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ON THE ABSOLUTE CONTINUITY OF ADDITIVE AND NON-ADDITIVE FUNCTIONS

Paola Cavaliere, Paolo de Lucia, Anna De Simone

Dedicated with great pleasure to
Lech Drewnowski

Abstract: We are concerned with the notions of absolute continuity in terms of 0-continuity, and of (ϵ, δ)-continuity, with respect to a finitely additive function taking values into a topological commutative group. A sufficient condition for their equivalence is established. The non-additive case of functions with values into Hausdorff topological spaces is also investigated.

Keywords: finitely additive function, non-additive function, absolute continuity, order-continuity
.

1. Introduction and main results

Let $\mathcal{A}$ be a σ-algebra, and let μ and ν be classical measures on $\mathcal{A}$, namely $[0, +\infty]$-valued σ-additive functions such that $\mu(0) = \nu(0) = 0$. In the literature on measure theory, the following notions of absolute continuity of ν with respect to μ are available:

(I) ν is 0-*continuous with respect to* μ, denoted by $\nu \ll \mu$, whenever $\mu(a) = 0$ for $a \in \mathcal{A}$ implies $\nu(a) = 0$;

(II) ν is (ε, δ)-*continuous with respect to* μ, denoted by $\nu[AC]\mu$, whenever for every $\varepsilon > 0$ there exists some $\delta > 0$ such that if $\mu(a) < \delta$ for $a \in \mathcal{A}$, then $\nu(a) < \varepsilon$.

A measure μ fulfilling *(II)* is called a *control function for* ν.

Clearly, property *(II)* always implies *(I)*. The two properties are equivalent provided that ν is finite. If this assumption is dropped, the equivalence may fail. A trivial example is given by the case when $\mathcal{A}$ is the σ-algebra $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$, μ is the Lebesgue measure on $\mathcal{B}(\mathbb{R})$ and $\nu(E) = \int_E |x| dx$ for $E \in B(\mathbb{R})$.

On enlarging the class of target spaces of the measures taken into account - including, for instance, Banach spaces or Hausdorff topological groups - both notions can be analogously formulated. Accordingly, it is natural to seek conditions for the equivalence of *(I)* and *(II)*. This piece of information can be of interest in

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view of various applications. For instance, property *(II)* is, via Radon-Nikodým Theorem, a very useful tool in integration and distribution theory (see e.g. [9, 6]), but *(I)* is easier to be checked.

For Hausdorff group-valued σ-additive functions on σ-rings, it was proved in [11] that the mentioned equivalence holds provided that the target space of the control function μ is metrizable. The same situation occurs if such requirement is replaced with the assumption that μ satisfies the countable chain condition, as shown in [7]. Recall that μ is said to satisfy the *countable chain condition* (briefly, the *ccc*) if each family of pairwise disjoint non-μ-negligible elements is (at most) countable.

Our first aim in the present paper is to extend these results in the framework of additive functions along two different directions. Firstly, we allow the control function μ to be just finitely additive. Secondly, we exhibit a weaker condition on μ ensuring that for each group-valued σ-additive function the notions of (ε, δ)-continuity and 0-continuity with respect to μ are actually equivalent. Let us mention that here we do not require topological groups to be Hausdorff.

To be more specific, let $\mu : \mathcal{R} \to G$ be a finitely additive function, where $\mathcal{R}$ is a σ–ring and G is a topological commutative group, additively written. The *kernel* of μ is defined by

$$\mathcal{N}(\mu) \;=\; \Big\{a \in \mathcal{R} :\; \mu([0,a]) \;\subseteq \bigcap_{U\in\tau[0]} U = \overline{\{0\}}^{\,\tau}\Big\}, \tag{1.1}$$

where $[0,a] = \{x \in \mathcal{R} : 0 \leqslant x \leqslant a\}$, $\tau[0]$ denotes the collection of all 0-neighbourhoods in G and " $\overline{\cdot}^{\,\tau}$ " stands for closure operation in G.

