APPLICATIONS OF BANACH IDEALS OF OPERATORS

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For Stacy and Dana

1. **Apologies.** Since much of the recent work in the Banach space aspects of Functional Analysis, especially the geometry of Banach spaces, could, by a bit of chicanery, be construed as applications of Banach ideals, the title does not indicate a complete survey. This work is surely not exhaustive of the subject matter. Thus, many good papers are totally ignored. This is somewhat compensated for by the monograph [1.1] of Lindenstrauss and Tzafriri on the geometry of the classical Banach spaces and the "pre-book" [1.2] of A. Pietsch on the general theory of ideals of operators.

Since a lecture should have a central theme, I have chosen a fundamental result of Grothendieck which asserts that there are Banach spaces E and F for which every bounded linear operator from E to F is 2-absolutely summing. (Definitions will be forthcoming.) This result and the local structure of Banach spaces are the unifying topics of this paper. For the numerous topics this unification omits, again, apologies.

I have addressed myself to the material at hand twice before [1.3], [1.4], [1.5], the latter in collaboration with Y. Gordon and D. R. Lewis. I apologize for mentioning, again, the beautiful result of Stegall and Lewis [1.6] and tramping once again over ground covered in [1.3]–[1.5]. However, I feel, perhaps with prejudice, that these results are worthy of further discussion.

An additional apology of sorts is needed. I have included many definitions which are old hat to experts in Banach space theory. It is hoped that the material will thus be accessible to a larger audience, perhaps even to some persons completely outside Functional Analysis.

Finally, many of the results stated have their natural setting in Probability Theory. I have avoided the probabilistic language entirely. Thus "random variable" becomes "measurable function" etc. This is an editorial judgment on my part, and apologies are extended to anyone this may offend.

A remark on the Bibliography is in order. Instead of listing bibliographical references alphabetically we have listed the results as they appear in the text, e.g. [5.3] means reference [3] of §5. Thus, one interested in the proof of results stated in a given section can go to the source immediately. Due to the length of the paper almost all proofs are omitted; hopefully, this unorthodox bibliography will help compensate for that.

2. **Notation.** Most of our notation is standard. All spaces considered are Banach spaces. The word *operator* means bounded linear transformation. By an *isomorphism* we mean a one-to-one, open operator. The *Banach-Mazur distance*, $d(E, F)$, between Banach spaces E and F, is given by

$$d(E, F) = \inf \|T\| \|T^{-1}\|$$

where the infimum is taken over all isomorphisms between E and F. If E and F are not isomorphic set $d(E, F) = +\infty$.

A *projection* P is an operator from E to E with $P^2 = P$. If A is a *subspace* (=closed linear manifold) then A is complemented in E if there is a projection P with $P(E) = A$. We will denote the identity operator on a Banach space E by id_E.

A sequence (x_n) in a Banach space E is a *Schauder basis* for E (*unconditional Schauder basis* for E) if for each $x \in E$ there is a unique sequence of scalars (a_n) such that $\sum_{n=1}^{\infty} a_n x_n$ converges to x in norm (such that $\sum_{n=1}^{\infty} \varepsilon_n a_n x_n$ converges for all choices of signs $\varepsilon_n = \pm 1$). The functionals f_i, defined by $f_i(x) = a_i$, are called the coefficient functionals of the basis (x_i).

For $1 \leq p \leq \infty$ we denote by l_p the Banach space of scalar sequences (a_i) with

$$\|a\| = \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} \quad \text{if} \quad 1 \leq p < \infty,$$

$$= \sup_i |a_i| \quad \text{if} \quad p = \infty.$$

Similarly, for $l_p(\Gamma)$ where Γ is any discrete set. In particular, we denote by l_p^n the space of n-tuples with the above norm. We denote by c_0 the closed subspace of l_∞ consisting of those sequences which tend to 0. Given p in $[1, \infty)$ we will always denote by p' the number satisfying $1/p + 1/p' = 1$.

If E' denotes the conjugate of a Banach space E then $(l_p)' = l_{p'}$.

We will use tensor notation in §§11, 12, and 14. By $l_p \hat{\otimes} l_q$ we mean the closure of the finite rank operators $T : l_{p'} \to l_q$ in the norm $\inf\{\sum_{i=1}^{n} \|f_i\| \|x_i\| : Tx = \sum_{i=1}^{n} f_i(x)x_i\}$. By $l_p \otimes l_q$ we mean the closure of the same finite rank operators in the usual operator norm. A similar statement holds for $l_p^n \otimes l_q^n$ and $l_p^n \hat{\otimes} l_q^n$.

If (S, Σ, μ) is a measure space then as usual $L_p(S, \Sigma, \mu)$ or $L_p(\mu)$ denotes the Banach space of equivalence classes of almost everywhere equal functions under the norm

$$\|f\|_p = \left(\int_S |f(s)|^p \, \mu(ds) \right)^{1/p} \quad \text{for} \quad 1 \leq p < \infty,$$

$$= \operatorname*{ess\,sup}_{s \in S} |f(s)| \quad \text{for} \quad p = \infty.$$

By a probability measure space (Ω, μ) we mean a positive measure with $\mu(\Omega) = 1$. For μ Lebesgue measure on $[0, 1]$ we will suppress μ (or dt) and write $L_p[0, 1]$.

By $C(K)$ we mean the Banach space of continuous scalar valued functions under the sup norm.

If (E_n) is a sequence of Banach spaces then

$$(\bigoplus E_n)l_p = \left\{ (x_n), x_n \in E_n \,|\, \|(x_n)\|^p = \sum_{n=1}^{\infty} \|x_n\|^p < +\infty \right\}.$$

For a Banach space E let

$$\delta_E(t) = \inf\{1 - \tfrac{1}{2}\|x + y\| : \|x\| = \|y\| = 1, \|x - y\| \geq t > 0\}.$$

The function δ_E is called the *modulus of convexity of E*. If $\delta_E(t) > 0$ for $0 < \varepsilon \leq 2$ then E is said to be *uniformly convex*. The spaces $L_p(\mu)$, $1 < p < \infty$, are uniformly convex.

Let $\lambda \geq 1$ and $1 \leq p \leq \infty$. A Banach space E is an $\mathscr{L}_{p,\lambda}$-space if for each finite dimensional subspace $F \subset E$ there is a finite dimensional subspace B with $F \subset B \subset E$ such that $d(B, l_p^n) \leq \lambda$ where $n = \dim B$. A space E is an $\mathscr{L}_p$-space if it is an $\mathscr{L}_{p,\lambda}$-space for some $\lambda \geq 1$. These spaces, introduced in [2.1], generalize and include the $L_p(S, \Sigma, \mu)$ and $C(K)$-spaces above.

An operator T from E to F is *compact* if the image of the unit ball of E is relatively compact in F.

A space E has the *approximation property* if every compact operator from F to E is the limit of finite rank operators. The space E has the *bounded approximation property* if there is a constant $C > 0$ such that if B is a finite dimensional subspace of E there is an operator on E with finite dimensional range such that $\|T\| \leq C$ and T restricted to B is the identity. If C can be taken to be 1 then E is said to have the *metric approximation property*. A remarkable result of Enflo [2.2] (see also [2.3]) asserts that not every Banach space has the approximation property.

A sequence of subspaces (E_n) in a Banach space E is *uniformly complemented* if there is an $M > 0$ and a sequence of projections (P_n) with $P_n(E) = E_n$ and $\|P_n\| \leq M$ for all positive integers n.

Following [2.4] we will say that a Banach space E is an $\mathscr{S}_{p,\lambda}$-space if for each n E contains subspaces E_n which are uniformly complemented and $d(E_n, l_p^n) \leq \lambda$; and E is a $\mathscr{D}_{p,\lambda}$-space if for each n E contains E_n with $d(E_n, l_p^n) \leq \lambda$. Finally, E is an $\mathscr{S}_p$- or a $\mathscr{D}_p$-space if it is a $\mathscr{D}_{p,\lambda}$- or $\mathscr{F}$-space for some $\lambda \geq 1$. For the relationships between $\mathscr{L}_p^{-1}$-, $\mathscr{S}_p$- and $\mathscr{D}_p$-spaces we refer the reader to [2.4].

3. History. An ideal in the ring $\mathscr{L}(H)$ of all bounded linear operators on a separable infinite dimensional Hilbert space is a subset $A(H)$ with the properties: if $S, S_1, S_2 \in A(H)$ and $R, T \in \mathscr{L}(H)$ then $S_1 + S_2 \in A(H)$ and $RST \in A(H)$.

The oldest ideal known to the author is the ideal $\sigma_2(H)$ of "Hilbert-Schmidt" operators. This ideal originated with the work of D. Hilbert [3.1] and E. Schmidt [3.2].

In considering the question "What operators on Hilbert space have a

trace?", F. J. Murray and J. von Neumann [3.3] found the ideal $\sigma_1(H)$ of "trace class" operators. Later J. von Neumann and R. Schatten [3.4] generalized the Hilbert-Schmidt and trace class operators to the ideals $\sigma_p(H)$ $(0 < p < \infty)$. We will have more to say about these important ideals later.

The final work in the ideal theory in $\mathscr{L}(H)$, in the sense that a "complete" characterization of all two-sided ideals was given, was done by J. W. Calkin [3.5]. In particular, Calkin showed that there is a one-to-one correspondence between the ideals $A(H)$ and the permutation invariant ideals in the ring l_∞ of bounded sequences. (Further results along these lines were obtained by Schatten [3.6] and Gohberg and Kreĭn [3.7]. More recently, the situation on nonseparable Hilbert spaces has been considered by Oosterbrink [3.8] and his colleagues.)

On the other hand, as we will see, to obtain an ideal theory on Banach spaces that is suitable for applications, it is not sufficient to consider only the ring $\mathscr{L}(E)$ of bounded linear operators on a Banach space E. We must consider the space $\mathscr{L}(E, F)$ of bounded linear operators between arbitrary Banach spaces E and F.

Roughly speaking, a subset $\mathscr{A}$ of the class $\mathscr{L}$ of all bounded linear operators between all Banach spaces is an ideal if whenever $S, S_1, S_2 \in \mathscr{A}$ and $R, T \in \mathscr{L}$, then $S_1 + S_2 \in \mathscr{A}$ and $RST \in \mathscr{A}$ (whenever $S_1 + S_2$ and RST are defined). We will give precise definitions later.

The ideal of compact operators was introduced by F. Riesz [3.9] in 1918. This is the first example of a "Banach" ideal known to the author. (Recall that Banach spaces were not introduced until circa 1932!) Still other examples are the ideals of weakly compact operators (S. Kakutani [3.10]), nuclear operators (A. F. Ruston [3.11], A. Grothendieck [3.12]), and the strictly singular operators (T. Kato [3.13]).

Other special classes of operators were considered by numerous authors. However, the general theory of Banach ideals of operators began, I believe, with the fundamental work on tensor products of Schatten [3.14] and Grothendieck [3.12]. However, the nontrivial translation from the language of tensor products to operators on Banach spaces was accomplished in a series of papers by A. Pietsch [3.15]–[3.21] and, in particular, [3.22] which influenced much of the subsequent work in the area.

4. **Diversity of applications.** The applications of the theory of Banach ideals have been numerous but mainly in three directions:

I—Classifying types of locally convex spaces (e.g. Schwartz spaces, nuclear spaces); papers related to applications of type I include [4.1]–[4.4] and the numerous references given in [4.3] and [4.4]. II—Measure theory on Banach spaces (linear stochastic processes); here the work is mainly by L. Schwartz and the French school. Principal works are [4.5]–[4.7]. See also [4.8] and the Seminaire Maurey-Schwartz 1972–1973, 1973–1974. We will have something to say about the important Schwartz duality theorem later. III—The structure theory of Banach spaces; applications of type III will be our

concern in this paper. Appropriate references will be given in the subsequent sections.

5. **Banach ideals.** Throughout the remainder of the paper $\mathcal{L}$ denotes the class of all bounded linear operators between arbitrary Banach spaces and $\mathcal{L}(E, F)$ the set of all such operators between specific Banach spaces E and F. We now define an *ideal* in the sense of A. Pietsch [**5.1**]. We say that a class A of bounded linear operators is an ideal if for each set $A(E, F) = A \cap \mathcal{L}(E, F)$ one has

(a) if $x' \in E'$, $y \in F$ then $x' \otimes y \in A(E, F)$ ($x' \otimes y$ denotes the rank one operator given by $x' \otimes y(x) = \langle x, x' \rangle y$. Clearly, every rank one operator has this form.);

(b) $A(E, F)$ is a linear subset of $\mathcal{L}(E, F)$ for each E and F; and

(c) if $U \in \mathcal{L}(X, E)$, $T \in A(E, F)$, $V \in \mathcal{L}(F, Y)$, then $VTU \in A(X, Y)$.

