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On a Characteristic Function of the Tensor K-module of Inner Type Noncompact Real Simple Groups

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1. Introduction

Let $\mathbf{C}$ (resp. $\mathbf{R}$) denote the complex (resp. real) number field. We consider a connected simply connected complex simple Lie group $G_{\mathbf{C}}$ and a connected noncompact inner type simple real form G of $G_{\mathbf{C}}$. Let K be a maximal compact subgroup of G. We denote the Lie algebras of G and K respectively by $\mathfrak{g}$ and $\mathfrak{k}$. Let θ be the Cartan involution of $\mathfrak{g}$ corresponding to $\mathfrak{k}$. Let's denote the eigensubspace of θ of $\mathfrak{g}$ with the eigenvalue -1 by $\mathfrak{p}$. Then we have a Cartan decomposition: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Consequently the Lie algebra $\mathfrak{g}_{\mathbf{C}}$ of $G_{\mathbf{C}}$ is also decomposed by $\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_{\mathbf{C}}$, where $\mathfrak{k}_{\mathbf{C}}$ (resp. $\mathfrak{p}_{\mathbf{C}}$) is the complexification of $\mathfrak{k}$ (resp. $\mathfrak{p}$) in $\mathfrak{g}_{\mathbf{C}}$. Canonically K acts on the space $\mathfrak{p}_{\mathbf{C}}$. Let B be a maximal abelian subgroup of K. Since K is connected and G is an inner type simple Lie group, B is also a maximal abelian subgruop of G. Therefore B is a Cartan subgroup of G and K. Let $\mathfrak{b}_{\mathbf{C}}$ be the complexification of the Lie algebra $\mathfrak{b}$ of B. Let Σ be the root system of the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$. Then we have $\Sigma = \Sigma_K \cup \Sigma_n$, where Σ_K (resp. Σ_n) is the set of all compact (resp. noncompact) roots of Σ . We shall fix a positive root system P_K of Σ_K. Let (π_μ, V_μ) be a simple K-module with the highest weight μ. Then the tensor space $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ is a unitary K-module. Let ν be a P_K-dominant integral form on $\mathfrak{b}_{\mathbf{C}}$ and V_ν a simple K-module corresponding to ν. We define a projection operator P_ν on $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ by

$$P_\nu(Z) = \deg \pi_\nu \int_K kZ\overline{\operatorname{trace} \pi_\nu(k)}dk \quad \text{for } Z \text{ in } \mathfrak{p}_{\mathbf{C}} \otimes V_\mu\,,$$

where dk is the Haar measure on K normalized as $\int_K dk = 1$. Let Γ_K be the set of all P_K-dominant integral form on $\mathfrak{b}_{\mathbf{C}}$. Then we have the following decomposition:

$$\mathfrak{p}_{\mathbf{C}} \otimes V_\mu = \oplus_{\omega \in \Sigma_n, \mu+\omega \in \Gamma_K} P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu)\,, \tag{1.1}$$

where $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) = \{0\}$ or is a simple K-module. The purpose of this paper is to characterize nontrivial K-module $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$ by using a rational function. Let us state our results more precisely. We can prove that $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$ is nontrivial if and only if

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$|P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 \neq 0$, where $|*|$ is the norm on $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$, X_ω is the root vector corresponding to a noncompact root ω and $v(\mu)$ is the highest weight vector of V_μ normalized as $|v(\mu)| = 1$. Assume that $2(\mu, \alpha)|\alpha|^{-2} \geq 3$ for all α in P_K. Then we can prove (see Lemma 4.7) that $|P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2$ is given by a rational function $f(\lambda + \omega; \omega)$ in $\lambda = \mu + \rho_K$, where ρ_K is one half the sum of all roots in P_K. Let $(\sqrt{-1}\mathfrak{b})^*$ be the dual space of the real vector space $\sqrt{-1}\mathfrak{b}$. Let $f(\eta; \omega)$ be the rational function in $\eta \in (\sqrt{-1}\mathfrak{b})^*$ satisfying $f(\lambda+\omega; \omega) = |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2$. We can calculate $f(\eta; \omega)$ explicitely (see Theorem 6.5) by using the functional equations in Theorem 5.4. Finally in §7 we shall prove the following main theorem.

MAIN THEOREM. *Let μ be a P_K-dominant integral form on $\mathfrak{b}_{\mathbf{C}}$ and V_μ the simple K-module with the highest weight μ. Suppose that $\mu + \omega$ is P_K-dominant for a noncompact root ω in Σ. Then the K-submodule $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$ of $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ in* (1, 1) *is nontrivial if and only if $f(\lambda + \omega; \omega) > 0$.*

The tensor K-modules $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ and $\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ are closely related with the classification of irreducible infinitesimal unitary representations of G. For example, by using the Clebsch-Gordan coefficients of these tensor K-modules, the complete classifications are obtained for the groups : $SL(2, \mathbf{R})$ in [1], De Sitter group in [2] and [10], $SO(2n, 1)$ in [5], [6] and $SU(n, 1)$ in [8] and etc. In the subsequent paper we shall apply the main theorem to determine the multiplicity of V_μ in $\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu$.

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2. Preliminalies

Let G be the connected inner type noncompact real simple Lie group. We shall always fix a maximal compact subgroup K and the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let B be the maximal abelian subgroup of K. Since G is inner, B is a Cartan subgroup of K and G. A linear form α on $\mathfrak{b}_{\mathbf{C}}$ is said to be a root if there exists a nontrivial element X in $\mathfrak{g}_{\mathbf{C}}$ such that $[H, X] \equiv ad(H)X = \alpha(H)X$ for all H in $\mathfrak{b}_{\mathbf{C}}$. Let Σ be the set of all roots on $\mathfrak{b}_{\mathbf{C}}$. Then Σ is a finite set. Furthermore, we have the following decomposition.