We say that $\mathcal{N}(\mu)$ *is identified by a sequence* $(U_n)_{n\in\mathbb{N}}$ in $\tau[0]$ whenever

$$\mathcal{N}(\mu) \;=\; \Big\{a \in \mathcal{R} : \mu([0,a]) \;\subseteq\; \bigcap_{n\in\mathbb{N}} U_n\Big\}\,. \tag{1.2}$$

Note that, in particular, $\mathcal{N}(\mu)$ fulfils such a condition when G is pseudometrizable or when $\mathcal{R}$ is a σ–ring and μ is a σ–additive function on $\mathcal{R}$ satisfying the countable chain condition (see Remark 2.2 below, and [14] for related results).

Our first result reads as follows.

Theorem 1.1. *Let $\mathcal{R}$ be a σ–ring, and let G and G' be topological commutative groups. Assume that $\nu : \mathcal{R} \to G'$ is σ–additive, and that $\mu : \mathcal{R} \to G$ is finitely additive. If $\mathcal{N}(\mu)$ is identified by some sequence $(U_n)_{n\in\mathbb{N}}$ of 0-neighbourhoods in G -i.e.* (1.2) *holds-, then*

$$\nu[\mathrm{AC}]\mu \Longleftrightarrow \nu \ll \mu\,. \tag{1.3}$$

Next we deal with the same problem, but in the more general (non-additive) context of quasi-triangular functions. In fact, additivity is nowadays a too much restrictive requirement in several theoretical and applied problems (see e.g. [10, 13, 16] and the references therein).

Specifically, let $\mathcal{R}$ be a Boolean ring, and $\mathcal{S} = (S, \tau)$ a Hausdorff topological space. A function $\eta : \mathcal{R} \to \mathcal{S}$ is said to be *quasi–triangular* if for every $U \in \tau[\eta(0)]$ there exists some $V \in \tau[\eta(0)]$ such that the following property holds: when any two of the quantities $\eta(a)$, $\eta(b)$, $\eta(a \vee b)$ belong to V for disjoint $a, b \in \mathcal{R}$, then the remaining one does belong to U. Here, $\tau[\eta(0)]$ stands for the collection of $\eta(0)$-neighbourhoods in $\mathcal{S}$.

Notions *(I)* and *(II)* can be further strengthened in the quasi-triangular framework, as will be shown in Sect. 3. This follows from the fact that *(I)* and *(II)* actually involve just negligible sets of 'measures' and topologies of their target spaces, respectively.

Let us highlight that target spaces of quasi-triangular functions are Hausdorff topological spaces, where no algebraic structure is required. As a consequence, neither monotonicity, nor pseudo-additivity are meaningful notions in this setting. Interestingly, this non-additive class includes classical finitely additive functions as well as various families of non-additive functions extensively studied in the literature. For instance, outer measures, k-triangular functions, decomposible functions with respect to a t-conorm, and quasi-submeasures are quasi-triangular functions (see [1] and [3, Examples 2.1]).

Our generalization of Theorem 1.1 to quasi-triangular functions involves the notion of order-continuous functions. Recall that $\eta : \mathcal{R} \to \mathcal{S}$ is said to be *order-continuous* whenever $\lim \eta(b_n) = \eta(0)$ for every sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathcal{R}$ decreasing to *0*.

Theorem 1.2. *Let $\mathcal{R}$ be a σ–ring and let $\mathcal{S}$ and $\mathcal{S}'$ be Hausdorff topological spaces. Assume that $\upsilon : \mathcal{R} \to \mathcal{S}'$ is an order-continuous quasi–triangular function, and that $\eta : \mathcal{R} \to \mathcal{S}$ is quasi–triangular. If $\mathcal{N}(\eta)$ is identified by some sequence $(U_n)_{n\in\mathbb{N}}$ of $\eta(0)$-neighbourhoods in $\mathcal{S}$ -i.e.* (3.2) *holds-, then*