The finite rank operators $\mathcal{F}$ obviously form the smallest ideal.

A function α on the operators T in an ideal A to the nonnegative real numbers is an *ideal norm* if

(d) $x' \in E'$, $y \in F$ then $\alpha(x' \otimes y) = \|x'\| \|y\|$;

(e) $S, T \in A(E, F)$ then $\alpha(S + T) \leq \alpha(S) + \alpha(T)$; and

(f) if $U \in \mathcal{L}(X, E); T \in A(E, F)$ and $V \in \mathcal{L}(F, Y)$, then

$$\alpha(VTU) \leq \|V\| \alpha(T) \|U\|.$$

An ideal A with norm α, $[A, \alpha]$, is a *Banach ideal* if each *component* $A(E, F)$ is a Banach space under α.

To any linear normed ideal $[A, \alpha]$ one can associate three normed ideals in a more or less natural fashion:

(I) *The dual ideal* $[A', \alpha']$: An operator T is in $A'(E, F)$ if and only if $T' \in A(F', E')$. Here $\alpha'(T) = \alpha(T')$;

(II) *The conjugate ideal* $[A^{\triangle}, \alpha^{\triangle}]$: $A^{\triangle}(E, F)$ is the class of all operators $T \in \mathcal{L}(E, F)$ for which there is a $\rho > 0$ such that for any $L \in \mathcal{F}(F, E)$

$$|\text{trace } LT| \leq \rho \alpha(L).$$

The norm $\alpha^{\triangle}(T)$ is defined by $\inf \rho$, ρ satisfying the above inequality. $[A^{\triangle}, \alpha^{\triangle}]$ is always a Banach ideal.

(III) *The adjoint ideal* $[A^*, \alpha^*]$: $A^*(E, F)$ is the class of all $T \in \mathcal{L}(E, F)$ for which there is a $\rho > 0$ such that for all finite dimensional Banach spaces X, Y and for all $V \in \mathcal{L}(X, E)$, $U \in A(Y, X)$, $W \in \mathcal{L}(F, Y)$,

$$|\text{trace } VUWT| \leq \rho \|W\| \|V\| \alpha(U).$$

Here the norm α^* is also given by $\inf \rho$, ρ satisfying the above inequality. The ideal $[A^*, \alpha^*]$ is also always a Banach ideal.

The ideals $A^{\triangle}$ and A^* are intimately related. Indeed, for any $T \in \mathcal{L}(E, F)$, $\alpha^*(T) \leq \alpha^{\triangle}(T)$ and equality holds if both E and F have the metric approximation property.

We now give a few examples of Banach ideals. These ideals are due to several different authors. Bibliographical information and a table showing the relationships between various ideals is given in [**5.2**].

(1) Let $C(E, F)$ denote the closure of $\mathscr{F}(E, F)$ in $\mathscr{L}(E, F)$, and $K(E, F)$ the compact operators from E to F. Then $[\mathscr{L}, \|\cdot\|]$, $[K, \|\cdot\|]$ and $[C, \|\cdot\|]$ are Banach ideals.

For a finite or denumerably infinite set $\{x_1, \cdots, x_N\}$ in a Banach space E, let

$$\varepsilon_p(\{x_i\}) = \sup\left\{\left(\sum_{i=1}^{N} |\langle x_i, f\rangle|^p\right)^{1/p} : \|f\| = 1\right\}, \quad \text{if} \quad 1 \leqq p < +\infty,$$

$$\varepsilon_\infty(\{x_i\}) = \sup\left\{\sup_{i \leqq N} |\langle x_i, f\rangle| : \|f\| = 1\right\};$$

$$\alpha_p(\{x_i\}) = \left(\sum_{i=1}^{N} \|x_i\|^p\right)^{1/p} \quad \text{if} \quad 1 \leqq p < +\infty,$$

$$\alpha_\infty(\{x_i\}) = \sup_{i \leqq N} \|x_i\|;$$

and

$$\sigma_p(\{x_i\}) = \sup\left\{\left|\sum_{i=1}^{N} \langle x_i, f_i\rangle\right| : \varepsilon_{p'}(\{f_i\}) \leqq 1\right\}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

(2) $[\Pi_p, \pi_p]$ denotes the ideal of p-*absolutely summing* operators: $T \in \Pi_p(E, F)$ if there is a $\rho > 0$ such that $\alpha_p(\{Tx_i\}) \leqq \rho \varepsilon_p(\{x_i\})$, for all finite sets $\{x_1, \cdots, x_N\}$ in E. The norm π_p is given by $\pi_p(T) = \inf \rho$, ρ as above.

The ideal $[\Pi_p, \pi_p]$ will be extensively used throughout the remainder of the paper.

(3) $[D_p, d_p]$ denotes the ideal of *strongly p-summing* operators: $T \in D_p(E, F)$ if there is a $\rho > 0$ such that $\sigma_p[\{Tx_i\}] \leqq \rho \alpha_p(\{x_i\})$ for all finite sets $\{x_1, \cdots, x_N\}$ in E; $d_p(T) = \inf \rho$.

(4) $[I_p, i_p]$ denotes the ideal of p-*integral* operators: $T \in I_p(E, F)$ if there is a probability measure μ and operators $V \in \mathscr{L}(E, L_\infty(\mu))$ and $W \in \mathscr{L}(L_\infty(\mu), F'')$, F'' the bidual of F, such that $WjV = iT$, where j is the canonical injection of $L_\infty(\mu)$ into $L_p(\mu)$ and i the canonical injection of F into F'', i.e.,

$$\begin{array}{ccc} E & \xrightarrow{\ T\ } T & \overset{i}{\lhook\joinrel\longrightarrow} F'' \\ {\scriptstyle V}\Big\downarrow & & \Big\uparrow{\scriptstyle W} \\ L_\infty(\mu) & \xrightarrow{\qquad j \qquad} & L_p(\mu) \end{array}$$

The norm i_p is given by $i_p(T) = \inf\|V\|\,\|W\|$, the infimum taken over all probability measures μ and operators V, W.

(5) $[N_p, \nu_p]$ denotes the ideal of p-*nuclear* operators: $T \in N_p(E, F)$ if T has a representation $T = \sum_{i=1}^{\infty} f_i \otimes y_i$, $f_i \in E'$, $y_i \in F$ and $\alpha_p(\{f_i\}) < +\infty$, and $\varepsilon_{p'}(\{y_i\}) < +\infty$ $(1/p + 1/p' = 1)$. If $p = \infty$ there is the additional requirement that $f_i \to 0$ as $i \to \infty$. The p-nuclear norm is given by $\nu_p(T) = \inf \alpha_p(\{f_i\})\varepsilon_{p'}(\{y_i\})$, where the infimum is taken over all such representations of T.

An operator T in the class N_1 will be called a nuclear operator. This class of course generalizes the "trace class operators" on Hilbert space.

(6) $[C_p, c_p]$ denotes the ideal of operators *factoring compactly through* l_p: $T \in C_p(E, F)$ if there are $A \in C(E, l_p)$, $B \in C(l_p, F)$ with $T = BA$. The norm

c_p is given by $c_p(T) = \inf\|A\|\,\|B\|$, where the infimum is over all such factorizations of T.

(7) $[I_{p,q}, i_{pq}]$, $1 \le q \le p \le \infty$, denotes the ideal of operators *factoring through a diagonal* $B \in \mathcal{L}(L_p(\mu), L_q(\mu))$: $T \in I_{p,q}(E, F)$ if for some positive measure μ there are operators $A \in \mathcal{L}(E, Lp(\mu))$, $B \in \mathcal{L}(L_p(\mu), L_q(\mu))$, where B is of the form $Bf = f \cdot g$ for some fixed $g \in L_r(\mu)$, $1/r = 1/q - 1/p$, and $C \in \mathcal{L}(L_q(\mu), f'')$, such that $iT = CBA$, where i is the canonical injection of F into F''. The norm i_{pq}, is given by $i_{pq}(T) = \inf\|A\|\,\|B\|\,\|C\|$.

Observe that $I_q(E, F) = I_{\infty,q}(E, F)$ with equality of norms.

(8) $[J_{p,q}, j_{pq}]$, $1 \le q \le p \le \infty$, denotes the ideal of operators *factoring through* $D_q \circ \Pi_p$: $T \in J_{p,q}(E, F)$ if iT admits a factorization as follows:

$$E \xrightarrow{\;T\;} F \xrightarrow{\;i\;} F''$$

with U and V through G.

where $U \in \Pi_p(E, G)$ and $V \in D_q(G, F'')$. Here $j_{pq}(T) = \inf \pi_p(U) d_q(V)$, the infimum taken over all U, V, G.

The last ideal we will discuss is a generalization of an ideal introduced by Kwapien.

(9) $[\Gamma_{p,q}, \gamma_{pq}]$ is defined as in $[I_{p,q}, i_{pq}]$, the difference being that B ranges over all members of $\mathcal{L}(L_p(\mu), L_q(\mu))$.

The adjoints, conjugates and duals of these ideals (and several others) are computed in [**5.2**].

We remark that in the ideals (8) and (9) whenever $p = q$ we will index the ideal and its norm by p alone.

6. **Ideal characterizations of $\mathcal{L}_p$-spaces.** We first present some characterizations of $\mathcal{L}_p$-spaces via Banach ideals. We begin with a characterization of $\mathcal{L}_\infty$-spaces.

THEOREM 1 [**6.1**]. *The following assertions are equivalent:*
(a) $\Pi_1(E, F) = I_1(E, F)$ *for all* F; *and*
(b) E *is an* $\mathcal{L}_\infty$-*space.*

Using this result Lewis and Stegall [**6.2**] proved

THEOREM 2. *Let E be a Banach space. Then* $\Pi_1(E, F) = N_1(E, F)$ *for all F if and only if E' is isomorphic to $l_1(\Gamma)$ for suitable Γ.*

There are some beautiful applications of Theorem 2.

APPLICATIONS. (1) If E is a complemented subspace of $L_1[0, 1]$, isomorphic to a subspace of a separable conjugate space, it is isomorphic to l^1. In particular. (2) $L_1[0, 1]$ is not isomorphic to a subspace of a separable conjugate space (Gel'fand [**6.3**], Pelczyński [6.4]. (3) Any separable $\mathcal{L}_1$-space which is isomorphic to a conjugate space is isomorphic to l_1. (We remark that there are many separable $\mathcal{L}_1$-spaces. Indeed let $\Phi_0 : l_1 \to L_1[0, 1]$ be a surjection and let $X_1 = \Phi_0^{-1}(0)$. Let Φ_1 be a surjection from l_1 onto X_1 and

$X_2 = \Phi_1^{-1}(0)$. In general let $X_n = \Phi_{n-1}^{-1}(0)$. Then all of the X_n are separable, nonisomorphic $\mathcal{L}_1$-spaces [6.5].)

We now give an omnibus result which includes results of Cohen [6.6], Holub [6.7], Johnson [6.8], Kwapien [6.9], Lewis [6.10] and Persson [6.11], as well as some new results. This result was first proved in [6.12].

THEOREM 3. *The following are equivalent* $(1 \leqq p \leqq \infty)$:

(a) I_E, *the identity on E, factors through L_p*;

(b) $\Gamma_p(F, E) \supseteq C(F, E)$ *for all F*;

(c) $\Gamma_{p'}(E', F') \supseteq C(E', F')$ *for all F*;

(d) $\Gamma_p(E, E) \supseteq C(E, E)$ *and E has the metric approximation property*;

(e) $\Gamma_p^*(E, F) = I_1(E, F)$ *for all F*;

(f) $\Gamma_p^*(F, E) = I_1(F, E)$ *for all F*;

(g) $\Gamma_p^*(E, E) = I_1(E, E)$ *and E has the metric approximation property*;

(h) *for every Banach space G, and every adjoint operator, if $W' \in \Pi_p(E', G')$ then $W \in I_p(G, E)$*;

(i) *if $V \in \Pi_{p'}(E, G)$ then $V' \in I_{p'}(G', E')$; and*

(j) $\Gamma_p(E, F) \supseteq C(E, F)$ *for all F*.

We mention that some other characterizations of c_0, l_1, $\mathcal{L}_\infty$-spaces and $\mathcal{L}_1$-spaces are given in [6.13].

It would be of considerable interest to know the situation whenever the range space in Theorem 2 is fixed. We conjecture that if $\Pi_1(F, E) = N_1(F, E)$ for all Banach space F then E must be finite dimensional.

7. **More on $\mathcal{L}_p$-spaces.** In [7.1] Grothendieck outlined the theory of tensor products of Banach spaces. This was, in fact, the "beginning" of the theory of ideals of operators on Banach spaces. Indeed, Grothendieck showed the importance of factoring techniques which will be emphasized over and over in this paper.