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{b}_{\mathbf{C}} \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha ,$$

where $\mathfrak{g}_\alpha$ is a one dimensional eigenspace corresponding to α. The real subalgebra $\mathfrak{g}_u = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$ of $\mathfrak{g}_{\mathbf{C}}$ is said to be a compact real form of $\mathfrak{g}_{\mathbf{C}}$. We choose a Weyl basis $X_\alpha \in \mathfrak{g}_\alpha, \alpha \in \Sigma$, satisfying the followings (cf. the proof of Theorem 6.3 in [4]).

$$X_\alpha - X_{-\alpha},\ \sqrt{-1}(X_\alpha + X_{-\alpha}) \in \mathfrak{g}_u \quad \text{and} \quad \phi(X_\alpha, X_{-\alpha}) = 1\,, \tag{2.1}$$

where ϕ is the Killing form on $\mathfrak{g}_{\mathbf{C}}$. For the element $H_\alpha = ad(X_\alpha)X_{-\alpha}$ in $\sqrt{-1}\mathfrak{b}$, we have $\phi(H_\alpha, H) = \alpha(H)$ for all H in $\mathfrak{b}_{\mathbf{C}}$. Let μ be a linear form on $\sqrt{-1}\mathfrak{b}$. Then there exists a unique H_μ in $\sqrt{-1}\mathfrak{b}$ such that $\phi(H_\mu, H) = \mu(H)$ for all H in $\sqrt{-1}\mathfrak{b}$. Let $(\sqrt{-1}\mathfrak{b})^*$ be the dual space of $\sqrt{-1}\mathfrak{b}$. We define a positive definite bilinear form (λ, μ) by $(\lambda, \mu) = \phi(H_\mu, H_\lambda)$ for $\lambda, \mu \in (\sqrt{-1}\mathfrak{b})^*$. We put, for each pair of α and β in Σ, a complex number $\langle\alpha, \beta\rangle$ by

$$\langle\alpha, \beta\rangle = \begin{cases} \phi(ad(X_\alpha)X_\beta, X_{-\alpha-\beta}) & \text{if } \alpha + \beta \in \Sigma \\ 0 & \text{otherwise}. \end{cases} \tag{2.2}$$

Then $\langle\alpha, \beta\rangle$ is a pure imaginary number. Let p and q be two nonnegative integers such that $\beta + j\alpha \in \Sigma$ if and only if $-q \leq j \leq p$. $\beta + j\alpha, -q \leq j \leq p$, is said to be the α-series containing β. We have (cf. the proof of Lemma 4.3.8 in [11])

$$2(\beta, \alpha)|\alpha|^{-2} = q - p \quad \text{and} \quad \beta - 2(\beta, \alpha)|\alpha|^{-2}\alpha \in \Sigma\,. \tag{2.3}$$

Furthermore, we have

$$|\,\langle\alpha, \beta\rangle\,|^2 = q(p+1)\frac{|\alpha|^2}{2}\,, \tag{2.4}$$

and $p + q \leq 3$ (cf. Corollary 4.3.12 in [11]). Suppose that $|\alpha| \geq |\beta|$. Then

$$2(\alpha, \beta)|\beta|^{-2} \in \{0, \pm 1, \pm 2, \pm 3\}\,. \tag{2.5}$$

We remark that if $|\alpha| > |\beta|$, then $|\alpha|^2 = 2|\beta|^2$ or $|\alpha|^2 = 3|\beta|^2$. Especially if $2(\alpha, \beta)|\beta|^2 = \pm 2$ (resp. ± 3), then $|\alpha|^2 = 2|\beta|^2$ (resp. $|\alpha|^2 = 3|\beta|^2$).

A root in Σ is compact (resp. noncompact) if $X_\alpha \in \mathfrak{k}_{\mathbf{C}}$ (resp. $X_\alpha \in \mathfrak{p}_{\mathbf{C}}$). Since $\mathfrak{k}_{\mathbf{C}}$ and $\mathfrak{p}_{\mathbf{C}}$ are invariant under $ad(\mathfrak{b})$, Σ is a disjoint union of the set of all compact roots Σ_K and the set of all noncompact roots Σ_n. Σ_K is also the root system of the pair $(\mathfrak{k}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$. Let P be a positive root system of Σ. Then $P_K = \Sigma_K \cap P$ is a positive root system of Σ_K. A linear form μ on $\mathfrak{b}_{\mathbf{C}}$ is integral if $2(\mu, \alpha)|\alpha|^{-2}$ is an integer for all $\alpha \in P$, and μ is P-dominant (resp. P_K-dominant) if $2(\mu, \alpha)|\alpha|^{-2} \geq 0$ for all $\alpha \in P$ (resp. P_K). We shall denote the set of all P-dominant (resp. P_K-dominant) integral forms on $\mathfrak{b}_{\mathbf{C}}$ by Γ (resp. Γ_K).

Let σ (resp. τ) be the conjugation of $\mathfrak{g}_{\mathbf{C}}$ with respect to the real form $\mathfrak{g}$ (resp. $\mathfrak{g}_u$). By our choice for the Weyl basis of $\mathfrak{g}_{\mathbf{C}}$ we have

$$\sigma(X_\alpha) = -X_{-\alpha} \quad \text{for } \alpha \in \Sigma_K\,, \quad \sigma(X_\alpha) = X_{-\alpha} \quad \text{for } \alpha \in \Sigma_n \tag{2.6}$$

and

$$\tau(X_\alpha) = -X_{-\alpha} \quad \text{for } \alpha \in \Sigma\,. \tag{2.7}$$

The inner types noncompact real simple Lie groups (i.e. rank G = rank K) are classified into three (cf. Table II, p. 354 in [4]):

(2.8)
(1) all noncompact roots have the same length,
(2) $S_p(n, \mathbf{R})$ and $SO(2m, 2n+1)$,
(3) the type G_2.

We shall use in §6 and §7 the following Dynkin diagrams of the simple root systems Ψ for the groups in (2) and (3), where a white circle indicates a compact root and a black circle does a noncompact root.

