NOTE ON THE DIVISION OF A PLANE BY A POINT SET*

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A plane set of points K is said to divide a plane S if the set $S - K$ is composed of two mutually exclusive domains S_1, S_2, of which K is a common boundary, where by domain is meant a connected open set. The condition that K be a simple closed curve or an open curve has been stated by J. R. Kline† in terms of the concept "connected im kleinen." In proving that the set K is a connected set, Kline employs the condition "connected im kleinen."

If we assume that K is bounded and that we have at our disposal the parallel and perpendicular straight lines of a number plane, then the connectedness of K is established by other writers, for example Hausdorff, *Grundzüge der Mengenlehre*, page 346, Theorem XII. Hausdorff calls attention, in a footnote to page 342, to the difficulty of extending his argument to the case of unbounded sets.

It seems in view of the importance of the theory of open curves as indicated by R. L. Moore‡ and of the importance in general of the fundamental theorems of plane analysis situs that it is of interest to show that the set K is connected, whether bounded or not, without the use of the restriction employed by Kline or of the properties of straight lines and rectangles. The present note is concerned, therefore, with the proof of the following theorem.

THEOREM. *Let K be a plane point set, S, the set of all points of the plane, and denote by S_1, S_2 two mutually exclusive domains such that*

$$S - K = S_1 + S_2.$$

Then if every point of K is a limit point of both S_1 and S_2, the set K is closed and connected.

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† *Concerning approachability of simple closed and open curves.* TRANSACTIONS OF THIS SOCIETY, vol. 21 (1920), pp. 451–458.

‡ R. L. Moore, *On the foundations of plane analysis situs.* TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), pp. 131–164. This paper will be referred to as "Foundations."

Proof. The set K is closed. For any limit point of K is evidently a common limit point of S_1 and S_2. But no limit point of S_1 can belong to the domain S_2, and likewise no limit point of S_2 can belong to S_1. Such a point must belong to K.

Assume that the set K is not connected. Then there exist two mutually exclusive closed subsets K_1 and K_2 of K such that $K = K_1 + K_2$. Let P_i denote a point of K_i $(i = 1, 2)$. We may enclose P_i in a region R_i which contains no point of K_{i+1}.* Let J_i be a simple closed curve lying in R_i and containing P_i as an interior point. Let P_{ij} be a point of S_j lying within J_i. Since S_j is a domain, there is an arc

$$P_{1j}\,X\,P_{2j}$$

lying entirely in S_j. Let A_{1j} be the last point which this arc has in common with J_1 and let A_{2j} be the first point which the arc has in common with J_2 after A_{1j} on $P_{1j}\,X\,P_{2j}$.

Since the point A_{ij} lies on the boundary of J_i, it may be connected with P_i by an arc $P_i\,X\,A_{ij}$ which, except for the end-point A_{ij}, lies entirely within J_i. Furthermore the arcs $P_i\,X\,A_{i1}$ and $P_i\,X\,A_{i2}$ may be constructed so that they have no common point besides P_i.

From the arcs so defined we construct a simple closed curve J,

$$J\colon\ P_1A_{11}A_{21}P_2A_{22}A_{12}P_1.$$

Let H_1 denote the set of all points of K_1 which are interior to J but not interior to J_1. The set H_1 is closed.

CASE I. No point of H_1 lies on J_1. Then the points of J_1 which lie in J lie in $S_1 + S_2$. There exist points of J_1 within J, since by Theorem 40 of the *Foundations* it is possible to join P_1 and P_2 by an arc $P_1\,X\,P_2$ such that $\underline{P_1\,X\,P_2}$† lies entirely within J. This arc must meet J_1 in at least one point. It follows readily that there is an arc $A_{11}\,X\,A_{12}$ of J_1 which lies in J and therefore in $S_1 + S_2$. This is contrary to Lemma A of the paper of J. R. Kline.‡

* The subscripts are to be reduced modulo 2.

† The symbol $\overline{A\,X\,B}$ denotes the set $A\,X\,B - A - B$, that is, the set of all points of the arc $A\,X\,B$ except its end-points.

‡ Loc. cit., p. 452. Every arc joining a point of S_1 to a point of S_2 contains a point of K.

CASE II. The curve J_1 contains a point of H_1. From the Heine-Borel property we may assume a finite set of regions

$$(1) \qquad \overline{R}_1, \overline{R}_2, \cdots, \overline{R}_m,$$

covering H_1. We may without loss of generality assume that the boundary of each region $\overline{R}_k$ ($k = 1, 2, \cdots, m$) is a simple closed curve $\overline{J}_k$. We will also assume that no point of J, J_2, or K_2 lies in or on the boundary of any of the regions $\overline{R}_k$. By hypothesis some of the regions $\overline{R}_k$ have points in common with the interior of J_1. Let

$$(2) \qquad \overline{R}_1, \overline{R}_2, \cdots, \overline{R}_p, \quad (p \leqq m),$$

be a subset of the regions (1) such that $\overline{R}_1 + \cdots + \overline{R}_p$ forms, with the interior of J_1, a connected set. Then the curves $J_1, \overline{J}_1, \overline{J}_2, \cdots, \overline{J}_p$ form a finite family G of closed curves whose interiors form a connected set. By theorem 42 of the *Foundations* there is a simple closed curve $\overline{J}$ which satisfies these conditions: every point of $\overline{J}$ belongs to some curve of G; the interior of $\overline{J}$ contains the interiors of all the curves of the set G.

The curve $\overline{J}$ meets J in the points A_{1j} and in no other points. We may show as in Case I that there is a point X on $\overline{J}$ which is interior to J, and that the arc $\underline{A_{11}\,X\,A_{12}}$ of the curve $\overline{J}$ lies within J.

We will show that no point of $A_{11}\,X\,A_{12}$ is in K. By construction no point of K_2 lies on any curve of the set G. Suppose Q, a point of K_1, lies on the arc $A_{11}\,X\,A_{12}$. Then Q must lie on one of the curves of the set G. If Q is on J_1, it must belong to H_1 since Q is interior to J. Consequently Q is in H_1 in any case and must lie in some region $\overline{R}_k$ ($k \leqq m$) of the regions (1). Suppose that Q lies on $\overline{J}_q$ ($q \leqq p$). Then R_k must contain an interior point of $\overline{J}_q$. But the interior of $\overline{J}_q$ is connected with R_1. Consequently R_k is connected with R_1 and is interior to $\overline{J}$. It follows that Q is an interior point of $\overline{J}$, contrary to hypothesis.

Since no point of K lies on $A_{11}\,X\,A_{12}$, we have obtained an arc connecting a point of S_1 with a point of S_2 which contains no point of K. This again contradicts Lemma A.

The proof that the set K is connected is completed.

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