A NEW CONGRUENCE PROPERTY OF RAMANUJAN'S FUNCTION $\tau(n)$

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In a joint sitting we guessed, with the help of Gupta's tables, that

$$\tau(n) \equiv n^2\sigma(n) \pmod 9, \tag{1}$$

where

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\prod_{n=1}^{\infty}(1 - x^n)^{24} \qquad (|x| < 1)$$

and

$$\sigma_k(n) = \sum_{d|n} d^k, \qquad \sigma(n) = \sigma_1(n).$$

Each of us obtained a separate proof of (1). But we publish only one.

PROOF OF (1). Writing

$$P = 1 - 24\sum_{1}^{\infty} \sigma(n)x^n,$$

$$Q = 1 + 240\sum_{1}^{\infty} \sigma_3(n)x^n,$$

$$R = 1 - 504\sum_{1}^{\infty} \sigma_5(n)x^n,$$

Ramanujan proved that

$$1728\sum_{1}^{n} \tau(n)x^n = Q^3 - R^2.$$

Writing J_γ ($\gamma = 1, 2, 3, \cdots$) for an integral power series in x with integral coefficients, we derive from relations 8, 1, and 3, Table III, p. 142 of Ramanujan's *Collected papers*, Cambridge, 1927,

$$1728\sum_{1}^{\infty} \tau(n)x^n = Q^3 - R^2 = 5184\sum_{1}^{\infty} (n^3\sigma_5(n)x^n)$$
$$- 2R^2 - 6P^2Q^2 + 2P^3R + 6PQR$$

Received by the editors September 11, 1946.

$$= 5184\sum_{1}^{\infty}(n^3\sigma_5(n)x^n)$$
$$+ R(3PQ - 2R - P^3) - 3P(2PQ^2 - P^2R - QR)$$
$$= 5184\sum_{1}^{\infty} n^3\sigma_5(n)x^n$$
$$+ (1 - 504J_1)\left(1728\sum_{1}^{\infty} n^2\sigma(n)x^n\right)$$
$$- 3(1 - 24J_2)\left(1728\sum_{1}^{\infty} n^2\sigma_5(n)x^n\right)$$
$$= 5184\sum_{1}^{\infty} n^3\sigma_5(n)x^n + 1728\sum_{1}^{\infty} n^2\sigma(n)x^n$$
$$- 5184\sum_{1}^{\infty} n^2\sigma_5(n)x^n + 243J_3.$$

Comparing the coefficients of x^n, we have

$$27\tau(n) \equiv 81n^3\sigma_5(n) + 27n^2\sigma(n) - 81n^2\sigma_5(n) \pmod{243}$$

or

$$\begin{aligned}\tau(n) &\equiv n^2\sigma(n) + 3(n^3 - n^2)\sigma_5(n) \pmod 9\\ &\equiv n^2\sigma(n) + 3(n^3 - n^2)\sigma(n) \pmod 9\\ &\equiv n^2\sigma(n) \pmod 9,\end{aligned}$$

for if

$$n \equiv 0, 1 \pmod 3 \quad \text{then} \quad n^3 - n^2 \equiv 0 \pmod 3$$

while if

$$n \equiv 2 \pmod 3 \quad \text{then} \quad \sigma(n) \equiv 0 \pmod 3.$$

COROLLARY.

$\tau(n) \equiv 0 \pmod 9$ *for almost all n.*

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