(2.9)
$$G = S_p(n, \mathbf{R}) : \overset{\alpha_1}{\circ} — \overset{\alpha_2}{\circ} — \cdots \overset{\alpha_{n-2}}{\circ} — \overset{\alpha_{n-1}}{\circ} \Longleftarrow \overset{\alpha_n}{\bullet},$$
$$K = U(n) : \overset{\alpha_1}{\circ} — \overset{\alpha_2}{\circ} — \cdots \overset{\alpha_{n-2}}{\circ} — \overset{\alpha_{n-1}}{\circ}.$$

(2.10)
$$G = SO(2m, 2n+1) : \overset{\alpha_1}{\circ} — \overset{\alpha_2}{\circ} \cdots — \overset{\alpha_{m-1}}{\circ} — \overset{\alpha_m}{\bullet} — \overset{\alpha_{m+1}}{\circ} \cdots \overset{\alpha_{m+n-1}}{\circ} \Longrightarrow \overset{\alpha_{m+n}}{\circ},$$
$$K = SO(2m) \otimes SO(2n+1) : \overset{\alpha_1}{\circ} — \overset{\alpha_2}{\circ} \cdots \underset{\substack{| \\ \circ\ \alpha_0}}{\overset{\alpha_{m-2}}{\circ}} — \overset{\alpha_{m-1}}{\circ} \quad \overset{\alpha_{m+1}}{\circ} — \cdots \overset{\alpha_{m+n+1}}{\circ} \Longrightarrow \overset{\alpha_{m+n}}{\circ},$$

where $\alpha_0 = \alpha_{m-1} + 2\alpha_m + \cdots + 2\alpha_{m+n}$.

(2.11)
$$G = G_2 : \overset{\alpha_1}{\bullet} \Rrightarrow \overset{\alpha_2}{\circ}, \ K = SU(2) \otimes SU(2) : \overset{\alpha_0}{\circ} \ \overset{\alpha_2}{\circ},$$

where $\alpha_0 = 2\alpha_1 + 3\alpha_2$.

3. Decomposition of a tensor K-module

For the simplicity of our notations, the adjoint action $\mathrm{Ad}(k)$ $(k \in K)$ on $\mathfrak{p}_{\mathbf{C}}$ will be denoted by kX for X in $\mathfrak{p}_{\mathbf{C}}$. We define a hermitian structure (X, Y) on $\mathfrak{p}_{\mathbf{C}}$ by

(3.1)
$$(X, Y) = -\phi(X, \tau(Y)) \quad \text{for } X, Y \in \mathfrak{p}_{\mathbf{C}}.$$

Then $\mathfrak{p}_{\mathbf{C}}$ is a unitary K-module with respect to this hermitian structure. For $\mu \in \Gamma_K$, there exists a unitary simple K-module (π_μ, V_μ) with the highest weight μ. We also denote the action $\pi_\mu(k)$ $(k \in K)$ of K on V_μ by kv for $v \in V_\mu$. Let dk be the Haar measure on K normalized as $\int_K dk = 1$. We define a character χ_μ of the K-module (π_μ, V_μ) by

(3.2)
$$\chi_\mu(k) = \deg(\pi_\mu) \operatorname{trace} \pi_\mu(k), \quad k \in K,$$

where $\deg \pi_\mu = \dim V_\mu$. Then we have

(3.3)
$$\int_K \chi_\mu(k^{-1}k')\chi_\mu(k)dk = \chi_\mu(k'), \quad k' \in K.$$

For a finite dimensional K-module V, we define a projection operator P_μ on V by

(3.4)
$$P_\mu(v) = \int_K kv\overline{\chi_\mu(k)}dk, \quad v \in V,$$

where $\overline{\chi_\mu(k)}$ is the complex conjugate of $\chi_\mu(k)$.

LEMMA 3.1. *The projection operator P_μ on V satisfies the followings.*

$$(P_\mu)^2 = P_\mu \quad \textit{and} \quad kP_\mu = P_\mu k \quad \textit{for all } k \in K\,.$$

PROOF. Changing the variables and the order of integrals, we have for $v \in V$,

$$\begin{aligned}(P_\mu)^2(v) &= \int_K \int_K k'kv\overline{\chi_\mu(k')\chi_\mu(k)}dkdk' \\ &= \int_K \int_K kv\overline{\chi_\mu((k')^{-1}k)\chi_\mu(k')}dkdk' \\ &= \int_K kv \int_K \overline{\chi_\mu((k')^{-1}k)\chi_\mu(k')}dk'dk\,.\end{aligned}$$

Hence by the formula (3.3), we have $(P_\mu)^2 = P_\mu$. For $k \in K$ and $v \in V$ we have

$$\begin{aligned}kP_\mu(v) &= \int_K kk'v\overline{\chi_\mu(k')}dk' \\ &= \int_K (kk'k^{-1})(kv)\overline{\chi_\mu(kk'k^{-1})}dk' \\ &= P_\mu(kv)\,.\end{aligned}$$

Thus we can prove that $kP_\mu v = P_\mu kv$.

We now define an action of $\mathfrak{k}_{\mathbf{C}}$ on V_μ by

$$Xv = \frac{d}{dt}\exp(tX)v|_{t=0} \quad \text{for } X \in \mathfrak{k}_{\mathbf{C}} \text{ and } v \in V_\mu\,.$$

By the choice of X_α in (2.1) we have

(3.5) $$(X_\alpha v, w) = (v, X_{-\alpha}w) \quad \text{for all } \alpha \in \Sigma_K \text{ and } v, w \in V_\mu\,.$$

We define a unitary K-module structure on $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ by

(3.6) $$\begin{aligned}k(X \otimes v) &= kX \otimes kv \quad \text{for } k \in K\,, \\ (X \otimes v, Y \otimes w) &= (X, Y)(v, w) \quad \text{for } X, Y \in \mathfrak{p}_{\mathbf{C}} \text{ and } v, w \in V_\mu\,.\end{aligned}$$

Thereby $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ is a finite dimensional unitary K-module. Let ω be a noncompact root in Σ. Assume that $\mu + \omega$ is P_K-dominant. By the second property in Lemma 3.1 we have

(3.7) $$\begin{gathered}YP_{\mu+\omega}(X \otimes v) = P_{\mu+\omega}(ad(Y)X \otimes v) + P_{\mu+\omega}(X \otimes Yv) \\ \text{for all } Y \in \mathfrak{k}_{\mathbf{C}},\ X \in \mathfrak{p}_{\mathbf{C}} \text{ and } v \in V_\mu\,.\end{gathered}$$

DEFINITION 3.2. Let p be a nonnegative integer. We define a set Π_p by

$$\Pi_0 = \{\tilde{\phi}\}\,, \quad \Pi_p = \{(\alpha_1, \alpha_2, \cdots, \alpha_p) : \alpha_i \in P_K\} \quad \text{for } p > 1\,, \quad \text{and put } \Pi = \bigcup_{p=0}^{\infty} \Pi_p\,.$$

Let $I = (\alpha_1, \alpha_2, \cdots, \alpha_p)$ and $J = (\beta_1, \beta_2, \cdots, \beta_q)$ be two elements in Π. We define a multiplicative operation $\star$ in Π by

$$I \star J = (\alpha_1, \alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, \cdots, \beta_q)\,.$$

Then Π is a semigroup with the identity $\tilde{\phi}$.