The crowning achievement of this work of Grothendieck was called by him "the fundamental theorem of tensor products." In terms of matrices this theorem can be stated as follows:

Let $(a_{ij})_{i,j=1}^n$ be a finite matrix of real numbers such that $|\sum_{i,j=1}^n a_{ij} t_i s_j| \leqq 1$ whenever $|t_i| \leqq 1$, $|s_j| \leqq 1$. Then, for every set of unit vectors $(x_i)_{i=1}^n$ and $(y_j)_{j=1}^n$ in a Hilbert space

$$\left| \sum_{i,j} a_{ij}(x_i, y_j) \right| \leqq K_G,$$

where K_G is an absolute constant. Here $(\cdot, \cdot)$ denotes the inner product in the Hilbert space. We remark that the exact value of K_G is not known.

Surprisingly this fundamental paper of Grothendieck lay dormant for many years but was finally taken up again by Lindenstrauss and Pełczyński in 1968 [7.2]. Lindenstrauss and Pełczyński were persuaded to write their paper avoiding the notion of tensor products because "the paper of Grothendieck was quite hard to read and its results were not generally known even to experts in Banach space theory." This remarkable paper of Lindenstrauss and Pełczyński contains the seeds of the application of

Banach ideals of operators. (Earlier [**7.3**] Grothendieck essentially used the theory to obtain the Dvoretzky-Rogers theorem.) Before stating a few of their results let us mention that a new proof of the Grothendieck inequality has recently been given by Maurey [**7.4**]. One of the achievements of the Lindenstrauss-Pełczyński paper was the introduction of the classes of spaces called the $\mathcal{L}_p$-spaces defined in the introduction. The proofs of the theorems below depend heavily on the Grothendieck inequality.

THEOREM 1.   *Let X be an $\mathcal{L}_1$-space and H an $\mathcal{L}_2$-space (=isomorph of a Hilbert space). Then $\mathcal{L}(X, H) = \Pi_1(X, H)$.*

As remarked by Lindenstrauss and Pełczyński it is conceivable that Theorem 1 actually characterizes $\mathcal{L}_1$ and $\mathcal{L}_2$-spaces. Indeed they obtained the following partial converse to Theorem 1.

THEOREM 2.   *Let X and Y be Banach spaces such that X has an unconditional basis and such that $\mathcal{L}(X, Y) = \Pi_1(X, Y)$. Then X is isomorphic to l_1 and Y is isomorphic to a Hilbert space.*

THEOREM 3.   *Let X be an $\mathcal{L}_\infty$-space and Y an $\mathcal{L}_p$-space, $1 \leq p \leq 2$. Then $\mathcal{L}(X, Y) = \Pi_2(X, Y)$.*

Of the numerous applications of these results the following are striking examples.

APPLICATION 1.   Let X be a complemented subspace of an $\mathcal{L}_1(\mathcal{L}_\infty)$-space Y and let (x_i) be a normalized unconditional basis in X. Then the basis (x_i) is equivalent to the unit vector basis (e_i) in $l_1(c_0)$, i.e. the operator T defined by $Tx_i = e_i$ is an isomorphism.

For the next application we consider the complex Banach space $L_1(\mu)$, where μ is Haar measure on $\{z : |z| = 1\}$. Let H_1 be the closure of the polynomials $\sum_{k=0}^{n} a_k z^k$ in $L_1(\mu)$.

PROPOSITION. $\mathcal{L}(H_1, l_2) \neq \Pi_1(H_1, l_2)$.

We thus obtain the classic result of D. J. Newman (see [**7.5**, p. 154]).

APPLICATION 2.   Every isomorphic image of H_1 in an arbitrary $\mathcal{L}_1$-space X is uncomplemented.

Another application is Grothendieck's characterization of Hilbert spaces.

APPLICATION 3.   A Banach space X is isomorphic to a Hilbert space if and only if it is isomorphic to a subspace of an $\mathcal{L}_1$-space and to a quotient of an $\mathcal{L}_\infty$-space.

Of course, if the roles of the $\mathcal{L}_1$- and $\mathcal{L}_\infty$-spaces are interchanged, *every* Banach space meets the requirement.

Finally if $\mathcal{L}(X, Y) = \Pi_1(X, Y)$ there is a bit one can say about X. More precisely,

PROPOSITION.   *If $\mathcal{L}(X, Y) = \Pi_1(X, Y)$ then*
$1°$ $\mathcal{L}(X, l_2) = \Pi_1(X, l_2)$;
$2°$ *if $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent in X then $\sum_{n=1}^{\infty} \|x_n\|^2 < +\infty$; and,*
$3°$ *if Z is any $\mathcal{L}_\infty$-space, $\mathcal{L}(Z, X) = \Pi_2(Z, X)$.*

For a detailed study of this proposition, see [**7.6**].

8. **Schwartz duality theorem.** Following Kwapien [8.1] we present the Schwartz duality theorem without the theory of cylindrical measures and radonifying operators.

We first need to extend the notion of p-absolutely summing operator to include the interval $[0, 1)$. For $0 < p < +\infty$ define a p-absolutely summing operator in the obvious way, i.e., T is p-absolutely summing if there is a C such that for $x_1 \cdots x_n \in E$

$$\sum_{i=1}^{n} \|Tx_i\|^p \leq C \sup_{\|f\|=1} \sum_{i=1}^{n} |\langle x_i, f \rangle|^p.$$

For $p = 0$ we say that T is 0-absolutely summing (and write $T \in \Pi_0(E, F)$, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x_1 \cdots x_n \in E$ and

$$\sup_{\|f\|=1} \sum_{i=1}^{n} \frac{1}{n} \min[1, |\langle x_i, f \rangle|] < \delta,$$

it follows that $\sum_{i=1}^{n} n^{-1} \min[1, \|Tx_i\|] < \varepsilon$. This last definition is a reformulation of Schwartz's definition of a radonifying operator.

Now let μ be a probability measure on a Hausdorff space Ω and $0 \leq p \leq \infty$. A linear operator $\nu : E \to L_p(\Omega, \mu)$ is p-*decomposable* if there is a $\varphi : \Omega \to E'$ such that

(a) for each $x \in E$, $\langle x, \varphi(\cdot) \rangle$ is μ-measurable and equal to $\nu(x)$ μ—a.e.; and

(b) there is $f \in L_p(\Omega, \mu)$ such that $\|\varphi(\omega)\| \leq f(\omega)$ μ—a.e.

THEOREM (THE SCHWARTZ DUALITY THEOREM). *Let E be a Banach space and $0 \leq p < +\infty$. If $\nu : E \to L_p(\Omega, \mu)$ is p-decomposable then ν is p-absolutely summing.*

For $0 \leq q < p < 2$ there is an isomorphic embedding of $L_p[0, 1]$ into $L_q(\Omega, \mu)$. If $p = 2$ the same is true for all $0 \leq q < +\infty$. We denote such an isomorphism by ν_p.

THEOREM 2. *Let $1 < p \leq \infty$ and $T : L_p[0, 1] \to L_2[0, 1]$. The following are equivalent:*

(a) *$\nu_2 T$ is q-decomposable for all $q < \infty$;*
(b) *$\nu_2 T$ is 0-decomposable;*
(c) *T is 0-absolutely summing; and,*
(d) *T is p'-absolutely summing.*

THEOREM 3. *Let either $1 \leq p < 2$ and $0 \leq q < p$ or $p = 2$ and $0 \leq q < +\infty$. Let F be an $\mathcal{L}_p$-space. If $T \in \mathcal{L}(E, F)$ and $T' \in \Pi_q(F', E')$ then $T \in \Pi_0(E, F)$.*

APPLICATION 1. Let F be an $\mathcal{L}_r$-space where $1 < r \leq 2$.
(a) If $0 \leq p \leq 2$ then $\Pi_p(F, E) = \Pi_0(F, E)$;
(b) If $2 \leq p < \infty$ then $\Pi_p(E, F) = \Pi_2(E, F)$.
In particular for an isomorph of a Hilbert space (i.e., an $\mathcal{L}_2$-space), H, an operator is p-absolutely summing for some p, $0 \leq p < \infty$ if and only if it is 2-absolutely summing ($=$Hilbert-Schmidt).

APPLICATION 2.   Let F be an $\mathscr{L}_r$-space, $1<r<\infty$. Then $\amalg_1(F, E)=\amalg_0(F, E)$ for any Banach space E.

APPLICATION 3.   Let $2<p<\infty$. Let E be an $\mathscr{L}_\infty$-space and F an $\mathscr{L}_p$-space. Then

(a) $\mathscr{L}(E, F)=\Pi_q(E, F)$ for $q>p$; but,

(b) $\mathscr{L}(E, F)\neq\Pi_p(E, F)$.

(We mention that Saphar [**8.2**], [**8.3**] has obtained (a) of Application 3 replacing Π_q with I_q.)

APPLICATION 4.   Let F be a subspace of an $\mathscr{L}_1$-space. Then

(a) $\Pi_1(F, E)=\Pi_2(F, E)$; and,

(b) if $p\geqq2$, $\Pi_p(E, F)=\Pi_2(E, F)$ for any Banach space E.

Many authors have considered the question: "When is the adjoint of an operator p-absolutely summing?" The most satisfying answer, we feel, was given by Cohen [**8.4**] since it retains the series interpretations of these operators. Stated in the language of ideals Cohen's result is $\Pi'_p=D_{p'}$ and $D'_p=\Pi_{p'}$.

Aside from the work of Kwapien here and the fundamental result of Schwartz we mention the papers of Garling [**8.5**] and Nielsen [**8.6**].

9. **Banach spaces X for which $\Pi_2(\mathscr{L}_\infty, X)=\mathscr{L}(\mathscr{L}_\infty, X)$.**   The Grothendieck-Lindenstrauss-Pełczyński theorem and the previous results of Kwapien indicate that perhaps a space X for which $\mathscr{L}(E, X)=\Pi_2(E, X)$ for an $\mathscr{L}_\infty$-space E is necessarily a subspace of some $L_1(\mu)$. This problem was considered by Dubinsky, Pełczyński and Rosenthal [**9.1**]. They show that the above conjecture is false. Moreover they give a complete characterization of such spaces whenever X has an unconditional basis. First we remark that if $\mathscr{L}(E, X)=\Pi_2(E, X)$ for some $\mathscr{L}_\infty$-space E then the equality holds for all $\mathscr{L}_\infty$-spaces; thus, the notation $\mathscr{L}(\mathscr{L}_\infty, X)=\Pi_2(\mathscr{L}_\infty, X)$.

EXAMPLE 1.   Let $X=(\bigoplus E_n)_{l_2}$ where each $E_n=l_1$.

(a) $\Pi_2(\mathscr{L}_\infty, X)=\mathscr{L}(\mathscr{L}_\infty, X)$, and

(b) X is not isomorphic to any subspace of an $\mathscr{L}_1$-space.

Actually one can obtain a reflexive space with properties (a) and (b) of Example 1. Indeed let $X=(\bigoplus E_n)_{l_2}$ where $E_n=l_1^n$.

Although there are no known necessary and sufficient conditions on a general space X guaranteeing $\Pi_2(\mathscr{L}_\infty, X)=\mathscr{L}(\mathscr{L}_\infty, X)$, there are rather simple conditions guaranteeing this property for X in the class of spaces having unconditional bases.

THEOREM 1.   *Let X have an unconditional basis (u_n). Then $\mathscr{L}(\mathscr{L}_\infty, X)=\Pi_2(\mathscr{L}_\infty, X)$ if and only if every operator from c_0 into X, which takes the nth unit vector of c_0 into a multiple of u_n for each n (i.e., a diagonal mapping), is 2-absolutely summing.*

It is obvious that if $\mathscr{L}(\mathscr{L}_\infty, X)=\Pi_2(\mathscr{L}_\infty, X)$ the same is true for any subspace of X. However, it is easily seen that the property is not preserved by quotients.

The following result gives a property which preserves the above equality for quotients.

THEOREM 2. *Let X be a Banach space. Suppose that there exists a probability space (Ω, Σ, μ) and a sequence of functions (f_n) in $L_1(\Omega, \Sigma, \mu)$, and a constant $C > 0$ such that*

(a)
$$C^{-1}\left(\sum_{i=1}^{m} |a_i|^2\right)^{1/2} \le \int_\Omega \left|\sum_{i=1}^{m} a_i f_i(\omega)\right| \mu \, d(\omega)$$

for any scalars $a_1, a_2, \cdots, a_m$ $(m = 1, 2, \cdots)$, and

(b)
$$\int_\Omega \left\|\sum_{i=1}^{m} x_i^* f_i(\omega)\right\| \mu \, d(\omega) \le C\left(\sum_{i=1}^{m} \|x_i^*\|^2\right)^{1/2}$$

for any $x_1^*, \cdots, x_m^*$ in X^*.