$$\upsilon[AC]\eta \Longleftrightarrow \upsilon \ll \eta\,. \tag{1.4}$$

Let us mention that Theorem 1.2 in particular extends [8, Theorem 4.3], where η is a capacity on a σ–algebra $\mathcal{A}$ (thus, a $[0, +\infty]$-valued quasi-submeasure, namely, quasi-additive and quasi-monotone) and υ is a $[0, +\infty[$-valued σ-additive function on $\mathcal{A}$, with $\eta(0) = \upsilon(0) = 0$. The proof of [8] exploits the theory of capacities, and specifically certain capacitary estimates, on semigruppoids (see [8, Sect. 2]).

A completely different approach is proposed here. Our proof is indeed based on standard methods of measure theory. Firstly, we prove Theorem 1.1, via arguments related to those of [7]. Then a crucial idea in the proof of Theorem 1.2 consists in calling into play the Fréchet-Nikodým topologies, which allows us to establish a precise connection between quasi-triangular functions and finitely additive functions valued in topological group [1, 2](see also Sect. 3). The Fréchet-Nikodým topologies, whose study was initiated by L. Drewnowski [5], turn out to be an elegant and powerful tool in non-additive measure theory as well.

2. The additive case: proof of Theorem 1.1

For functions $\mu : \mathcal{R} \to G$ and $\nu : \mathcal{R} \to G'$, where $\mathcal{R}$ is a Boolean ring and G, G' topological commutative groups (additively written), formulate *(I)* and *(II)* in Sect. 1 as follows

(I) ν is 0-*continuous with respect to* μ, denoted by $\nu \ll \mu$, whenever $\mathcal{N}(\mu) \subseteq \mathcal{N}(\nu)$;

(II) ν is (ε, δ)-*continuous with respect to* μ, denoted by $\nu[AC]\mu$, whenever for every $0'$-neighbourhood W' there exists some 0-neighbourhood V such that if $\mu([0,a]) \subseteq V$ for $a \in \mathcal{R}$, then $\nu([0,a]) \subseteq W'$.

Here, $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$ are the kernels of μ and ν, respectively, defined as in (1.1).

Let us observe

Lemma 2.1. *Let $\mathcal{R}$ be a Boolean ring, and G an Abelian topological group. If the kernel of a function $\mu : \mathcal{R} \to G$ is identified by some sequence of* 0*-neighbourhoods in G, then there exists a sequence $(U_n)_{n\in\mathbb{N}_o}$ fulfilling* (1.2) *and the following properties:*

$$U_n + U_n \subseteq U_{n-1} \qquad \text{for } n \in \mathbb{N}\,,\ \text{with } U_0 = G\,; \tag{2.1}$$

$$\sum_{p=n}^{n+k} U_p \subseteq U_{n-1} \qquad \text{for } n, k \in \mathbb{N}\ . \tag{2.2}$$

Proof. A standard approach (see e.g. [4, (1.7) in Chap. 1]) gives the statement. ■

In the proof of Theorem 1.1, we will use the fact that, as long as $\mathcal{R}$ is a σ-ring, every σ-additive function $\nu : \mathcal{R} \to G'$ is *exhaustive*, namely, $\lim \nu(d_n) = \nu(0)$ for each pairwise disjoint sequence $(d_n)_{n\in\mathbb{N}}$ in $\mathcal{R}$. Let us remark that such a property may fail for merely finitely additive functions on σ-rings.

Proof of Theorem 1.1. The implication

$$\nu[\mathrm{AC}]\mu \Longrightarrow \nu \ll \mu$$

is an immediate consequence of definitions *(I)*-*(II)* above.