DEFINITION 3.3. Let $U(\mathfrak{k}_{\mathbf{C}})$ be the universal enveloping algebra of $\mathfrak{k}_{\mathbf{C}}$. For each I in Π we define an element $Q(I)$ in $U(\mathfrak{k}_{\mathbf{C}})$ by

$$Q(I) = 1 \quad \text{for } I = \tilde{\phi} \quad \text{and} \quad Q(I) = X_{-\alpha_1} X_{-\alpha_2} \cdots X_{-\alpha_p} \quad \text{for } I = (\alpha_1, \alpha_2, \cdots, \alpha_p)\,.$$

Then Q is a semigroup homomorphism of Π to $U(\mathfrak{k}_{\mathbf{C}})$. Furthermore, $Q(I)$ acts on $\mathfrak{p}_{\mathbf{C}}$ by $Q(I)X = ad(Q(I))X$ for X in $\mathfrak{p}_{\mathbf{C}}$. We also define the adjoint operator $Q(I)^*$ of $Q(I)$ by $(Q(I)X, Y) = (X, Q(I)^*Y)$ for $X, Y \in \mathfrak{p}_{\mathbf{C}}$.

LEMMA 3.4. *Let $\mu \in \Gamma_K$ and V_μ a simple K-module with the highest weight μ. Then we have*

$$\mathfrak{p}_{\mathbf{C}} \otimes V_\mu = \bigoplus_{\omega \in \Sigma_n, \mu+\omega \in \Gamma_K} P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu)\,,$$

where $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) = \{0\}$ or is a simple K-module.

PROOF. By Peter-Weyl's theorem, we have (cf. Theorem 1.12 (c) in [7])

$$\mathfrak{p}_{\mathbf{C}} \otimes V_\mu = \bigoplus_{\lambda \in \Gamma_K} P_\lambda(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu)\,.$$

Let V_λ be a simple K-submodule of $P_\lambda(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$. We shall prove that $V_\lambda = P_{\mu+\gamma}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$ for a suitable noncompact root γ. We note that the simple K-module V_μ is generated by the set $\{Q(I)v(\mu) : I \in \Pi\}$, where $v(\mu)$ is the highest weight vector of V_μ normalized as $|v(\mu)| = 1$. Moreover, it follows from (3.7) that

$$X_\alpha P_\lambda(X \otimes v) = P_\lambda(ad(X_\alpha)X \otimes v) + P_\lambda(X \otimes X_\alpha v) \tag{3.8}$$

for all $X \in \mathfrak{p}_{\mathbf{C}}$, $v \in V_\mu$ and $\alpha \in \Sigma_K$. Let $v(\lambda)$ be the highest weight vector of V_λ. It follows from (3.8) that $v(\lambda)$ is written by

$$v(\lambda) = \sum_{\omega \in \Sigma_n} \sum_{I \in \Pi} c_{\omega, I}\, Q(I) P_\lambda(X_\omega \otimes v(\mu))\,, \tag{3.9}$$

where $c_{\omega, I}$ is a complex constant. Since $v(\lambda)$ is the highest weight vector, (3.5) implies that

$$(v(\lambda), v(\lambda)) = \sum_{\omega \in \Sigma_n} \overline{c_{\omega, \tilde{\phi}}}(v(\lambda), P_\lambda(X_\omega \otimes v(\mu)))\,.$$

Consequently, we have $\lambda = \mu + \gamma$ for a noncompact root γ. Again by (3.9) we have

$$v(\lambda) = \sum_{\omega \in \Sigma_n} \sum_{I \in \Pi} c_{\omega, I}\, Q(I) P_{\mu+\gamma}(X_\omega \otimes v(\mu))\,.$$

Let ω be a noncompact root. When $\omega > \gamma$, we have $P_{\mu+\gamma}(X_\omega \otimes v(\mu)) = 0$ because $\mu + \gamma$ is the highest weight in $P_{\mu+\gamma}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$. When $\omega < \gamma$, the weight of $Q(I)P_{\mu+\gamma}(X_\omega \otimes v(\mu))$ is distinct to $\mu + \gamma$. Hence we have $v(\lambda) = c_{\gamma,\tilde{\phi}} P_{\mu+\gamma}(X_\gamma \otimes v(\mu))$. This implies that $V_\lambda = P_{\mu+\gamma}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$.

In view of the proof of the above lemma we have the following.

COROLLARY 3.5. *Let ω be a noncompact root in Σ. If $\mu + \omega \in \Gamma_K$ and $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$, then we have $P_{\mu+\omega}(X_\omega \otimes v(\mu)) \neq 0$.*

LEMMA 3.6. *Let ω be a noncompact root in Σ, and suppose that $\mu + \omega \in \Gamma_K$, $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$. If $P_{\mu+\omega}(X_\gamma \otimes v(\mu)) \neq 0$ for a noncompact root γ, then we have*

$$(|\lambda + \omega|^2 - |\lambda + \gamma|^2)|P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 = \sum_{\alpha \in P_K} 2|\langle \alpha, \gamma \rangle|^2 |P_{\mu+\omega}(X_{\gamma+\alpha} \otimes v(\mu))|^2 ,$$

where $v(\mu)$ is the highest weight vector in V_μ, $\lambda = \mu + \rho_K$ and ρ_K is one half the sum of all roots in P_K.