Then $\Pi_2(\mathscr{L}_\infty, X) = \mathscr{L}(\mathscr{L}_\infty, x)$. Moreover, every quotient space Y of X has the same property (indeed the same (f_n) and C suffice). Consequently $\Pi_2(\mathscr{L}_\infty, Y) = \mathscr{L}(\mathscr{L}_\infty, Y)$ for all such Y.

Recall that a Banach space X is a $\mathscr{D}_1$-space if there is some $K > 1$ and a sequence of subspaces $X_n \subset X$ such that $d(X_n, l_1^n) \le K$ for $n = 1, 2, \cdots$.

THEOREM 3. *If X is a Banach space such that $\Pi_2(\mathscr{L}_\infty, Y) = \mathscr{L}(\mathscr{L}_\infty, Y)$ for every quotient space Y of X, then X is not a $\mathscr{D}_1$-space.*

Say that a Banach space X has the *Orlicz property* if for each unconditionally convergent series $\sum_{m=1}^{\infty} x_m$ in X, $\sum_{m=1}^{\infty} \|x_m\|^2 < +\infty$. (Orlicz [9.2] discovered that for $1 \le p \le 2$, the $L_p(\mu)$-spaces have this property.)

APPLICATION. If $\Pi_2(\mathscr{L}_\infty, X) = \mathscr{L}(\mathscr{L}_\infty, X)$ then X has the Orlicz property. It is not known if the converse is true.

10. Subspaces of L_p. In this section we consider a fundamental paper of H. Rosenthal [10.1], which in turn was motivated by a paper of J. Bretagnelle and D. Dachunha-Castelle [10.2]. Rosenthal's work is a beautiful application of the theory of p-absolutely summing operators.

We begin with a characterization of the π_p-norm of operators on l_∞^n. In what follows, $\{e_1 \cdots e_n\}$ will denote the natural basis of l_∞^n.

LEMMA 1. *Let X be a Banach space, n a positive integer; $K > 0$, $1 < q < \infty$ and $1/q + 1/p = 1$. Let $T \in \mathscr{L}(l_\infty^n, X)$ with $Te_i = x_i$. The following are equivalent:*

(1) $\pi_q(T) \le K$;

(2) *for all integers m and matrices (y_{ij}) $(1 \le j \le n, 1 \le i \le m)$ of scalars we have*

$$\left(\sum_{i=1}^{m} \left\|\sum_{j=1}^{n} y_{ij} x_j\right\|^q\right)^{1/q} \le K \sup_{1 \le j \le n}\left(\sum_{i=1}^{m} |y_{ij}|^q\right)^{1/q}.$$

Using Lemma 1 it is easy to prove the next result.

LEMMA 2. *Let $1 < q < \infty$, $1/p + 1/q = 1$, X a Banach space and $K > 0$. The following are equivalent:*

(1) *for any compact Hausdorff space S*

$$\mathscr{L}(C(S), X) = \Pi_q(C(S), X)$$

and

$$\pi_q(T) \leq K \|T\| \quad \text{for any } T \in \mathcal{L}(C(S), X);$$

(2) *for each n, $x_1 \cdots x_n \in X$ and matrices (y_{ij}) ($1 \leq i \leq n$, $1 \leq j \leq n$) of scalars,*

$$\left(\sum_i \left\|\sum_j y_{ij} x_j\right\|^q\right)^{1/q} \leq K \sup_j \left(\sum_j |y_{ij}|^2\right)^{1/q} \sup_{\|f\|=1} \sum_i |\langle x_i, f\rangle|;$$

(3) *if B is any Banach space and $T \in N_p(X, B)$ with $\nu_p(T) \leq 1$ then $T \in \Pi_1(X, B)$ and $\pi_1(T) \leq 1$.*

Our first theorem is the principal tool for the main results.

THEOREM 1. *Let R be a subspace of $L_1(\mu)$, $1 < p < +\infty$, $1/p + 1/q = 1$, $K < +\infty$. The following are equivalent:*
(1) *for any positive integer n and elements $r_1 \cdots r_n$ of R,*

$$\int \left(\sum |r_i|^p(t)\right)^{1/p} d\mu(t) \leq K \left(\sum \|r_i\|^p\right)^{1/p};$$

(2) *if $J: L^\infty(\mu)$ onto R^* is the natural map (i.e., the adjoint of the inclusion map $R \hookrightarrow L_1(\mu)$), then $\pi_q(J) \leq K$; and*
(3) *there is a nonnegative measurable function ϕ with $\int \phi \, d\mu \leq 1$, so that for all $r \in R$, $r(t) = 0$ for almost all $t \in \{t: \phi(t) = 0\}$ and such that*

$$\left(\int |r|^p(t) \phi^{1-p}(t) \, d\mu(t)\right)^{1/p} \leq K \int |r(t)| \, d\mu(t).$$

Moreover if $1 < p \leq 2$ and any of (1), (2), or (3) hold, then

$$\mathcal{L}(C(S), R^*) = \Pi_q(C(S), R^*)$$

and if $T \in \mathcal{L}(C(S), R^*)$, $\Pi_q(T) \leq K\|T\|$.

Using Theorem 1 we can now define a characteristic $I_p(R)$ for R a subspace of $L_1(\mu)$ and $1 < p < \infty$. Indeed let

$$I_p(R) = \pi_q(J) \quad \text{if } J \text{ is } q\text{-absolutely summing,}$$
$$= +\infty \quad \text{if } J \text{ is not } q\text{-absolutely summing.}$$

Here, again, J is the natural mapping from $L^\infty(\mu)$ onto R^*.

APPLICATIONS AND FACTS. Let R be a subspace of $L_1(\mu)$ and suppose $1 < p \leq p' < \infty$. Then:
(a) $I_p(R) \leq I_{p'}(R)$.
(b) If R is an $\mathcal{L}_2$-space then $I_2(R) < +\infty$; moreover, if $I_p(R) < +\infty$ for some $p \geq 2$, then R is an $\mathcal{L}_2$-space.
(c) If $\tilde{R}$ and R are isomorphic subspaces of $L_1(\mu)$ then $I_p(\tilde{R}) \leq d(R, \tilde{R}) I_p(R)$.
(d) $I_p(l^p) = +\infty$ for all $1 < p < 2$. (This last result was observed by S. Kwapien.)
(e) Given any subinterval A of $[2, \infty)$ containing 2, there is an infinite-dimensional subspace R of $L_1[0, 1]$ such that $A = \{p \geq 2: I_p(R) < +\infty\}$.
(f) Given any subinterval B of $[1, 2]$ containing 1, there is a subspace R of $L_1[0, 1]$ such that $B = \{p \leq 2: R \text{ imbeds in } L_p\}$.
In particular, (e) shows that for $\mathcal{L}_2$-subspaces of $L_1[0, 1]$, whether or not

$I_p(R)<+\infty$ for $p>2$ depends on the manner in which R is imbedded in $L_1[0, 1]$.

THEOREM 2. *Let* $1<p\leqq2$, $1/p+1/q=1$, *and* R *a subspace of* $L^p(\nu)$. *Then either*

(i) *there is a* $p'>p$ *such that* R *imbeds in* $L_{p'}(\nu)$; *or*

(ii) *for any* $\varepsilon>0$ *there is a subspace* Y *of* R, *an invertible operator* T *from* Y *onto* l_p, *and a projection* P *from* $L^p(\nu)$ *onto* Y *such that* $\|T\|\,\|T^{-1}\|\leqq 1+\varepsilon$ *and* $\|P\|\leqq1+\varepsilon$;

(iii) *reflexive quotients of* $C(S)$-*spaces are isomorphic to quotients of* $L_p(\mu)$-*spaces for some* $p<+\infty$. *In particular (Grothendieck) complemented reflexive subspaces of* $C(S)$-*spaces are finite dimensional; and*

(iv) *every subspace of* $L_1[0, 1]$ *contains a subspace with an unconditional basis.*

Our next result is analogous to Theorem 2.

THEOREM 3. *Let* $1<p<2$, *and* X *a subspace of* $L_p(\mu)$. *The following are equivalent:*

(1) $I_p(X)<\infty$;

(2) X *contains no subspace isomorphic to* l_p;

(3) X *imbeds in* $L_{p'}(\mu)$ *for some* $p'>p$;

(4) $\mathscr{L}(X, l_p)=K(X, l_p)$; *and*

(5) $\mathscr{L}(C(S), X^*)=\Pi_q(C(S), X^*)$ *where* $1/p+1/q=1$.

Maurey [**10.3**] has generalized some of the above results as follows: for a Banach space E let $I_E=\{q>0|$ for any F, $\Pi_q(E, F)=\Pi_p(E, F)$ for some $p\in(0, 1)\}$.

THEOREM 4. *For a Banach space* E *and* $q\in[1, 2)$ *the following are equivalent:*

(a) $q\notin I_E$;

(b) *for all* $\varepsilon>0$ *and for all* n *there are operators* U, V *with* $U\in\mathscr{L}(l_{q'}^n, E)$, $V\in\mathscr{L}(E, l_\infty^n)$ *such that* $\|U\|\leqq1+\varepsilon$, $\|V\|=1$ *and* $V\circ U$ *is the inclusion map.*

As a special case we obtain

COROLLARY 1. *For a Banach space* E *the following are equivalent:*

(a) $1\notin I_E$; *and,*

(b) *for each* $\varepsilon>0$ E *is an* $\mathscr{S}_{\infty,1+\varepsilon}$-*space.*

APPLICATION. If E is a Banach space and if there is a $q\geqq1$ such that for any summable sequence (x_n) in E, $\sum\|x_i\|^{q'}<+\infty$ then $(0, q)\subset I_E$.

The converse of this statement is false, e.g. if $q=2$ then for $1\leqq p<2$, $L_{p'}(\mu)$ provides a counterexample.

COROLLARY 2. *For any Banach space* E, I_E *is an open segment or is* $(0, 2]$.

The case $(0, 2)$ can occur, e.g. the Orlicz space $L^\phi([0, 1], dt)$ where $\phi(t)=t^2\log(t+1)$. (See [**10.4**] for the appropriate definitions.)

THEOREM 5. *If E is a Banach space and p, q satisfy $0<p<q$ and if for any F, $\Pi_q(E, F)=\Pi_p(E, F)$ then $q \in I_E$.*

We will continue with results on I_E in the next section.

Recently Nicole Tomczak has communicated still another application of the Rosenthal-Maurey work.

THEOREM 6. *If $1<p<2$ and E and F are Banach spaces such that $\mathcal{L}(E, F)=\Pi_p(E, F)$, i.e., $\langle E, F \rangle$ is p-trivial, then $\mathcal{L}(E, F)=\Pi_0(E, F)$.*

This answers a problem raised in [**10.5**],

11. **The weakly nuclear norm and LUST.** In studying the equality $\mathcal{L}(\mathcal{L}_\infty, X)=\Pi_2(\mathcal{L}_\infty, X)$ Dubinsky, Pełczyński and Rosenthal introduced the important notion of *local unconditional structure* (LUST). This concept is a generalization of the notion of an unconditional basis and has proved quite useful recently in gaining insight into the structure of certain Banach spaces.

First, suppose X is a Banach space with an unconditional basis. The *unconditional basis constant* of X is defined by

$$\chi(X) = \inf_{\{e_i\}} \sup_{\varepsilon_i = \pm 1, x_i} \left\| \sum_i \varepsilon_i x_i e_i \right\| \Big/ \left\| \sum_i x_i e_i \right\|$$

where the supremum is taken over all choices of signs $\varepsilon_i = \pm 1$ with $\varepsilon_i = 1$ for all but finitely many i, and over all vectors $\sum_i x_i e_i$ in E, and the infimum is over all possible unconditional bases $\{e_i\}$ of E.

A Banach space E has LUST if there is a family $\{E_i\}$ of finite dimensional spaces, each with unconditional basis constant 1, and a $C \geq 1$ such that for each finite dimensional subspace $F \subset E$ there is an i and an operator $T: E_i \to E$ such that $T(E_i) \supset F$ and $\|e\| \leq \|Te\| \leq C\|e\|$ for $e \in E_i$, and moreover, for each i there is an operator $S_i: E_i \to E$ with $\|e\| \leq \|S_i e\| \leq C\|e\|$ for $e \in E_i$.

Following Gordon and Lewis [**11.1**] we give a definition closely related to LUST, in terms of a certain Banach ideal.

An operator $U: E \to F$ is *weakly nuclear* if $U = \sum_{n=1}^\infty f_n \otimes y_n$ and the series is unconditionally convergent in the operator norm. The weakly nuclear norm η is defined by

$$\eta(U) = \inf \varepsilon_1 (f_n \otimes y_n)$$

where the infimum is over all possible representations of U.

