1,2}$$

are bounded in the interval $[0,T]$ by some constant $C(T)$ uniformly with respect to the indices $n_i,\ n_j,\ l$. Hence, applying the Cauchy inequality to the right-hand side of (2.20), we have

$$\begin{aligned}\int_0^T dt\int_\Omega \left|(u_l^{n_i} - u_l^{n_j})\partial_{x_l}u_k^{n_i}\right|^2 dx \le C(T)\left\{\int_0^T \left\|u_k^{n_i}(\cdot,t)\right\|_{2,2}^2 dt\right\}^{1/2} \\ \times\left\{\int_0^T \left\|(u_l^{n_i} - u_l^{n_j})(\cdot,t)\right\|_{1,2}^2 dt\right\}^{1/2}.\end{aligned} \tag{2.21}$$

By virtue of inequality (2.16), the numbers $\left\{\int_0^T \left\|u_k^{n_i}(\cdot,t)\right\|_{2,2}^2 dt\right\}^{1/2}$ are uniformly bounded by the constants $C(T)$ in the interval $[0,T]$, and it was proved above that

$$\left\{\int_0^T \left\|(u_l^{n_i} - u_l^{n_j})(\cdot,t)\right\|_{1,2}^2 dt\right\} \to 0$$

as $n_i,\ n_j\to\infty$. Therefore, the right-hand side in inequality (2.21) can be considered arbitrarily small as $n_i,\ n_j\to\infty$.

In a similar way we obtain the following inequalities:

$$\begin{aligned}
&\int_0^T dt\int_\Omega \left|u_k^{n_i}\partial_{x_k}(u_l^{n_i}-u_l^{n_j})\right|^2 dx\\
&\le c\max_{[0,T]}\left\|u_k^{n_i}(\cdot,t)\right\|_{1,2}^2\times\left\{\int_0^T \left\|(u_l^{n_i}-u_l^{n_j})(\cdot,t)\right\|_{1,2}\left\|(u_l^{n_i}-u_l^{n_j})(\cdot,t)\right\|_{2,2}dt\right\}\\
&\le C(T)\left\{\int_0^T \left\|(u_l^{n_i}-u_l^{n_j})(\cdot,t)\right\|_{2,2}^2 dt\right\}^{1/2}\left\{\int_0^T \left\|(u_l^{n_i}-u_l^{n_j})(\cdot,t)\right\|_{1,2}^2 dt\right\}^{1/2}\\
&\le C_1(T)\left\{\int_0^T \left\|(u_l^{n_i}-u_l^{n_j})(\cdot,t)\right\|_{1,2}^2 dt\right\}^{1/2}. \qquad (2.22)
\end{aligned}$$

Inequality (2.22) implies the convergence:

$$\left\{\int_0^T dt\int_\Omega \left|u_k^{n_i}\partial_{x_k}(u_l^{n_i}-u_l^{n_j})\right|^2 dx\right\}\to 0 \quad \text{as} \quad n_i,\ n_j\to\infty.$$

Combining inequalities (2.21) and (2.22), we infer that the sequence $\{(u^{n_i}\cdot\nabla)u^{n_i}\}$ strongly converges in $L_2(\mathbb{Q}_T)$ to a function

$$\psi := \lim_{n_j\to\infty}(u^{n_j}\cdot\nabla)u^{n_j}. \qquad (2.23)$$

b) *Here we prove the convergences of the sequence* $\{u^{n_i}\}$ *in the space* $Wr(T)$. From equation (2.15) we derive a Cauchy problem for the function (u^n-u^m),

$$\begin{aligned}
&\partial_t(u^n-u^m)-\nu\Delta(u^n-u^m)+\mu\Delta^2(u^n-u^m)\\
&=-(\mathbf{P}_n-\mathbf{P}_m)(u^n\cdot\nabla)u^n+(\mathbf{P}_n-\mathbf{P}_m)f+\mathbf{P}_m\{(u^m\cdot\nabla)u^m-(u^n\cdot\nabla)u^n\}, \qquad (2.24)\\
&(u^n-u^m)\big|_{t=0}=(\mathbf{P}_n-\mathbf{P}_m)u^0.
\end{aligned}$$