PROOF. Let Ω_K be the Casimir operator on K given by

$$\Omega_K = \sum_{i=1}^{\ell} (H_i)^2 + H_{2\rho_K} + \sum_{\alpha \in P_K} 2X_{-\alpha}X_\alpha ,$$

where $\{H_1, H_2, \cdots, H_\ell\}$ is an orthonormal basis of $\sqrt{-1}\mathfrak{b}$ with respect to the Killing form ϕ. Since $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$ is a simple K-module, Ω_K is a scalar operator on this space. We can verify $\Omega_K v(\mu + \omega) = (|\lambda + \omega|^2 - |\rho_K|^2)v(\mu + \omega)$, where $v(\mu + \omega)$ is the highest weight vector of $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$. Then we have for $\gamma \in \Sigma_n$,

$$\Omega_K P_{\mu+\omega}(X_\gamma \otimes v(\mu)) = (|\lambda + \omega|^2 - |\rho_K|^2) P_{\mu+\omega}(X_\gamma \otimes v(\mu)) .$$

On the other hand, since

$$\begin{aligned} \Omega_K P_{\mu+\omega}(X_\gamma \otimes v(\mu)) &= (|\lambda + \gamma|^2 - |\rho_K|^2) P_{\mu+\omega}(X_\gamma \otimes v(\mu)) \\ &\quad + \sum_{\alpha \in P_K} 2\langle \alpha, \gamma \rangle X_{-\alpha} P_{\mu+\omega}(X_{\gamma+\alpha} \otimes v(\mu)) , \end{aligned}$$

we have

$$(|\lambda + \omega|^2 - |\lambda + \gamma|^2) P_{\mu+\omega}(X_\gamma \otimes v(\mu)) = \sum_{\alpha \in P_K} 2\langle \alpha, \gamma \rangle X_{-\alpha} P_{\mu+\omega}(X_{\gamma+\alpha} \otimes v(\mu)) .$$

Consequently, by (3.5) we have

$$\begin{aligned}(|\lambda+\omega|^2-|\lambda+\gamma|^2)|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2 \\ &= \sum_{\alpha\in P_K} 2\,\langle\alpha,\gamma\rangle\,(X_{-\alpha}P_{\mu+\omega}(X_{\gamma+\alpha}\otimes v(\mu)),\,P_{\mu+\omega}(X_\gamma\otimes v(\mu))) \\ &= \sum_{\alpha\in P_K} 2|\,\langle\gamma,\alpha\rangle\,|^2|P_{\mu+\omega}(X_{\gamma+\alpha}\otimes v(\mu))|^2\,.\end{aligned}$$

DEFINITION 3.7. Let γ and ω be two noncompact roots. We put

$$\Pi(\gamma;\omega)=\{I\in\Pi : Q(I)^*X_\gamma\in\mathfrak{g}_\omega\backslash\{0\}\}\,.$$

LEMMA 3.8. *Suppose that $\mu+\omega\in\Gamma_K$ and $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu)\neq\{0\}$ for $\omega\in\Sigma_n$, and let γ be a root in Σ_n. Then $P_{\mu+\omega}(X_\gamma\otimes v(\mu))\neq 0$ if and only if $\Pi(\gamma;\omega)\neq\phi$. Moreover if $P_{\mu+\omega}(X_\gamma\otimes v(\mu))\neq 0$, then $|\lambda+\omega|^2-|\lambda+\gamma|^2>0$.*

PROOF. By Corollary 3.5 $P_{\mu+\omega}(X_\omega\otimes v(\mu))$ is the highest weight vector of the simple K-module $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes v(\mu))$. Assume that $P_{\mu+\omega}(X_\gamma\otimes v(\mu))\neq 0$. When $\omega=\gamma$ we have $\tilde{\phi}\in\Pi(\omega;\omega)$. Suppose $\omega\neq\gamma$. Since $P_{\mu+\omega}(X_\gamma\otimes v(\mu))$ is not the highest weight vector,

(3.10) there is $\beta\in P_K$ such that $X_\beta P_{\mu+\omega}(X_\gamma\otimes v(\mu))\neq 0$.

Similarly, since the dimension of the space of the highest vectors is one, we can choose $I\in\Pi$ satisfying $Q(I)^*P_{\mu+\omega}(X_\gamma\otimes v(\mu))\in\mathbf{C}P_{\mu+\omega}(X_\omega\otimes v(\mu))\backslash\{0\}$. Since $X_\alpha v(\mu)=0$ for $\alpha\in P_K$, we have

$$Q(I)^*P_{\mu+\omega}(X_\gamma\otimes v(\mu))=P_{\mu+\omega}(Q(I)^*X_\gamma\otimes v(\mu))\,,$$

and hence, $I\in\Pi(\gamma;\omega)$. Conversely assume that $I\in\Pi(\gamma;\omega)$. Since $Q(I)^*X_\gamma\in\mathfrak{g}_\omega\backslash\{0\}$, we have $Q(I)^*P_{\mu+\omega}(X_\gamma\otimes v(\mu))\neq 0$. This implies that $P_{\mu+\omega}(X_\gamma\otimes v(\mu))\neq 0$. The inequality $|\lambda+\omega|^2-|\lambda+\gamma|^2>0$ for $P_{\mu+\omega}(X_\gamma\otimes v(\mu))\neq 0$ follows from Lemma 3.6 and (3.10).

4. Rational function associated with the coefficient $|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2$

The purpose of this section is to prove Lemma 4.7. In order to prove this lemma we shall prepare three lemmas after the following two definitions.