LEMMA. *For any operator U, $\eta(U)=\inf \|\alpha\| \chi(X) \|\beta\|$ where the infimum is over all factorizations*

where $\alpha \in K(E, X)$, $\beta \in \mathcal{L}(X, F)$ and X has an unconditional basis.

From the lemma, it is clear that if $\dim E = n$ then $\eta(\mathrm{id}_E) \leq \chi(E)$.

DEFINITION. Let E be a Banach space. If $\sup\{\eta(j_F): F \subset E,\ \dim F < +\infty\} < +\infty$, we denote this constant by $\chi_u(E)$. Here j_F denotes the inclusion mapping $F \hookrightarrow E$.

FACTS. (a) $\chi_u(E) \leq \chi(E)$;

(b) if $\dim E < +\infty$, $\chi_u(E) = \eta(\mathrm{id}_E)$;

(c) if $\chi_u(E) = +\infty$ then E is not complemented in a space with an unconditional basis;

(d) if F is a finite dimensional subspace of E and $P : E \to F$ a projection, then $\chi_u(E) \|P\| \geq \chi_u(F)$. Clearly if E has LUST then $\chi_u(E) < +\infty$.

That LUST is a true generalization of the notion of an unconditional basis is immediate from the fact that any $C(K)$-space has LUST but, e.g., $C[0, 1]$ does not have an unconditional basis.

Before considering properties of spaces with LUST, we first give a fundamental result of Gordon and Lewis [**11.1**] which shows that not every Banach space has LUST.

THEOREM 1. $\eta(\mathscr{L}(l_2^n, l_2^n)) \geq (2/3\pi)\sqrt{n}$. *In particular* $\chi(\mathscr{L}(l_2^n, l_2^n)) \geq Kn^{1/2}$, *and since* $\mathscr{L}(l_2^n, l_2^n)$ *is complemented in* $K(l_2, l_2)$, $K(l_2, l_2)$ *does not have LUST.*

We should remark that the sequence $\{\mathscr{L}(l_2^n, l_2^n)\}$ was the first sequence of finite dimensional Banach spaces $\{E_n\}$ shown to have the property that $\chi(E_n) \to \infty$.

There are numerous applications of the theorem. Indeed, this result settled many outstanding problems. We mention only two applications here. Others are given in §12.

APPLICATIONS. (1) $\sigma_p(H)$ does not have LUST if $p \neq 2$; (2) there is a compact, nonweakly nuclear operator on $K(l_2, l_2)$ (problem of Pietsch).

The next result motivates the remainder of this section.

THEOREM 2. *If* $T \in \Pi_1(E, F)$ *then* $\nu_1(T) \leq \pi_1(T)\chi_u(E)$. *Thus, if* E *has LUST*, $\Pi_1(E, \cdot) \subset \Gamma_1(E, \cdot)$.

We now define the class GL. Roughly speaking, a Banach space X is in the class GL if absolutely summing operators from X factor locally through l_1. More precisely we make the following definition.

DEFINITION. A Banach space X is in the class GL if and only if for any $U \in \Pi_1(X, Y)$ and for any finite dimensional subspace X_1, of X there are operators $\alpha : X_1 \to l_1^n$, $\beta : l_1^n \to Y$, $\|\alpha\| = 1$, $\|\beta\| \leq \pi_1(U)$ such that $\beta\alpha = U|_{X_1}$.

THEOREM 3. *If* Y *is a conjugate space and* $X \in GL$ *then* $\Pi_1(X, Y) \subset \Gamma_1(X, Y)$.

Let us first list some spaces belonging to the class GL:

(a) quotients of $C(K)$-spaces and subspaces of L_1-spaces,

(b) all Banach lattices,

(c) any space with LUST.

Aside from the spaces mentioned earlier which are not in the class GL, we remark that recently Pełczyński [**11.2**] has shown that there are "classical" Banach spaces which are not in the class GL.

THEOREM 4. *The disk algebra* $A(D)$, *its dual space* A^*, *and* H^∞ *are not in the class* GL.

Here is an interesting application of this result:

APPLICATION. Let $n \geqq 2$ and let Ω be a domain in C^n. Then $A(D)$ is not isomorphic to $A(\Omega)$.

Recently, D. Lewis [11.3] has obtained the following result which shows the complete picture:

THEOREM 5. Let $[A, \alpha]$ be a Banach ideal such that $A(l_2, l_2)$ is not equivalent to the Hilbert-Schmidt operators. Then $[A(l_2, l_2), \alpha]$ does not have LUST. Indeed, such an ideal is not in the class GL.

We now discuss two papers, one by W. B. Johnson [11.4] and the other by Johnson, T. Figiel and L. Tzafriri [11.5].

We say that Y is *finitely representable* in X provided that for each $\varepsilon > 0$ and each finite dimensional subspace E of Y, there is a subspace F of X for which $d(E, F) \leqq 1 + \varepsilon$; and X is *superreflexive* [11.6], [11.7] if any Banach space which is finitely representable in X is reflexive. It is now known that X is superreflexive if and only if X admits an equivalent uniformly convex norm [11.8].

Our next result is a structural theorem for spaces with LUST.

THEOREM 6. Suppose X has LUST. Then either
(1) X is superreflexive, or
(2) $X \supset E_n$ with $d(E_n, l_\infty^n) \to 1$, or
(3) $X \supset E_n$ with $d(E_n, l_1^n) \to 1$ and E_n is $(1 + 1/n)$-complemented in X.

To again emphasize the relationships between LUST and unconditional bases we mention a result of Tzafriri [11.9]: If X has an unconditional basis then X is an $\mathscr{S}_p$-space for some $p = 1, 2$, or ∞.

There is an application of Theorem 6 to ideals.

APPLICATION. Let X and Y be Banach spaces and suppose that X is a subspace of a space with LUST which contains no sequence (E_n) with $d(E_n, l_\infty^n) \to 1$. Then

$$K(X, Y) \neq N(X, Y) \quad \text{and} \quad K(Y, X) \neq N_1(Y, X).$$

It is possible to characterize LUST using Banach lattices.

THEOREM 7. A Banach space X has LUST if and only if there is a Banach lattice L, a $\lambda < +\infty$, and a subspace Y of L, $d(Y, X) < +\infty$ such that for each finite dimensional subspace E of L there is an operator $T = T_E : E \to Y$ for which $T|_{E \cap Y} = \mathrm{id}_{E \cap Y}$ and $\|T\| \|T^{-1}\| \leqq \lambda$.

Thus,

COROLLARY 1. A Banach space X has $\chi_u(X) < +\infty$ if and only if X^{**} is complemented in a Banach lattice. Also X has LUST if and only if X^{**} has LUST.

COROLLARY 2. If X is complemented in a Banach lattice L and X is a $\mathscr{D}_\infty$-space $(= \mathscr{S}_\infty$-space) then X has LUST.

We next recall that every subspace of $L_1[0, 1]$ contains a subspace with an

unconditional basis. The next result shows that this remains true for certain Banach lattices. (See [**11.10**] for appropriate definitions.)

THEOREM 8. *Let L be a σ-complete and σ-order continuous Banach lattice. Then every subspace of L contains a subspace with an unconditional basis.*

COROLLARY 3. *If X has LUST and X is not a $\mathfrak{D}_\infty$-space, then every subspace of X contains a subspace with an unconditional basis.*

Finally, from [**11.11**] we obtain the following striking result:

THEOREM 9. *If X is a subspace of a Banach lattice which is not a $\mathfrak{D}_\infty$-space, then X contains uniformly complemented E_n with* $\sup d(E_n, l_1^n) < +\infty$ *or* $\sup d(E_n, l_2^n) < +\infty$.
Thus if X has LUST, X is either an $\mathscr{S}_1$-, $\mathscr{S}_2$- or $\mathscr{S}_\infty$-space.

It is conjectured that any Banach space is an $\mathscr{S}_1$-, $\mathscr{S}_2$- or $\mathscr{S}_\infty$-space!

12. **Parameters.** In studying the geometry of Banach spaces, numerous parameters have been introduced. Perhaps the "projection constant" is the best known and most often studied among these parameters. Ideal norms provide a unified approach for attaching parameters to a Banach space. Indeed, if $[A, \alpha]$ is a Banach ideal and E a Banach space, define $\alpha(E) = \alpha(\mathrm{id}_E)$.

Some of the most beautiful work done in recent years, utilizing ideal parameters to the fullest, has been done by Y. Gordon [**12.1**], [**12.2**], Gordon and Garling [**12.3**] and the superb papers of Gordon and Lewis [**12.4**], [**12.5**]. See also the related papers [**12.6**] of A. Pietsch.

In this section we give a few of the numerous results.

We first recall some definitions of asymmetry. For a study of the following concepts see, in addition to the papers above, [**12.7**].

Let E be a Banach space and G a multiplicative group of operators in $\mathscr{L}(E, E)$ and let $G' = \{t \in \mathscr{L}(E, E) : tg = gt, \ g \in G\}$, $G_E = \{J \in \mathscr{L}(E, E) : J$ an isometry}. We say E *has enough symmetries* if G_E' consists only of the scalar multiples of id_E.

The *asymmetry constant* $S(E)$ is defined to be the infimum of $\lambda > 0$ for which there is a group $G \subset \mathscr{L}(E, E)$ of invertible, onto operators such that

$$\sup\{\|g\| : g \in G\} \le \lambda \quad \text{and} \quad G' = \{\alpha \, \mathrm{id}_E\}.$$

It can be shown that, in fact,

$$S(E) = \inf\{d(E, F); \ F \text{ has enough symmetries}\}.$$

THEOREM 1. *If* $\dim E = n$ *and* $[A, \alpha]$ *is any Banach ideal then*

$$n \le \alpha(E)\alpha^\Delta(E) \le n(S(E))^2.$$

In particular if E has enough symmetries $\alpha(E)\alpha^\Delta(E) = n$.

Theorem 1 was generalized in [**12.8**]. Without going into technical details we state a special case of this generalization.

APPLICATION. For each injection map $I: l_p^n \to l_q^n$, $\alpha^\Delta(I)\alpha(I^{-1})=n$. Here p, q are arbitrary and α is any ideal norm.

If E has a basis $B=\{e_i\}$, and σ is a finite permutation of the integers, define $\sigma \in \mathcal{L}(E,E)$ by $\sigma(e_i)=e_{\sigma(i)}$. The *diagonal asymmetry constant* $\delta(B)$ is defined as $\sup_\sigma \|\sigma\|$ and the *diagonal asymmetry constant* of E, $\delta(E)$, is defined as $\inf\{\delta(B): B$ a basis for $E\}$.

If $\varepsilon=(\varepsilon_i)$ is a sequence of ± 1, with $\varepsilon_i=1$ except for a finite number of i then $g_\varepsilon \in \mathcal{L}(E,E)$ is defined by $g_\varepsilon(e_i)=\varepsilon_i e_i$ and the *coordinate asymmetry constant* $\chi(B)$ is given by $\sup_\varepsilon \|g_\varepsilon\|$. Thus $\chi(E)=\inf\{\chi(B): B$ a basis for $E\}$ is the unconditional basis constant of E defined in §11.

THEOREM 2. *If* $\dim E = n$, $[A, \alpha]$ *a Banach ideal, then*

$$n \leq \alpha(E)\alpha^\Delta(E) \leq 3n(\delta(E))^3.$$

A similar result for the parameter χ is a bit more difficult to state.

THEOREM 3. *Let* $\dim E = n$, $B=\{e_i\}_{i=1}^n$ *a basis for E and* $[A, \alpha]$ *a Banach ideal. For* $J \subset \{1, \cdots, n\}$ *let* $E_J = [e_i : i \in J]$ *and* $a_j = \min\{\max(\alpha^\Delta(E_I): I \subset J),$ *card* $J = j\}$, $j = 1, \cdots, n$. *Then*

$$\alpha(E) \leq \chi(B)^3 \sum_{i=1}^n a_j^{-1}.$$

APPLICATION. Let $E \subset l_p$ $(1 \leq p \leq 2)$, $\dim E = n$. Let $\mu(E)=\min\{S(E)^2, 3(\delta(E))^3, 2(\chi(E))^3\}$. Then $K_G\mu(E)\sqrt{n} \geq \pi_1(E) \geq \sqrt{n}$ where K_G is the Grothendieck constant. If $2<q<s<\infty$ and $E \subset l_q$, $\dim E = n$, then there is a constant $C_{q,s}$ such that

$$\sqrt{n} \leq \Pi_1(E) \leq C_{q,s}\mu(E)n^{1-1/s}.$$

From these facts we obtain the result that for

$$E_n = l_p^n \oplus l_q^n, \quad 1 \leq p \neq q \leq \infty, \quad \delta(E_n) \to \infty.$$

We now give some slightly deeper results alluded to in our section on LUST.