To prove the converse, assume by contradiction that ν fails to be (ε, δ)-continuous with respect to μ. Namely, an open $0'$-neighbourhood W' exists so that

$$\begin{gathered}\text{for any 0-neighbourhood } V \text{ there is some } a \in \mathcal{R} \setminus \mathcal{N}(\nu) \text{ such that}\\ \mu([0,a]) \subseteq V, \qquad \text{and} \qquad \nu(a) \notin W'\ .\end{gathered} \tag{2.3}$$

Since $\mathcal{N}(\mu)$ is identified by some sequence of 0-neighbourhoods in G, then Lemma 2.1 allows us to consider a sequence $(U_n)_{n\in\mathbb{N}}$ in $\tau[0]$ fulfilling conditions (1.2), (2.1) and (2.2). Then, by (2.3), one gets the existence of a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathcal{R} \setminus \mathcal{N}(\nu)$ such that

$$\mu([0,a_n]) \subseteq U_n, \qquad \text{and} \qquad \nu(a_n) \notin W'\ . \tag{2.4}$$

Set $b_n := \bigvee_{k \geqslant n} a_k$ for each n. Then $b_n \in \mathcal{R} \setminus \mathcal{N}(\nu)$ and

$$b_n \searrow \bigwedge_{n \in \mathbb{N}} b_n = \limsup_n a_n . \tag{2.5}$$

Firstly, we claim that

$$\nu([0, \limsup_n a_n]) \nsubseteq W' , \tag{2.6}$$

that is,

$$\text{there exists some } c \in \,]0, \limsup_n a_n] \text{ such that } \nu(c) \notin W' . \tag{2.7}$$

Note that, in particular, $c \notin \mathcal{N}(\nu)$.

To see this, notice that if (2.6) failed, combining (2.5) with the σ-additivity of ν would imply that

$$\lim_n \nu(x \wedge b_n) = \nu(x \wedge \limsup_n a_n)$$

uniformly with respect to $x \in \mathcal{R}$. Henceforth, $\nu([0, b_n]) \subseteq W'$ for sufficiently large n, and so $\nu(a_n) \in W'$. This clearly would contradict (2.4), thus (2.6) and (2.7) hold.

Now, on account of (2.7), set

$$c_{n,p} := c \wedge \left(\bigvee_{k=n}^{n+p} a_k \right) \qquad \text{for } \; n, p \in \mathbb{N} .$$

According to (2.5), for any fixed index n

$$c_{n,p} \nearrow \bigvee_{p \in \mathbb{N}} c_{n,p} = c \wedge b_n = c . \tag{2.8}$$

Owing to the σ-additivity of ν, for any $n \in \mathbb{N}$ one obtains that

$$\lim_p \nu(x \wedge c_{n,p}) = \nu(x \wedge c)$$

uniformly with respect to $x \in \mathcal{R}$. On making use of (2.8), this means that for any fixed index n

$$\lim_p \nu(x \wedge (c \setminus c_{n,p})) = 0 \tag{2.9}$$

uniformly with respect to $x \in \mathcal{R}$.

On the other hand, notice that

$$\mu([0, c_{n,p}]) \subseteq U_{n-1} \qquad \text{for } \; n, p \in \mathbb{N} . \tag{2.10}$$

In fact, from the first relation in (2.4) it follows that

$$\begin{aligned}\mu(x) &= \mu\left(x \wedge c \wedge \left(\bigvee_{k=n}^{n+p}\left(a_k \setminus \bigvee_{i=n}^{k-1} a_i\right)\right)\right) \\ &= \sum_{k=n}^{n+p} \mu\left(a_k \wedge x \wedge \left(c \setminus \bigvee_{i=n}^{k-1} a_i\right)\right) \in \sum_{k=n}^{n+p} U_k \subseteq U_{n-1}\end{aligned}$$

for each $x \in [0, c_{n,p}]$.