DEFINITION 4.1. For a generic point η in $(\sqrt{-1}\mathfrak{b})^*$, $\omega\in\Sigma_n$ and $I\in\Pi$, we define $R(\eta;I)$, $S(\eta;I)$, $T(\eta;I)$, $a_\omega(I)$ $(I\in\Pi)$ and $f(\eta;\omega)$ as follows:

$$R(\eta;\tilde{\phi})=S(\eta;\tilde{\phi})=T(\eta;\tilde{\phi})=a_\omega(\tilde{\phi})=1$$

and for $I = (\alpha_1, \alpha_2, \cdots, \alpha_p) \in \Pi$

$$
\begin{aligned}
R(\eta; I) &= (|\eta + \langle I \rangle|^2 - |\eta|^2)^{-1}, \\
S(\eta; I) &= \prod_{J, L \in \Pi, J \star L = I, J \neq \tilde{\phi}} R(\eta; J), \\
T(\eta; I) &= \prod_{J, L \in \Pi, J \star L = I} R(\eta + \langle J \rangle; L), \\
a_\omega(I) &= 2^{\sharp I} |\phi(Q(I)^* X_\omega, X_{-\omega - \langle I \rangle})|^2, \\
f(\eta; \omega) &= \sum_{I \in \Pi} (-1)^{\sharp I} a_\omega(I) S(\eta; I),
\end{aligned}
\tag{4.1}
$$

where $\sharp I = p$ and $\langle I \rangle = \sum_{i=1}^{p} \alpha_i$.

For $\gamma \in \Sigma_n$, $\alpha \in P_K$ and $J, L \in \Pi$, we have

$$a_\gamma(\alpha) a_{\gamma+\alpha}(J) = a_\gamma(\alpha \star J), \tag{4.2}$$

$$R(\eta; J) + R(\eta + \langle J \rangle; L) = R(\eta + \langle J \rangle; L) R(\eta; J) R(\eta; J \star L)^{-1}, \tag{4.3}$$

$$S(\eta; L \star \alpha) = S(\eta; L) R(\eta; L \star \alpha), \tag{4.4}$$

$$T(\eta; \alpha \star J) = T(\eta + \alpha; J) R(\eta; \alpha \star J). \tag{4.5}$$

DEFINITION 4.2. Let ω and γ be two noncompact roots. When $\Pi(\gamma; \omega) \neq \phi$ (see Definition 3.7), we define $n(\gamma; \omega)$ as the maximal integer of the set $\{\sharp I : I \in \Pi(\gamma; \omega)\}$.

LEMMA 4.3. *Assume that $\mu + \omega \in \Gamma_K$ and $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes v(\mu)) \neq \{0\}$ for a noncompact root ω. If $P_{\mu+\omega}(X_\gamma \otimes v(\mu)) \neq 0$, then we have*

$$|P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 = \sum_{I \in \Pi(\gamma; \omega)} a_\gamma(I) T(\lambda + \gamma; I) |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2. \tag{4.6}$$

PROOF. By Lemma 3.8 $P_{\mu+\omega}(X_\gamma \otimes v(\mu)) \neq 0$ if and only if $n(\gamma; \omega) \geq 0$. We shall prove (4.6) by using an induction on $n(\gamma; \omega) \geq 0$. When $n(\gamma; \omega) = 0$, our assertion is obvious. Assume that the lemma is true for all $\delta \in \Sigma_n$ satisfying $0 \leq n(\delta; \omega) < n(\gamma; \omega)$. Let α be an element in P_K satisfying $n(\gamma + \alpha; \omega) \geq 0$. Since $\alpha \star I \in \Pi(\gamma; \omega)$ for $I \in \Pi(\gamma + \alpha; \omega)$, we have $0 \leq n(\gamma + \alpha; \omega) < n(\gamma; \omega)$. By the hypothesis of our induction we have

$$
\begin{aligned}
&|P_{\mu+\omega}(X_{\gamma+\alpha} \otimes v(\mu))|^2 \\
&\quad = \sum_{I \in \Pi(\gamma+\alpha; \omega)} a_{\gamma+\alpha}(I) T(\lambda + \gamma + \alpha; I) |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2.
\end{aligned}
\tag{4.7}
$$

Since $|\mu+\omega|^2 - |\mu+\gamma|^2 > 0$ (see Lemma 3.8), (4.2) and (4.5) imply

$$\begin{aligned}&\frac{2|\langle \alpha, \gamma\rangle|^2}{|\lambda+\omega|^2 - |\lambda+\gamma|^2} a_{\gamma+\alpha}(I) T(\lambda+\gamma+\alpha; I)\\ &\quad = a_\gamma(\alpha) a_{\gamma+\alpha}(I) R(\lambda+\gamma; \alpha \star I) T(\lambda+\gamma+\alpha; I)\\ &\quad = a_\gamma(\alpha \star I) T(\lambda+\gamma; \alpha\star I)\,.\end{aligned}$$

Hence by Lemma 3.6 and (4.7), we have

$$\begin{aligned}&|P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2\\ &\quad = \sum_{\alpha\in P_K} \sum_{I\in \Pi(\gamma+\alpha;\omega)} a_\gamma(\alpha\star I) T(\lambda+\gamma; \alpha\star I) |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2\\ &\quad = \sum_{I\in\Pi(\gamma;\omega)} a_\gamma(I) T(\lambda+\gamma; I) |P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2\,.\end{aligned}$$

Let P be a positive root system containing P_K and $\Psi = \{\beta_1, \beta_2, \cdots, \beta_\ell\}$. We define $\lambda_i \in (\sqrt{-1}\mathfrak{b})^*$ by

$$2(\lambda_i, \beta_j)|\beta_j|^{-2} = \delta_{i,j}\,, \quad 1 \le i, j \le \ell\,, \tag{4.8}$$

where $\delta_{i,j}$ is Kronecker's delta. For $\eta \in (\sqrt{-1}\mathfrak{b})^*$ we have

$$\eta = \sum_{i=1}^{\ell} \eta_i \lambda_i\,,\ \eta_i = 2(\eta, \beta_i)|\beta_i|^{-2}\,.$$

Let $\mathbf{R}[\eta] = \mathbf{R}[\eta_1, \eta_2, \cdots, \eta_\ell]$ be the ring of all polynomials in $\eta_1, \eta_2, \cdots, \eta_\ell$ over the real number field $\mathbf{R}$. The quotient field of $\mathbf{R}[\eta]$ will be denoted by $\mathbf{R}(\eta)$.