THEOREM 4. *Let* J_n *(respectively, I_n) be the natural inclusion of* $l_2^n \hat{\otimes} l_2^n$ *(respectively, $l_2^n \check{\otimes} l_2^n$) to* $l_2^n \otimes^\sigma l_2^n$ $(\equiv l_2^{n^2})$. *Then* (1) $\nu_1(J_n) \sim n$ *and* $\pi_1(J_n) \sim n^{1/2}$; *and,* (2) $\nu_1(I_n) \sim n^{3/2}$ *and* $\pi_1(I_n) \sim n$ *(for two sequences (a_n) and (b_n) with both (a_n/b_n) and (b_n/a_n) bounded, we write $(a_n) \sim (b_n)$).*

THEOREM 5. *Let* $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$. *Then* $\chi(l_q^n \hat{\otimes} l_{p'}^n) = \chi(l_p^n \check{\otimes} l_q^n) \sim n^{1/2}$ *if* $\infty \geq q$, $p \geq 2$; $n^{1-1/q}$ *if* $p \geq 2 \geq q \geq 1$; $n^{1-1/p}$ *if* $q \geq 2 \geq p \geq 1$; $n^{3/2-1/p-1/q}$ *if* $2 \geq p$, $q \geq 1$.

APPLICATIONS. (1) $\chi(l_p^n \hat{\otimes} l_q^n) \to \infty$;

(2) $l_p \hat{\otimes} l_q$ and $l_q \hat{\otimes} l_{p'}$ are not isomorphic to a complemented subspace of a space with unconditional basis (problem of Kwapien-Pełczyński);

(3) the ideal Π_1 is not contained in the ideal Γ_1 (problem of Grothendieck);

(4) there is an operator T between certain Banach spaces E and F such that $T' \in \Pi_1(F', E')$ and $T \notin I_\infty(E, F)$ (problem of Grothendieck); and

(5) there is an operator of the form UV with U', $V \in \Pi_1$ yet $UV \notin I_1$ (problem of Grothendieck).

Concerning the weakly nuclear norm, D. R. Lewis [12.12] has recently obtained the following result.

THEOREM 5.   *Every finite dimensional space E satisfies $\eta(E) \leq 3\delta(E)$.*

From this result it follows that there is a sequence of finite dimensional spaces whose asymmetry constants are one but whose diagonal symmetry constants tend to ∞.

APPLICATION.   For each n, $n^{1/2}/18 \leq \delta(\mathcal{L}(l_2^n, l_2^n)) \leq n^{1/2}$.

We now discuss projection constants. The projection constant $\lambda(E)$ is defined to be the infimum of the numbers λ such that for every Banach space $Y \supset E$ there is a linear projection $P: Y \to E$, $\|P\| \leq \lambda$. In the language of ideals $\lambda(E) = i_\infty(E)$. Let $c_n = \max\{\lambda(E): \dim E = n\}$. It is known that $c_n \leq \sqrt{n}$ [12.10].

Using ideal theory (which we suppress) it is possible to construct spaces with the largest known projection constant: Let $E(\alpha)$, $1 < \alpha < n$, be the n-dimensional space of points $x = (x_1 \cdots x_n)$ with

$$\|x\| = \max\left\{ \max_{1 \leq i \leq n} |x_i|, \ \alpha^{-1} \sum_1^n |x_i| \right\}.$$

THEOREM   6. $\lambda(E(\sqrt{n})) = (n - \lambda(l_1^n))/(2\sqrt{n} - \lambda(l_1^n) - 1)$.   In   particular $\lim_{n \to \infty} \lambda(E(\sqrt{n}))/\sqrt{n} = (2 - \sqrt{2/\pi})^{-1}$.

It turns out that $c_2 \in [\frac{4}{3}, \sqrt{2})$. Actually Gordon [12.2] shows that $c_2 < 1.414211$.

We now turn to the ideals $I_{p,q}$ and $J_{p,q}$. First we observe that when $p = q$, $I_{p,q} = \Gamma_p$ and $i_{pp} = \nu_p$.

THEOREM 7.   (a) *If $1 \leq q \leq p \leq \infty$ then $j_{pq}(l_2^n) = \pi_p(l_2^n)\pi_{q'}(l_2^n)$;*

(b) *if $1 \leq q \leq p \leq \infty$ then $i_{pq}(l_2^n)\pi_p(l_2^n)\pi_{q'}(l_2^n) = n$;*

(c) *if $1 < q \leq p < +\infty$ and H is an infinite dimensional Hilbert space then*

$$i_{pq}(H) = 2\pi^{-1/2p - 1/2q'}\Gamma\left(\frac{p+1}{2}\right)^{1/p}\Gamma\left(\frac{q'+1}{2}\right)^{1/p'}.$$

It turns out that $\nu_p(H) = i_{pp}(H)$ is the relative projection constant of Hilbert space embedded isometrically in the L_p-spaces.

Lacunary sequences in the L_p-spaces have been studied extensively. Indeed for any probability measure space (Ω, Σ, μ) if E is a closed subspace of $L_p(\mu)$ ($2 \leq p < \infty$) isomorphic to a Hilbert space, then there is a constant C_E such that $\|f\|_p \leq C_E\|f\|_2$ for $f \in E$. Thus the following result is of some importance:

THEOREM 8.   (a) *Let $2 \leq p \leq \infty$; then*

$$\min_{\mu, H} \max_{f \in H} \|f\|_p/\|f\|_2 = i_{p2}(l_2^n)$$

where the minimum is taken over all subspaces $H \subset L_p(\mu)$ *isometric to* l_2^n *and probability measures* μ.

(b) *Let* $1 \leqq p \leqq 2$; *then*

$$\min_{\mu, H} \max_{f \in H} \|f\|_2 / \|f\|_p = \pi_p(l_2^n) / n^{1/2}$$

where the minimum ranges over all subspaces $H \subset L_p(\mu)$ *isometric to* l_2^n *complemented with* $\nu_p(l_2^n)$ *norm projections, and probability measures* μ.

In particular if $\infty > p \geqq 2$, *one obtains*

$$i_{p2}(l_2) = 2^{1/2} \left[\Gamma\left(\frac{p+1}{2}\right) \Big/ \pi^{1/2} \right]^{1/p} = \inf_{\mu, H} \sup_{f \in H} \frac{\|f\|_p}{\|f\|_2},$$

and for $1 \leqq p \leqq 2$,

$$\inf_{\mu, H} \sup_{f \in H} \frac{\|f\|_2}{\|f\|_p} = 2^{-1/2} \left(\pi^{1/2} \Big/ \Gamma\left(\frac{p+1}{2}\right) \right)^{1/p}.$$

See [**12.11**] for proofs of the above.

Theorem 5 admits a further application to projection constants:

Let E be an n-dimensional Banach space. There is a constant λ such that $d(E, l_1^n) \leqq \lambda i_\infty(E') \delta(E)^3$.

For a proof of this fact, see [**12.12**].

Finally we mention yet another result of Lewis [**12.12**].

THEOREM 9. *For each odd integer* $n \geqq 5$, *there is an* n-*dimensional space* E_n *with* $\chi(E_n) > 1$ *and* $\delta(E_n) = \eta(E_n) = 1$.

We end this section with some further remarks on ideal parameters on Hilbert space. For proofs of these results, see [**12.13**]. Above we gave the values of $i_{pq}(l_2^n)$ and $i_{pq}(l_2)$. To present further results of this nature we introduce the following ideas. For α an ideal norm and $T : E \to F$, let $/\alpha(T) = \alpha(T\phi)$ where $\phi : L_1(\mu) \to E$ is a surjection. The norm $/\alpha$ is called the *left injective envelop of* α. Similarly the *right injective envelop of* α, $\alpha\backslash$, is defined by $\alpha\backslash(T) = \alpha(JT)$, where $J : F \to L_\infty(\mu)$ is any isometric embedding. The *injective envelop of* α is $/\alpha\backslash = /(\alpha\backslash) = (/\alpha)\backslash$. Analogously, the *left projective envelop of* α, $\backslash\alpha$, is defined by $\backslash\alpha(T) = \alpha(U)$ where $j_F : F \to F''$ is the canonical map, $UT = j_F T$ and $J : E \to L_\infty(\mu)$ is any isometric imbedding. Similarly, the *right projective envelop of* α, $\alpha/$, is defined by $\alpha/(T) = \alpha(W)$ where $j_F T = \phi''W$ and $\phi : L_1(\mu) \to F$ is any surjection. The *projective envelop of* α is $\backslash\alpha/ = (\backslash\alpha)/ = \backslash(\alpha/)$.

For $1 \leqq q \leqq p \leqq \infty$, $T : E \to F$, $S : F \to G$,

$$\pi_{p'}((ST)') \leqq /i_{pq}(T)\pi_{q'}(S) \quad \text{and} \quad \pi_1((ST)') \leqq /\pi_p(T)_{p'}^i(S).$$

Using this result one can prove

THEOREM 10. *Let* $1 \leqq q \leqq p \leqq \infty$ *and* H *be a real Hilbert space.*

(a) *If* $\dim(H) = n$,

$$/i_{pq}(l_2^n) = \pi_{p'}(l_2^n)\pi_{q'}(l_2^n)^{-1}, \quad /i_q(l_2^n) = \pi_1(l_2^n)\pi_{q'}(l_2^n)^{-1},$$

and
$$/\pi_p(l_2^n) \gneqq n^{-1}\pi_1(l_2^n)\pi_p(l_2^n).$$

(b) *If* $\dim H = \infty$ *and* $q > 1$, *then*

$$/i_{pq}(H) = \pi^{(p-q)/2pq}\Gamma\left(\frac{p'+1}{2}\right)^{-1/p'}\Gamma\left(\frac{q'+1}{2}\right)^{1/q'}$$

and

$$/i_q(H) = \pi^{1/2q}\Gamma\left(\frac{a'+1}{2}\right)^{1/q'};$$

(c) *If* $\dim H = +\infty$,

$$/\pi_p(H) \gneqq 2^{-1}\pi^{(p+1)/2p}\Gamma\left(\frac{p+1}{2}\right)^{-1/p}.$$

We remark that geometrically $/i_{q'p'}(l_2)$ is a lower bound for the ratio of the L_p-norm and L_q-norm on infinite dimensional subspaces of L_q ($q \leq p < \infty$).

Moreover, the constant $/\pi_1(H)$, $\dim H = +\infty$, is just the Grothendieck constant K_G. While the exact value of this constant is still not known, $/\pi_p(H)$ can be calculated for values of $p > 1$.

FACTS.   (1) For any $n \geq 1$, $/\pi_2(l_2^n) = \pi_1(l_2^n)n^{-1/2}$ and $/\pi_2(l_2) = \sqrt{\pi/2}$.

(2) (Grothendieck). For any $n \geq 1$,

$$/\gamma_2^*\backslash(l_2^n) = (\pi_1(l_2^n))^2 n^{-1} \quad \text{and} \quad /\gamma_2^*\backslash(l_2) = \pi/2.$$

Also,

$$\backslash\gamma_2/(l_2^n) = n(/\gamma_2^*\backslash(l_2^n))^{-1}.$$

(3) Let H_1 and H_2 be two infinite dimensional Hilbert spaces, $1 \leq q \leq p < \infty$ and $T \in \mathcal{L}(H_1, H_2)$. Then $\pi_q(T) \leq /i_{q'p'}(l_2)\pi_p(T)$ and the constant $/i_{q'p'}(l_2)$ is the best possible. In particular ([**12.14**] and [**12.15**]), $\pi_1(H_1, H_2) = \pi_p(H_1, H_2)$ for all $p \geq 1$. (See Application 1, §8.)

13. **Rademacher averages, type, cotype and stability.**   Let (r_j) denote the Rademacher functions on $[0, 1]$, i.e., $r_j(t) = \text{sign} \sin 2^{j-1} 2\pi t$. One form of the classical Khinchine inequality asserts that for $p \in [1, \infty)$ there are constants A_p, B_p such that

$$B_p\left(\int_0^1\left|\sum_{j=1}^n a_jr_j(t)\right|^2 dt\right)^{1/2} \leq \left(\int_0^1\left|\sum_{j=1}^n a_jr_j(t)\right|^p dt\right)^{1/p}$$

$$\leq A_p\left(\int_0^1\left|\sum_{j=1}^n a_jr_j(t)\right|^2 dt\right)^{1/2}$$

for scalars $a_1 \cdots a_n$.

By a result of Kahane [**13.1**] the above inequalities remain true for suitable constants A_p, B_p if the scalars are replaced by elements in an arbitrary Banach space E (and absolute values, of course, replaced by norms).