Now, consider an arbitrary closed $0'$-neighbourhood W'_0, and a decreasing sequence $(W'_j)_{j\in\mathbb{N}}$ in $\tau'[0']$ such that

$$W'_j + W'_j \subseteq W'_{j-1} \qquad \text{for } j \in \mathbb{N}\,, \tag{2.11}$$

$$\sum_{n=j}^{j+k} W'_n \subseteq W'_{j-1} \qquad \text{for } j, k \in \mathbb{N}\,. \tag{2.12}$$

From (2.9) it follows that for each $n \in \mathbb{N}$ there exists some $p_n \in \mathbb{N}$ such

$$\nu([0, c \setminus c_{n,p_n}]) \subseteq W'_{n-1}\,.$$

Let us decompose c as disjoint join in the following way:

$$\left(\bigvee_{k\in\mathbb{N}} \bigwedge_{n\geqslant k} c_{n,p_n}\right) \vee \left(\bigwedge_{k\in\mathbb{N}} \bigvee_{n\geqslant k} (c \setminus c_{n,p_n})\right) = c \tag{2.13}$$

We claim that

$$\bigvee_{k\in\mathbb{N}} \bigwedge_{n\geqslant k} c_{n,p_n} \in \mathcal{N}(\nu)\,. \tag{2.14}$$

To see this, note that (2.10) implies that

$$\mu([0, \bigwedge_{n\geqslant k} c_{n,p_n}]) \subseteq \mu([0, c_{n,p_n}]) \subseteq U_{n-1} \qquad \text{for } n \geqslant k\,. \tag{2.15}$$

Since $(U_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $\tau[0]$ and identifies $\mathcal{N}(\mu)$, then it follows from (2.14) that

$$\mu([0, \bigwedge_{n\geqslant k} c_{n,p_n}]) \subseteq \bigcap_{n\geqslant k} U_{n-1} = \bigcap_{n\in\mathbb{N}} U_n$$

and

$$\bigwedge_{n\geqslant k} c_{n,p_n} \in \mathcal{N}(\mu) \qquad \text{for each } k \in \mathbb{N}\,.$$

Whence (2.14) follows from the assumption that $\nu \ll \mu$ and the σ–additivity of ν.

Next, we note that

$$\bigwedge_{k\in\mathbb{N}} \bigvee_{n\geqslant k} (c \setminus c_{n,p_n}) = \limsup_n (c \setminus c_{n,p_n})\,.$$

Moreover, each $x \in [0, \limsup_n (c \setminus c_{n,p_n})]$ may be written as

$$x = \bigvee_{n \geqslant k} (x \wedge (c \setminus c_{n,p_n})) = \bigvee_{n \geqslant k} (x \wedge d_n) \qquad \text{for each } \; k \in \mathbb{N}\,, \tag{2.16}$$

where $(d_n)_{n \geqslant k}$ is a pairwise disjoint sequence in $\mathcal{R}$, with $d_n \leqslant c \setminus c_{n,p_n}$, such that

$$\bigvee_{n \geqslant k} d_n = \bigvee_{n \geqslant k} (c \setminus c_{n,p_n}) \qquad \text{for } \; k \in \mathbb{N}\,.$$

Then, according to (2.16), the σ–additivity of the function ν allows us to deduce that

$$\nu(x) \;=\; \sum_{n \geqslant j} \nu(x \wedge d_n) \;=\; \lim_k \sum_{n=j}^{j+k} \nu(x \wedge d_n) \;\in W_0'\,. \tag{2.17}$$

Indeed, by (2.12),

$$\sum_{n=j}^{j+k} \nu(x \wedge d_n) \;\in\; \sum_{n=j}^{j+k} W_n' \;\subseteq\; W_{j-1}' \;\subseteq\; W_0'\,.$$

Thus, on account of (2.13), (2.14) and (2.17), one concludes that $\nu([0,c]) \subseteq W_0'$. Whence $c \in \mathcal{N}(\nu)$, being W_0' arbitrary. This contradicts (2.7) and ends the proof. ∎

Remark 2.2. The kernel of a function $\mu : \mathcal{R} \to G$ is identified by some sequence in $\tau[0]$ when G is pseudometrizable, or when $\mathcal{R}$ is a σ–ring and μ is a σ–additive function on $\mathcal{R}$ satisfying the countable chain condition. For the latter, arguing as in the proof of [7, Lemma] provides indeed the existence of a pseudometric p on G such that

$$\mathcal{N}(\mu) = \{a \in \mathcal{R} : \mu([0,a]) \subseteq \overline{\{0\}}^{\tau_p}\},$$

where τ_p is the group-topology induced by p.