LEMMA 4.4. *Let I $(\neq \tilde{\phi})$ be an element in Π. Then we have*

$$(-1)^{(\sharp I)-1} S(\eta; I) = \sum_{J,L\in\Pi, J\star L = I, J\neq\tilde{\phi}} (-1)^{\sharp L} T(\eta; J) S(\eta + \langle J\rangle; L)\,.$$

PROOF. We put $F(\eta; I) = \sum_{J\star L = I} (-1)^{\sharp L} T(\eta; J) S(\eta+\langle J\rangle; L)$. Then the identity of this lemma is equivalent to $F(\eta; I) = 0$ in $\mathbf{R}(\eta)$. We shall prove that $F(\eta; I) = 0$ by using an induction on $\sharp I$. When $\sharp I = 1$ our assertion is obvious. Suppose that $\sharp I > 1$ and $F(\eta; J) = 0$ for all J in $\Pi_{(\sharp I)-1}$. We put $I = (\alpha_1, \alpha_2, \cdots, \alpha_p)$. By the definition of F, we have

$$F(\eta; I) = \sum_{J\star L = I, J\neq\tilde{\phi}, L\neq\tilde{\phi}} (-1)^{\sharp L} T(\eta; J) S(\eta+\langle J\rangle; L) + (-1)^{\sharp I} S(\eta; I) + T(\eta; I)\,. \tag{4.9}$$

We now put $I' = (\alpha_1, \alpha_2, \cdots, \alpha_{p-1})$ and $I'' = (\alpha_2, \alpha_3, \cdots, \alpha_p)$. By (4.4) and (4.5) we have

$$S(\eta; I) = S(\eta; I') R(\eta; I)\,, \quad T(\eta; I) = T(\eta+\alpha_1; I'') R(\eta; I)\,. \tag{4.10}$$

By the hypothesis of our induction we have the followings.

$$(-1)^{p-2}S(\eta;\,I') = \sum_{J\star L=I',\,J\neq\tilde{\phi}} (-1)^{\sharp L}T(\eta;\,J)S(\eta+\langle J\rangle\,;\,L),$$

$$\begin{aligned}&(-1)^{p-2}S(\eta+\alpha_1;\,I'')\\ &\quad= \sum_{J\star L=I'',\,J\neq\tilde{\phi}} (-1)^{\sharp L}T(\eta+\alpha_1;\,J)S(\eta+\alpha_1+\langle J\rangle\,;\,L)\\ &\quad= T(\eta+\alpha_1;\,I'') + \sum_{J\star L=I'',\,J\neq\tilde{\phi},\,L\neq\tilde{\phi}} (-1)^{\sharp L}T(\eta+\alpha_1;\,J)S(\eta+\alpha_1+\langle J\rangle\,;\,L)\,.\end{aligned}$$

These two identities imply that

$$\begin{aligned}&(-1)^{p}S(\eta;\,I') + T(\eta+\alpha_1;\,I'')\\ &\quad= \sum_{J\star L=I',\,J\neq\tilde{\phi}} (-1)^{\sharp L}T(\eta;\,J)S(\eta+\langle J\rangle\,;\,L)\\ &\qquad+(-1)^{p-2}S(\eta+\alpha_1;\,I'') - \sum_{J\star L=I'',\,J\neq\tilde{\phi},\,L\neq\tilde{\phi}} (-1)^{\sharp L}T(\eta+\alpha_1;\,J)S(\eta+\alpha_1+\langle J\rangle\,;\,L)\\ &\quad= \sum_{J\star L=I',\,J\neq\tilde{\phi}} (-1)^{\sharp L}T(\eta;\,J)S(\eta+\langle J\rangle\,;\,L)\\ &\qquad- \sum_{J\star L=I'',\,L\neq\tilde{\phi}} (-1)^{\sharp L}T(\eta+\alpha_1;\,J)S(\eta+\alpha_1+\langle J\rangle\,;\,L)\,.\end{aligned}$$

We put $I''' = (\alpha_2, \alpha_3, \cdots, \alpha_{p-1})$. Then by (4.4) and (4.5) we have

$$\begin{aligned}&(-1)^{p}S(\eta;\,I') + T(\eta+\alpha_1;\,I'')\\ &\quad= \sum_{J'\star L=I'''} (-1)^{\sharp L}T(\eta;\,\alpha_1\star J')S(\eta+\alpha_1+\left\langle J'\right\rangle;\,L)\\ &\qquad+ \sum_{J\star L'=I'''} (-1)^{\sharp L'}T(\eta+\alpha_1;\,J)S(\eta+\alpha_1+\langle J\rangle\,;\,L'\star\alpha_p)\\ &\quad= \sum_{J'\star L'=I'''} (-1)^{\sharp L'}T(\eta+\alpha_1;\,J')R(\eta;\,\alpha_1\star J')S(\eta+\langle\alpha_1\star J'\rangle;\,L')\\ &\qquad+ \sum_{J'\star L'=I'''} (-1)^{\sharp L'}T(\eta+\alpha_1;\,J')S(\eta+\langle\alpha_1\star J\rangle;\,L')R(\eta+\langle\alpha_1\star J'\rangle;\,L'\star\alpha_p)\\ &\quad= \sum_{J'\star L'=I'''} (-1)^{\sharp L'}T(\eta+\alpha_1;\,J')S(\eta+\langle\alpha_1\star J'\rangle;\,L')\\ &\qquad\times\{R(\eta;\,\alpha_1\star J') + R(\eta+\langle\alpha_1\star J'\rangle;\,L'\star\alpha_p)\}\,.\end{aligned}$$

By (4.3) and (4.10) we have

$$
\begin{aligned}
&(-1)^{\sharp I} S(\eta; I) + T(\eta; I) \\
&\quad = \sum_{J'\star L' = I'''} (-1)^{\sharp L'} T(\eta + \alpha_1; J') R(\eta; \alpha_1 \star J') \\
&\qquad \times S(\eta + \langle \alpha_1 \star J' \rangle; L') R(\eta + \langle \alpha_1 \star J' \rangle; L' \star \alpha_p) \\
&\quad = - \sum_{J\star L = I, J \neq \tilde{\phi}, L \neq \tilde{\phi}} (-1)^{\sharp L} T(\eta; J) S(\eta + \langle J \rangle; L)\,.
\end{aligned}
$$

Consequently, by (4.9) we have $F(\eta; I) = 0$ as claimed.