DEFINITION.   If $(x_1, \cdots, x_n)$ is a finite set of elements in a Banach space E then the expression $(\int\|\sum x_ir_i(t)\|^p dt)^{1/p}$ is called the pth *Rademacher average* of the sequence $(x_1, \cdots, x_n)$.

We now present some fundamental work of Maurey [**13.1**]–[**13.5**] and

Pisier [**13.6**]–[**13.9**]. (Let us mention that in this section we have been influenced by the lecture [**13.10**] of Pełczyński and the paper of Krivine [**13.11**].)

For $p \in (0, 2]$ let

$$p^* = p \quad \text{if} \quad p < 2,$$
$$= \infty \quad \text{if} \quad p = 2.$$

An operator $T \in \mathcal{L}(E, F)$ is said to be *type p-Rademacher* if there is an $r \in (0, \infty)$ and $C > 0$ such that

$$\left[\int \int \left\| \sum_{n=1}^{k} T(x_n) r_n(t) \right\|^r dt \right]^{1/r} \leq C \left(\sum_{n=1}^{k} \|x_n\|^p \right)^{1/p}.$$

An operator is said to be of *type p-stable* if there is a $C > 0$ and $r \in (0, p^*)$ such that

$$\left[\int \int \left\| \sum f_n(t) T(x_n) \right\|^r dt \right]^{1/r} \leq C \left(\sum \|x_n\|^p \right)^{1/p}.$$

Here (f_n) is a stable sequence of order p on $[0, 1]$ i.e., there are $C_n > 0$ and $p \in (0, 2]$ such that $\int \exp(if_n(t)) \, dt = \exp(-c_n |\lambda|^{1/p})$. A space E is said to be of *type p-Rademacher* (*type p-stable*) if the identity mapping on E is of type p-Rademacher (resp. type p-stable). It is clear what this says about Rademacher averages and the Khinchine inequality. If $p \in [2, \infty)$ then E is said to be of *cotype p-Rademacher* if $(\sum_{i=1}^{n} \|x_i\|^p)^{1/p}$ is dominated by some constant multiple of the 2-Rademacher average of the $x_1 \cdots x_n$. We remark that Pełczyński [**13.10**] calls spaces of type 2 (cotype 2) spaces with sub-quadratic (superquadratic) Rademacher averages. We could use functions other than the Rademacher functions (e.g. the Gaussian averages) but we choose not to do this. Thus we will say type p for type p-Rademacher; similarly for cotype p.

Finally a Banach space E is of *infratype p*, $p \in (0, 2]$, if there is a constant C such that for all $x_n \in E$

$$\inf_{\varepsilon_n = \pm 1} \left\| \sum \varepsilon_n x_n \right\| \leq C \left(\sum \|x_n\|^p \right)^{1/p}.$$

Some remarks relating these concepts are in order.

(i) If T is of type p then T is of type q for any $q \leq p$.

(ii) Subspaces and quotients of spaces of type p (resp. type p-stable) are of type p (type p-stable).

(iii) Subspaces of spaces of cotype p are of cotype p, but cotype, in general, is not preserved by quotients.

(iv) $L_p(\mu)$ is of cotype 2 for $1 \leq p \leq 2$ and any measure μ.

(v) If $p \in [1, 2]$, $L_p(\mu)$ is q-stable for $q \in (0, p)$. If $p \in [2, \infty)$, $L_p(\mu)$ is q-stable for $q \in [0, 2]$.

(vi) If $p \in (0, 2]$ and E is p-stable then E is q-stable for $q \in (0, p]$.

(vii) If $p \in (1, 2]$ and E is p-stable then E is of type p; the converse is false. Indeed l_p is of type p but not p-stable.

(viii) If E is of type p, $p \in (0, 2]$, then E is q-stable for $q < p$. In particular, for $p \in (0, 2]$, $L_p(\mu)$ is q-stable for $q \in (0, p)$ and if $p \in [2, \infty)$, $L_p(\mu)$ is q-stable for $q \in [0, 2]$.

(ix) If $p \in (0, 2]$ and E is p-stable then E is q-stable for $q \in (0, p]$.

(x) If E is of type p then E is of infratype p.

(xi) If E is of type p then E' is of cotype p'. However, l_1 is of cotype 2 and l^∞ is not of type p for any $p > 1$.

(xii) If E is 1-stable then there is a $p \in [2, \infty)$ such that both E and E' are of cotype p.

We should mention that there are nontrivial examples of the various types of spaces discussed above. Part 2 below is due to N. Tomczak-Jaegermann [**13.12**]. For part 1, see [**13.13**].

THEOREM 1.   (1) *If* $p \in [1, 2]$ *every subspace of an* $\mathscr{L}_p$-*space is of cotype* 2; *if* $p \in [2, \infty)$ *every subspace and quotient space of an* $\mathscr{L}_p$-*space is of type* 2.

(2) *If* $1 \leq p \leq 2$, *then* $\sigma_p(H)$ *is of cotype* 2; *and, if* $2 \leq p < \infty$, $\sigma_p(H)$ *is of type* 2.

Of course, as we have seen, the spaces $\sigma_p(H)$, $p \neq 2$, lack LUST.

We now consider the relationships of these concepts to ideal theory.

THEOREM 2.   (a) *If* E *is of cotype* 2 *then* $2 \in I_E$. *In particular if* E *is an* $\mathscr{L}_{1,\lambda}$-*space there is a constant* C_p *such that* $\pi_p(T) \leq \lambda C_p \pi_2(T)$ *for any* $T \in \mathscr{L}(E, F)$. *If* E *is an* $\mathscr{L}_{\infty,\lambda}$-*space and* $1 \leq p \leq 2$, $\mathscr{L}(E, L_p(\mu)) \subset \Gamma_2(E, L_p(\mu))$.

(b) *If* F *is of cotype* 2 *and* $T \in \Pi_q(E, F)$ *for some* $q < \infty$ *then* $T \in \Pi_2(E, F)$.

A Banach space E has the *Grothendieck property* [**13.10**] if $\mathscr{L}(c_0, E) = \pi_2(c_0, E)$. Pełczyński has conjectured that a Banach space E is of cotype 2 if and only if E has the Grothendieck property. The following results are known concerning this conjecture.

THEOREM 3.   *If* E *is of cotype* 2 *then* E *has the Grothendieck property.*

The proof of this result rests on a profound result due to Maurey [**13.14**] concerning certain ideals.

LEMMA.   *If* X *is of type* 2 *and* Y *of cotype* 2 *then* $\mathscr{L}(X, Y) \subset \Gamma_2(X, Y)$.

For a wide class of spaces the conjecture is proved.

THEOREM 4.   *If* X *has LUST then* X *is of cotype* 2 *if and only if* X *has the Grothendieck property.*

THEOREM 5.   *If* X *has LUST then* X *is of type* 2 *if and only if* X' *has the Grothendieck property and* X' *is not a* $\mathscr{D}_1$-*space.*

Another result related to ideals is the following

PROPOSITION.   *Let* $p \in (0, 1)$. *Then* E *is 1-stable if and only if there is a* $C > 0$ *such that for any quotient* G *of* E' *and any* $T \in \mathscr{L}(G, F)$, F *arbitrary, one has* $\pi_p(T) \leq C \pi_1(T)$.

In this connection we remark that if $p \in [1, 2)$ and E is of type p-stable,

then E is of type $p+\varepsilon$-stable for some ε in $(0, 2-p)$. Since a subspace E of L_p is of type p-stable if and only if $l_p \not\subset E$ the above result generalizes the theorem of Rosenthal given in §9.

B-CONVEXITY. In this subsection we consider the work of Pisier [13.8]. A Banach space E is B-(k, ε)-convex (for ε in $[0, 1]$) provided for all $x_1 \cdots x_n$ in E

$$\inf_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{k} \varepsilon_i x_i \right\| \leq k(1 - \varepsilon) \sup \|x_i\|.$$

Then E is B-convex if there is a k and $\varepsilon \in [0, 1]$ such that E is B-(k, ε)-convex. Let

$$\lambda_k(E) = \inf\left\{ \lambda : \inf_{\varepsilon_i = \pm 1} \left\| \sum \varepsilon_i x_i \right\| \leq \lambda k \sup \|x_i\| \right\}.$$

If E is such that $\lambda_N(E) = 1/N^{1/p'}$ for some $N > 1$ and $p' \in [1, \infty)$, then E is of infratype q for all $q < p$ $(1/p + 1/p' = 1)$.

THEOREM 6. *A Banach space E is B-convex if and only if there is a $p \in (1, 2]$ such that E is of infratype P. Moveover,*

$$\Lambda_E = \sup\{p : E \text{ is of infratype } p\} = \lim_{n \to \infty} \frac{\text{Log } n}{\text{Log } n\lambda_n(E)}.$$

For a Banach space E let

$$\mu_n(E) = \inf\left\{ \mu : \left(\int \left\| \sum_1^n r_i(t) x_i \right\|^2 dt \right)^{1/2} \leq \mu_n \sup_{1 \leq i \leq n} \|x_i\| \right\}$$

and

$$\nu_n(E) = \inf\left\{ \nu : \left(\int \left\| \sum_1^n r_i(t) x_i \right\|^2 dt \right)^{1/2} \leq \nu \sqrt{n} \left(\sum_1^n \|x_i\|^2 \right)^{1/2} \right\}.$$

Here $(r_i(t))$ denotes the Rademacher functions. Clearly

$$\lambda_n(E) \leq \mu_n(E) \leq \nu_n(E) \leq 1$$

and

$$\mu_n(E) \geq 1/\sqrt{n}, \qquad \nu_n(E) \geq 1/\sqrt{n}.$$

Also, if E is such that $\nu_N(E) = 1/N^{1/p'}$ for some $N > 1$ and $p' \in [2, \infty)$, then E is of type q for any $q < p$.

PROPOSITION. *Let $R_E = \sup\{p : E \text{ is of type } p\}$. Then*

$$R_E = \lim_{n \to \infty} \frac{\text{Log } n}{\text{Log}[n\nu_n(E)]}.$$

THEOREM 7. *The following statements about the Banach space E are equivalent:*

(1) *E is not a $\mathcal{D}_1$-space;*
(2) *E is of infratype p for some $p \in (1, \infty)$ (equivalently $\Lambda_E > 1$);*
(3) *E is of type p for some $p \in (1, 2]$ (equivalently $R_E < 1$);*
(4) *E is of type $1+\varepsilon$-stable for some ε in $(0, 1]$; and*
(5) *E is of type 1-stable.*

Finally, let us observe that if $p \in (0, 2)$ and there is a constant C such that for all n there are elements $x_1^n \cdots x_n^n$ in E such that

$$\left(\sum_{i=1}^n |a_i|^p \right)^{1/p} \leq \left\| \sum a_i x_i^n \right\| \leq C \sum_1^n |a_i|$$

for all scalars (a_i), then E is not p-stable. Thus from results of R. C. James [**13.15**] we have

THEOREM 8. *Any uniformly convex space is of type p (type p-stable) for some $p > 1$.*

Also,

COROLLARY. *A Banach space E is of type 1-stable if and only if E' is of type 1-stable.*

LATTICES. When a lattice structure is present, more information can be obtained. Thus let L be a Banach lattice, $V \in \mathcal{L}(L, F)$, q real. Then V is said to be of *type* $\leq q$ if there is a C such that

$$\left(\sum \|V(x_i)\|^q \right)^{1/q} \leq C \left\| \left(\sum |x_i|^q \right)^{1/q} \right\|_L.$$

Let $1 \leq p \leq q < \infty$. Then V is said to be of *type* $\leq(p, q)$ if there is a C such that for $x_1 \cdots x_n \in L$

$$\left(\sum \|Vx_i\|^q \right)^{1/q} \leq C \left\| \left(\sum |x_i|^p \right)^{1/p} \right\|_L.$$

Let $K_{p,q}(v)$ be the smallest C satisfying the above.

THEOREM 9. *For a Banach lattice L and $V \in \mathcal{L}(L, F)$ the following statements are equivalent:*
(1) *for any positive operator $T \in \mathcal{L}(C(K), L)$, $VT \in \Pi_{p,q}(C(K), X)$ and $\pi_{p,q}(VT) \leq C \|T\|$; and*
(2) *$K_{p,q}(V) \leq C$.*

(Let us recall that an operator $T \in \Pi_{p,q}(E, F)$ if there is a constant C such that

$$\left(\sum_{i=1}^n \|Tx_i\|^q \right)^{1/q} \leq C \left(\sup_{\|f\|=1} \sum_1^n |f(x_i)|^p \right)^{1/p}$$

for arbitrary $x_1 \cdots x_n \in E$. In this case inf C, C satisfying the above, is denoted by $\pi_{p,q}(T)$. We have suppressed a growing and interesting series of papers on p,q-summing operators!)