Consequently, Theorem 1.1 improves [11, Theorem] and [7, Theorem 2], respectively. Interestingly, our result allows the control function μ to be just finitely additive.

Remark 2.3. In [11, Example], T. Traynor says that the requirement of metrizability for the target space G of the control (σ-additive) function $\mu : \mathcal{R} \to G$ cannot be eliminated in order to establish the equivalence of *(I)* and *(II)* above. The mentioned example does exhibit a control function whose kernel is identified by no sequence in $\tau[0]$; according to Remark 2.2, G could not be metrizable.

Specifically, consider the σ-algebra Σ of Lebesgue measurable subsets of $[0,1]$, and the collection $B[0,1]$ of bounded real function on $[0,1]$ endowed with the topology of the pointwise convergence. For each $A \in \Sigma$, denote by χ_A the characteristic function of A. Then the Lebesgue measure on Σ is 0–continuous with respect to the function $\mu : \Sigma \to B[0,1]$ defined by $\mu(A) = \chi_A$ for $A \in \Sigma$, but it fails to be (ε, δ)-continuous with respect to μ. Note that $\mathcal{N}(\mu)$ is identified by no sequence in $\tau[0]$.

3. The non-additive case of quasi-triangular functions: proof of Theorem 1.2

Let $\eta : \mathcal{R} \to \mathcal{S}$ be quasi-triangular, where $\mathcal{R}$ is a Boolean ring and $\mathcal{S} = (S, \tau)$ an Hausdorff topological space. Then

(i) η generates a Fréchet-Nikodým topology (namely Γ_η) on the underlying ring $\mathcal{R}$ having as neighbourhood base at each $a \in \mathcal{R}$ the family

$$\Gamma_\eta(a) := \big(\{x \in \mathcal{R} : \eta([0, x \triangle a]) \subseteq U\}\big)_{U \in \mathcal{B}[\eta(0)]},$$

where $\triangle$ stands for the symmetric difference in $\mathcal{R}$ and $\mathcal{B}[\eta(0)]$ is a $\eta(0)$-neighbourhood base in $\mathcal{S}$ ([1, Theorem 3.2]);

(ii) there exists a finitely additive function μ, acting on the same Boolean ring $\mathcal{R}$ and valued in some topological commutative group G, which is equivalent to η in the sense that $\mu \ll$ [AC] and $\eta \ll$ [AC] ([1, Corollary 3.5]). Thus, by [12, Remarks 1.5 (2)], Γ_η and Γ_μ coincide;

(iii) η is order-continuous if, and only if, there exists a σ-additive function ν, acting on the same Boolean ring $\mathcal{R}$ and valued in some Abelian topological group G, which is equivalent to η ([2, Lemma 3.1.3]). Recall that η is said to be *order-continuous* whenever $\lim \eta(b_n) = \eta(0)$ for every sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathcal{R}$ decreasing to 0.

Note that a σ-additive function on a σ-ring may fail to be order-continuous. Clearly, the same conclusion holds for quasi-triangular functions. A trivial example is the the Lebesgue measure η on $[0, +\infty]$. It suffices to observe that

$$\bigcap_{n\in\mathbb{N}} [n, +\infty[= \emptyset\,, \qquad \text{and} \qquad \lim_n \eta([n, +\infty[) = +\infty\,.$$

We refer the reader to [4, 5, 15] for a comprehensive treatment of Fréchet-Nikodým topologies in the additive context and to [1] for a similar analysis in the non-additive setting of quasi-triangular functions.