By standard calculations from (2.24) we have the inequality

$$\begin{aligned}
\left\|u^n-u^m\right\|_{Wr(T)}^2 \le{}& c\left\|(\mathbf{P}_n-\mathbf{P}_m)u^0\right\|_{2,2}^2\\
&+c\int_0^T\Big\{\left\|\{(\mathbf{P}_n-\mathbf{P}_m)(u^n\cdot\nabla)u^n\}(\cdot,t)\right\|^2+\|\{(\mathbf{P}_n-\mathbf{P}_m)f\}(\cdot,t)\|^2\\
&+\left\|\{(u^m\cdot\nabla)u^m-(u^n\cdot\nabla)u^n\}(\cdot,t)\right\|^2\Big\}dt. \qquad (2.25)
\end{aligned}$$

Evidently $\left\|(\mathbf{P}_n-\mathbf{P}_m)u^0\right\|_{2,2}^2\to 0$ and $\int_0^T\|\{(\mathbf{P}_n-\mathbf{P}_m)f\}(\cdot,t)\|^2dt\to 0$ as $n,m\to\infty$. Above we proved that $\int_0^T\|\{(u^{n_j}\cdot\nabla)u^{n_j}-(u^{n_i}\cdot\nabla)u^{n_i}\}(\cdot,t)\|^2dt\to 0$ as $i,j\to\infty$. Further note that

$$\begin{aligned}
&\int_0^T\left\|\{(\mathbf{P}_n-\mathbf{P}_m)(u^n\cdot\nabla)u^n\}(\cdot,t)\right\|^2 dt \qquad (2.26)\\
&\le 4\int_0^T\left\|\{(u^{n_j}\cdot\nabla)u^{n_j}-\psi\}(\cdot,t)\right\|^2 dt+4\int_0^T\|\{(\mathbf{P}_n-\mathbf{P}_m)\psi\}(\cdot,t)\|^2dt,
\end{aligned}$$

where $\psi := \lim_{n_j\to\infty}(u^{n_j}\cdot\nabla)u^{n_j}$. Therefore the right-hand side of inequality (2.25) for $n = n_j$, $m = n_i$ tends to zero as $i, j \to \infty$. Hence Galerkin's approximations $\{u^{n_j}\}$ converge in the norm $\|\cdot\|_{Wr(T)}$ to the function

$$u := \lim_{j\to\infty} u^{n_j} \in Wr(T). \tag{2.27}$$

If in inequalities (2.21) and (2.22) we substitute the expressions $(u_l^{n_i} - u_l^{n_j})\partial_{x_l}u_k^{n_i}$ and $u_k^{n_i}\partial_{x_k}(u_l^{n_i} - u_l^{n_j})$ by $(u_l^{n_i} - u_l)\partial_{x_l}u_k^{n_i}$ and $u_k\partial_{x_k}(u_l^{n_i} - u_l)$, respectively, then similarly to part a) of the proof we obtain in the space $L_2(\mathbb{Q}_T)$ the convergence

$$\lim_{n_j\to\infty}(u^{n_j}\cdot\nabla)u^{n_j} = (u\cdot\nabla)u = \psi.$$

Note that linear combinations of the functions a^j, $j = 1,\dots$ with time dependent coefficients $d_j(t)$ are dense in $\overset{\circ}{L}_2(\mathbb{Q}_T)$. Thus, integrating the scalar product of the right-hand and left-hand sides of equality (2.15) with a function $g \in \overset{\circ}{L}_2(\mathbb{Q}_T)$ and passing to limit at $n = n_j \to \infty$ we deduce that function (2.27) satisfies an integral equality

$$\int_0^T \left(\left\{\frac{\partial u}{\partial t} - \nu\Delta u + \mu\Delta^2 u + \mathbf{P}(u\cdot\nabla)u - \mathbf{P}f\right\}(\cdot,t),\ g(\cdot,t)\right)dt = 0 \tag{2.28}$$