COROLLARY. (a) *Let $1 \leq p_0 < q$. If $V \in \mathcal{L}(L, F)$ is of type $\leq(p_0, q)$ then V is also of type $\leq(p, q)$ for all $p \in [1, q)$.*
(b) *If V is of type $\leq(p, q)$ it is also of type $\leq r$ for any $r > q$.*

We can now describe the lattices of cotype q, $q > 2$.

THEOREM 10. *Let L be a Banach lattice, $q > 2$ and $V \in \mathcal{L}(L, F)$. The following are equivalent:*
(1) *$V \in \Pi_{1,q}(L, F)$;*
(2) *V is of type $\leq(p, q)$ for any $p \in [1, q)$;*

(3) *V is of cotype q; and*

(4) *there is a constant C such that if $x_1 \cdots x_n$ are pairwise disjoint elements of L then $(\sum \|Vx_i\|^q)^{1/q} \leqq C \|\sum x_i\|$.*

Letting V be the identity on L yields the following result.

THEOREM 11. *For a Banach lattice L the following are equivalent:*
(1) *L is of cotype 2;*
(2) *L is of type $\leqq 2$; and*
(3) $\mathscr{L}(C(K), L) = \Pi_2(C(K), L)$.

We now give some dual results. An operator $V \in \mathscr{L}(E, L)$, L a Banach lattice, is of type $\geqq(q, p)$, $1 \leqq p \leqq q < \infty$, if there is a constant C such that for $x_1, \cdots, x_n \in E$,

$$\|(\sum |Vx_i|^q)^{1/q}\|_L \leqq C(\sum \|x_i\|^p)^{1/p}.$$

Clearly V is of type $\geqq(q, p)$ if and only if V' is of type $\leqq(q', p')$.

PROPOSITION. *If an operator $U \in \mathscr{L}(E, L)$ is of type $\geqq(2, p)$ and $V \in \mathscr{L}(L, F)$ is of some finite type, then VU is of type p.*

THEOREM 12. *Let E be a Banach space such that E' is of cotype 2 and F a Banach space of cotype 2. If $U \in \mathscr{L}(E, F)$ factors through a lattice, then U factors through a Hilbert space.*

PROPOSITION. *If L is a Banach lattice then either L is a $\mathscr{S}_\infty$-space, or there is a $q < \infty$ such that the identity on L is $(1, q)$-summing.*

Thus,

THEOREM 13. *Let X be a Banach space. The following are equivalent:*
(a) *c_0 is not finitely representable in X;*
(b) *the identity on X is $(1, q)$-summing;*
(c) *there is an $r < \infty$ such that $\mathscr{L}(C(K), X) = \Pi_r(C(K), X)$; and*
(d) *there is a $q < \infty$ such that X is of cotype q.*

THEOREM 14. *Let E be a space with LUST and $q > 2$. The following are equivalent:*
(a) *E is of cotype q;* (b) *the identity on E is $(1, q)$-summing and* (c) *E has the Grothendieck property.*

COROLLARY. *If E has LUST and c_0 is not finitely representable in E then E is of type p if and only if E' is of cotype p'.*

14. **Basis theory and ideals of operators on Hilbert space.** Holub in a series of three papers [14.1]–[14.3] reexamined the fundamental work of Calkin, Schatten and Gohberg and Kreĭn and showed that many problems concerning ideals of operators on Hilbert space could be restated (and solved) in the standard language of Schauder bases and Köthe sequence spaces.

Of particular interest are the *minimal ideals* introduced by Schatten and the duals of such ideals. An ideal is *minimal* if it is the completion of $l_2 \otimes l_2$

with respect to some unitarily invariant cross norm. (See [14.2] and the references there for appropriate definitions.) If α denotes such a cross norm on $l_2 \otimes l_2$ it generates a symmetric gauge function ψ on the finitely nonzero sequences by defining

$$\psi(a_i, \cdots, a_n) = \left\| \sum_i^n a_i \varphi_i \otimes \psi_i \right\|_\alpha$$

for any orthonormal sets (φ_i) and (ψ_i). If (μ, ψ) denotes the completion of the finitely nonzero sequences under ψ then the unit vectors (e_i) form a basis for (μ, ψ) which is similar to $(\varphi_i \otimes \psi_i)$ in $l_2 \otimes_\alpha l_2$.

While this correspondence was well known, Holub utilized properties of bases to settle several problems. We give a few of those results. A unitarily invariant cross norm is *significant* if every operator in $(l_2 \otimes_\alpha l_2)^*$ is compact.

THEOREM 1. *Let α be a unitarily invariant cross norm. The following are equivalent.*
  (a) *α is significant;*
  (b) *α is not equivalent to ν (the greatest cross norm = nuclear norm); and*
  (c) *the basis (e_i) for (μ, ψ) is weakly convergent to 0.*

THEOREM 2. *If α and (μ, ψ) are as above then the basis (e_i) for (μ, ψ) is boundedly complete if and only if $l_2 \otimes_\alpha l_2 = (l_2 \otimes_{\alpha'} l_2)^*$. The basis is shrinking if and only if $(l_2 \otimes_\alpha l_2)^* = l_2 \otimes_{\alpha'} l_2$. (Recall that a basis (x_i) with coefficient functionals (f_i) is boundedly complete if $\sup_n \left\| \sum_{i=1}^n a_i x_i \right\| < +\infty$ implies $\sum_{i=1}^\infty a_i x_i$ converges, and is shrinking if (f_i) forms a basis for E'.*

Thus, if α is a unitarily invariant cross norm, $(l_2 \otimes_\alpha l_2)^* = l^2 \otimes_{\alpha'} l_2$ if and only if $(l_2 \otimes_\alpha l_2)^*$ is separable.

Let us return to minimal ideals. Clearly an ideal is minimal if it contains no proper closed subspace which is a norm ideal in $\mathscr{L}(H)$. An ideal is *maximal* if it is not itself a closed proper subspace of another norm ideal. If J is a maximal ideal then $\overset{\circ}{J}$, the closure of the finite rank operators under the ideal norm, is the unique minimal ideal contained in J. An ideal I such that $\overset{\circ}{J} \subset I \subset J$ is called *intermediate*.

Consider the simplest pair of minimal-maximal ideals, i.e., $\mathscr{L}(H)$ and $K(H)$. The results of Calkin show that for this pair there is no intermediate norm ideal (indeed, no two-sided ideal). On the other hand Mitiagin [14.4] exhibited an intermediate ideal between a certain pair of minimal-maximal ideals. Holub showed the true picture with the following remarkable result.

THEOREM 3. *Let $\overset{\circ}{J} - J$ be a minimal-maximal pair of norm ideals with $J \neq \mathscr{L}(H)$. Then there exists an intermediate ideal between $\overset{\circ}{J}$ and J.*

The idea of the construction, as above, is to reduce the problem to problems concerning bases in Köthe sequence spaces.

In our final section we will continue with applications of Banach space theory to ideal theory.

15. **Loose ends.** In this final section we mention some work on the geometry of certain ideals.

The geometry of $\sigma_p(H)H$ a Hilbert space has been studied by C. A. McCarthy [15.1] who proved, among other things, that the modulus of convexity of $\sigma_p(H)$ was the same as that of l_p for $1<p<\infty$. We have previously mentioned the work of N. Tomczak-Jaegermann [15.2] concerning Rademacher averages in $\sigma_p(H)$. Holub [15.3], [15.4] has discussed the subspaces and metric geometry (e.g. extreme points and smooth points) of the trace class operators $\sigma_1(H)$.

More recently J. Arazy and J. Lindenstrauss [15.5] have discussed several linear topological properties of $\sigma_p(H)$. They make some striking comparisons with known results concerning the structure of the L_p-spaces.

Many years ago Schatten observed that $K(H, H)$ is not a conjugate space. Of course, there are spaces (e.g. $E=l_p$ and $F=l_q$, $p>q>1$) for which $K(E, F)$ is reflexive. The following result of Saphar and Feder [15.6] gives a complete answer to the question raised by the above results.

THEOREM 1. *Let E and F be reflexive Banach spaces. Then $K(E, F)$ is either reflexive or not conjugate.*

It would be of considerable interest to ascertain if the isomorphic version of the theorem is also valid.

It is only fitting that we close with applications of Banach space theory to Banach ideals. We will mention only a profound result of Rosenthal [15.7] and two equally remarkable results of Johnson and Figiel [15.8] and Johnson [15.9].

THEOREM 2 (ROSENTHAL). *A Banach space E contains l_1 if and only if in E there is a uniformly bounded sequence (x_n) such that no subsequence of (x_n) is weakly Cauchy.*

A sequence (x_n) in E is weakly Cauchy if $(f(x_n))$ is a convergent sequence for each $F \in E'$. A weakly Cauchy sequence need not be weakly convergent, e.g. the unit vectors in c_0.

We now give a result of Pełczyński and Ovsepian [15.10] which connects this result with Banach ideals. First we make a definition: Let μ be a probability measure on a compact Hausdorff space S. A subspace Z of $C(S)$ is *fat with respect to* μ if the natural injection $I_\mu : L_\infty(\mu) \to L_2(\mu)$ when restricted to Z is not a compact operator.

THEOREM 3. *For any Banach space E the following are equivalent:*
(a) *E contains l_1;*
(b) *there is a 2-absolutely summing operator from X onto l_2;*
(c) $\Pi_2(E, l_2) \backslash K(E, l_2) \neq \varnothing$; *and*
(d) *for every isometric embedding j of E into C(S) there is a probability measure μ on S such that $j(E)$ is fat with respect to μ.*

The difficult step in the proof, (d)$\Rightarrow$(a), follows from the Rosenthal theorem.

The same reasoning used to prove Theorem 3 yields the following observation of Y. Gordon (private communication).

THEOREM 4.   *Let E be a Banach space. Then either*

(a) $\Pi_1(E, l_2) \subset K(E, l_2)$; *or*

(b) E *contains a complemented copy of* l_1; *or*

(c) E *lacks LUST.*

We now show how a construction of Figiel and Johnson [**15.8**] solves a problem of A. Pietsch [**15.11**]. We begin with an interesting example of Johnson [**15.9**]. Let (E_n) be a sequence of finite dimensional Banach spaces such that, for $\varepsilon > 0$ and any finite dimensional Banach space F, there is an m such that $d(E_m, F) < 1 + \varepsilon$. (A simple compactness argument yields the existence of such E_n.) Let $E = (\bigoplus E_n)_{l_1}$. Then the conjugate of any separable Banach space is isomorphic to a complemented subspace of E'. Since there are separable conjugate spaces which fail the approximation property, E' fails the approximation property. However, it is not difficult to see that E has the approximation property.

THEOREM 5 (FIGIEL AND JOHNSON).   *For each positive integer n there is an equivalent renorming of E, $\|\cdot\|_n$, such that if $F_n = (E, \|\cdot\|_n)$ then F_n does not have the bounded approximation property with constant less than n.*

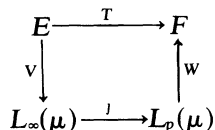APPLICATION.   An operator $T: E \to F$ is *dualizable* if T factors through a conjugate space:

$$
\begin{array}{ccc}
E & \xrightarrow{\;\;T\;\;} & F \\
& \searrow_{A} \quad \nearrow^{B} & \\
& X' &
\end{array}
$$
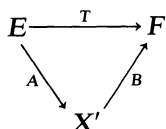
The dualizable norm $\delta(T)$ is defined by $\delta(T) = \inf \|A\| \|B\|$, the infimum taken over all such factorizations. If Δ denotes the class of all dualizable operators then $[\Delta, \delta]$ is a Banach ideal. Clearly $\|T\| \leq \delta(T)$ for each $T \in \Delta$. A. Pietsch [**15.11**] has asked for examples of $T \in \Delta$ with $\|T\| < \delta(T)$.

THEOREM 6.   *For each integer n there is an operator $T_n \in \Delta$ such that $\|T_n\| = 1$ and $\delta(T_n) \geq n$.*

Indeed let T_n denote the identity on F_n of the Figiel-Johnson theorem. This operator meets the requirements.

There are at least two other interesting applications of the Johnson-Figiel construction. It is clear from the definition that the adjoint of a nuclear operator is also nuclear. The results of [**15.8**] show that the converse is false!

Also Pietsch [**15.12**] has defined p-integral operator via the following diagram:

$$
\begin{array}{ccc}
E & \xrightarrow{\;\;T\;\;} & F \\
\downarrow{\scriptstyle v} & & \uparrow{\scriptstyle w} \\
L_\infty(\mu) & \xrightarrow{\;\;I\;\;} & L_p(\mu)
\end{array}
$$

with meanings as in §5. Again the Figiel-Johnson construction shows that the Pietsch definition and the Grothendieck definition given in §5 are not the same.

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