For quasi-triangular functions definitions *(I)* and *(II)* in Sects. 1- 2 may be naturally generalized. Indeed, if $\eta : \mathcal{R} \to \mathcal{S}$ is a quasi-triangular, where $\mathcal{S} = (S, \tau)$ is a Hausdorff topological space, the *kernel* of η is defined by

$$\mathcal{N}(\eta) \;=\; \Big\{a \in \mathcal{R} : \eta([0, a]) = \{\eta(0)\}\Big\} \tag{3.1}$$

and $\mathcal{N}(\eta)$ *is idenfied by a sequence* $(U_n)_{n\in\mathbb{N}}$ in $\tau[\eta(0)]$ whenever

$$\mathcal{N}(\eta) \;=\; \Big\{a \in \mathcal{R} : \eta([0, a]) \subseteq \bigcap_{n\in\mathbb{N}} U_n\Big\}\,. \tag{3.2}$$

Then for functions $\eta : \mathcal{R} \to \mathcal{S}$ and $\upsilon : \mathcal{R} \to \mathcal{S}'$, where $\mathcal{S}$ and $\mathcal{S}'$ are Hausdorff topological spaces, *(I)* and *(II)* in Sects. 1- 2 read as follows

(I) υ is 0-*continuous with respect to* η, denoted by $\upsilon \ll \eta$, whenever $\mathcal{N}(\eta) \subseteq \mathcal{N}(\upsilon)$;

(II) υ is (ε, δ)-*continuous with respect to* η, denoted by $\upsilon[\mathrm{AC}]\eta$, whenever for every $\upsilon(0)$-neighbourhood W' there exists some $\eta(0)$-neighbourhood U such that if $\eta([0, a]) \subseteq U$ for $a \in \mathcal{R}$, then $\upsilon([0, a]) \subseteq W'$.

Proof of Theorem 1.2. The implication

$$\upsilon[\mathrm{AC}]\eta \Longrightarrow \upsilon \ll \eta$$

is an immediate consequence of definitions *(I)*-*(II)* above.

To prove the converse, notice that [1, Corollary 3.5] and [2, Lemma 3.1.3] tell us that there exist a finitely additive $\mu : \mathcal{R} \to G$ and a σ-additive function $\nu : \mathcal{R} \to G'$, with G and G' topological commutative groups, such that

$$\begin{array}{ccc} \nu[\mathrm{AC}]\upsilon & \text{and} & \upsilon[\mathrm{AC}]\nu\,, \\ \mu[\mathrm{AC}]\eta & \text{and} & \eta[\mathrm{AC}]\mu\,. \end{array} \tag{3.3}$$

Since $\mathcal{N}(\mu) = \mathcal{N}(\eta)$ and $\mathcal{N}(\nu) = \mathcal{N}(\upsilon)$, assumption $\upsilon \ll \eta$ actually implies that $\mathcal{N}(\mu) \subseteq \mathcal{N}(\nu)$, namely $\nu \ll \mu$.

Moreover, observe that the kernel $\mathcal{N}(\mu)$ of the finitely additive function μ is actually identified by some sequence of 0-neighbourhoods in G. In fact, taking a sequence $(W_n)_{n\in\mathbb{N}}$ of $\eta(0)$-neighborhoods in $\mathcal{S}$ identifying $\mathcal{N}(\eta)$, [1, Theorem 3.2] yields that

$$(\{a \in \mathcal{R} : \eta([0, a]) \subseteq W_n\})_{n\in\mathbb{N}}$$

is a sequence of 0-neighborhoods in G identifying $\mathcal{N}(\mu)$.

On applying Theorem 1.1 provides

$$\nu[\mathrm{AC}]\mu\,. \tag{3.4}$$

Coupling (3.3) with (3.4) concludes the proof. ■

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Addresses: Paola Cavaliere: Dipartimento di Matematica, Università di Salerno, Via Giovanni Paolo II, 84084 Fisciano (Sa), Italy;
Paolo de Lucia and Anna De Simone: Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università "Federico II", Via Cinthia, 80126 Napoli, Italy.

E-mail: pcavaliere@unisa.it, padeluci@unina.it, annades@unina.it

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