for every $g \in \overset{\circ}{L}_2(\mathbb{Q}_T)$. *Evidently, the function u has all properties of the SRNS solution. By* [1, p. 144]*, the SRNS solution is unique.*

c) *Now we define the SRNS in the cylinder Q_∞.* Fix $T_1 > 0$. We proved that for initial data (1.7) there exists a unique SRNS u on $[0, kT_1+\varepsilon]$, $\varepsilon > 0$ and $u \in Wr(kT_1+\varepsilon)$. It follows from the definition of the $Wr(T)$ norm that the mapping

$$[0, kT_1] \ni t \mapsto u(\cdot,t) \in W^{2,2}(\Omega) \tag{2.29}$$

is continuous in t. Hence $u(\cdot, kT_1) \in H(\Omega)\cap W^{2,2}(\Omega)$ and by parts a) and b) of the proof there exists a unique SRNS $\widetilde{u}$ on the interval $[kT_1, (k+1)T_1+\varepsilon]$ with the initial data $u(\cdot, kT_1)$. On the other hand, on the interval $[0, (k+1)T_1+\varepsilon]$ there exists a unique SRNS $\widehat{u}$ with initial data (1.7). Evidently, $\widetilde{u}(\cdot,t) = \widehat{u}(\cdot,t)$ on $[kT_1,\ (k+1)T_1+\varepsilon]$. Thus by induction we continue the SRNS u in the cylinder $\mathbb{Q}_{T_1}$ to the SRNS in the cylinder $\mathbb{Q}_\infty = \Omega\times[0,+\infty)$. Estimates for the norms $\|u\|_{Wr(t)}$, $t \ge 0$, of this global solution u give inequality (2.5).

d) Let us prove that the sequence $\{(u^n\cdot\nabla)u^n\}$ converges in $L_2(\mathbb{Q}_T)$, and hence obtain the convergence of Galerkin's approximations u^n in the space $Wr(T)$ to the SRNS. Note that the sequence $\{u^{n_j}\}$ converges in the norm $\|\cdot\|_{Wr(T)}$ to the unique SRNS u and

$$\int_0^T \left\|\{(u^{n_j}\cdot\nabla)u^{n_j} - (u\cdot\nabla)u\}(\cdot,t)\right\|^2 dt \to 0 \quad \text{as} \quad j\to\infty.$$

Now suppose the opposite, i.e. that the sequence $\{(u^n,\nabla)u^n\}$ does not converge in $L_2(\mathbb{Q}_T)$ to the function $(u\cdot\nabla)u$. Then there exists $\varepsilon_0 > 0$ and such a subsequence $\{\widetilde{n}_q\}$ that

$$\int_0^T \left\|\left\{(u^{\widetilde{n}_q}\cdot\nabla)u^{\widetilde{n}_q} - (u\cdot\nabla)u\right\}(\cdot,t)\right\|^2 dt \ge \varepsilon_0 \ \text{ for all } \ \{\widetilde{n}_q\}.$$

Applying considerations of parts a) and b) we can find a subsequence $\{\widehat{n}_i\} \subset \{\widetilde{n}_q\}$ such that

$$\int_0^T \left\|\left\{(u^{\widehat{n}_i}\cdot\nabla)u^{\widehat{n}_i} - (u\cdot\nabla)u\right\}(\cdot,t)\right\|^2 dt \to 0 \ \text{ as } \ j\to\infty.$$

The obtained contradiction proves that the sequence $\{(u^n\cdot\nabla)u^n\}$ converges in $L_2(\mathbb{Q}_T)$ to the function $(u,\nabla)u$. As all the sequence $\{(u^n\cdot\nabla)u^n\}$ converges in $L_2(\mathbb{Q}_T)$ to the function $(u\cdot\nabla)u$, then it follows from (2.25) that $\|u^n-u\|_{Wr(T)}\to 0$ as $n\to\infty$. $\square$

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