We now choose a positve root system P of Σ as follows. If (G, K) is a hermitian pair, then we choose P for which (cf. Proposition 7.2 in [4]) $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, where $\mathfrak{p}^\pm$ is the subspace of $\mathfrak{p}_{\mathbf{C}}$ generated by the root vectors corresponding to the noncompact positive (resp. negative) roots. If (G, K) is nonhermitian, then we choose a positive root system P containing P_K.

DEFINITION 4.5. We put, for the hermitian case, $\mathfrak{v} = \mathfrak{p}^\pm$ and for nonhermitian case $\mathfrak{v} = \mathfrak{p}_{\mathbf{C}}$. $\mathfrak{v}$ is a simple K-module. The set of all weights (roots) in $\mathfrak{v}$ will be denoted by $\Sigma_{\mathfrak{v}}$.

REMARK. If (G, K) is hermitian, then we have $\mathfrak{p}_{\mathbf{C}} = \mathfrak{v} \oplus \tau(\mathfrak{v})$ and $U(\mathfrak{k}_{\mathbf{C}})\mathfrak{v} \subset \mathfrak{v}$. These imply that $\Pi(\omega; \gamma) = \phi$ for $\omega \in \Sigma_{\mathfrak{v}}$ and $\gamma \in \Sigma_{\tau(\mathfrak{v})}$. Moreover since $\mathfrak{v} \otimes V_\mu$ and $\tau(\mathfrak{v}) \otimes V_\mu$ are orthogonal with respect to the hermitian product in (3.6), $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu = \mathfrak{v} \otimes V_\mu \oplus \tau(\mathfrak{v}) \otimes V_\mu$ as K-modules. By these properties the conclusions of Lemma 3.8 and Lemma 4.3, replacing $\mathfrak{p}_{\mathbf{C}}$ and Σ_n respectively with $\mathfrak{v}$ and $\Sigma_{\mathfrak{v}}$, are also true. Let W_K be the Weyl group of the pair $(\mathfrak{k}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$. Each s in W_K is realized by $s = \mathrm{Ad}(k)|_{\mathfrak{b}_{\mathbf{C}}}$, $k \in N_K(B)$, where $N_K(B)$ is the normalizer of B in K and $\mathrm{Ad}(k)|_{\mathfrak{b}_{\mathbf{C}}}$ the restriction of $\mathrm{Ad}(k)$ to $\mathfrak{b}_{\mathbf{C}}$. Thereby $\Sigma_{\mathfrak{v}}$ is W_K-invariant.

LEMMA 4.6. *Let (π_μ, V_μ) be a unitary simple K-module with the highest weight μ. Assume that $\mu + \omega \in \Gamma_K$ for all noncompact root ω in $\Sigma_{\mathfrak{v}}$. Then we have*

$$
\mathfrak{v} \otimes V_\mu = \oplus_{\omega \in \Sigma_{\mathfrak{v}}} P_{\mu+\omega}(\mathfrak{v} \otimes V_\mu),\ \ P_{\mu+\omega}(\mathfrak{v} \otimes V_\mu) \neq \{0\}\,.
$$

PROOF. There exists a finite covering group K^* of K such that the function $\xi_{\rho_K}(\exp H) = e^{\rho_K(H)} (H \in \mathfrak{b})$ is well- defined, where $X \to \exp(X)$ is the exponential mapping of $\mathfrak{k}$ to K^*. Let B^* be the Cartan subgroup of K^* corresponding to $\mathfrak{b}$. Define a function Δ_K on B^* by

$$
\Delta_K(\exp H) = \prod_{\alpha \in P_K} (e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)})\,.
$$

Applying Weyl's character formula to π_μ (cf. Theorem 4.46 in [7]), we have

$$
(\Delta_K \mathrm{trace}(\mathrm{Ad}|_{\mathfrak{v}} \otimes \pi_\mu))(\exp H) = \left(\sum_{\omega \in \Sigma_{\mathfrak{v}}} e^{\omega(H)} \right) \left(\sum_{t \in W_K} \varepsilon(t) e^{t(\mu+\rho_K)(H)} \right),
$$

where $\varepsilon(t)$ is the signature of t and $\mathrm{Ad}|_{\mathfrak{v}}$ is the restriction of the adjoint representation of K to $\mathfrak{v}$. Since

$$\sum_{\omega\in\Sigma_{\mathfrak{v}}} e^{\omega(H)} = \sum_{\omega\in\Sigma_{\mathfrak{v}}} e^{t\omega(H)} \quad \text{for all } \ t \in W_K,$$

it follows that

$$(\Delta_K \mathrm{trace}(\mathrm{Ad}|_{\mathfrak{v}} \otimes \pi_\mu))(\exp H) = \sum_{\omega\in\Sigma_{\mathfrak{v}}} \sum_{t\in W_K} \varepsilon(t) e^{t(\mu+\omega+\rho_K)(H)} . \tag{4.11}$$

We now assume that $\mu + \omega \in \Gamma_K$ for all $\omega \in \Sigma_{\mathfrak{v}}$, and let $\pi_{\mu+\omega}$ be the simple K-module with the highest weight $\mu + \omega$. By (4.11) we have

$$\mathrm{trace}(\mathrm{Ad}|_{\mathfrak{v}} \otimes \pi_\mu)(k) = \sum_{\omega\in\Sigma_{\mathfrak{v}}} \mathrm{trace}\, \pi_{\mu+\omega}(k) \quad \text{for all } \ k \in K \ ,$$

and thus, the assertion of this lemma.

LEMMA 4.7. *Assume that $\mu + \delta \in \Gamma_K$ for all noncompact roots δ. Then for ω in Σ_n we have*

$$|P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 = f(\lambda + \omega; \omega) ,$$

where $v(\mu)$ is the highest weight vector of V_μ normalized as $|v(\mu)| = 1$ and $\lambda = \mu + \rho_K$.

PROOF. We choose a K-module $\mathfrak{v}$ satisfying $\omega \in \Sigma_{\mathfrak{v}}$, and let γ_0 be the highest root in $\Sigma_{\mathfrak{v}}$. Since $\mathfrak{v}$ is a simple K-module, we have $n(\omega; \gamma_0) \geq 0$. We shall prove the identity in this lemma by using an induction on $n(\omega; \gamma_0)$. By Lemma 4.6 we have

$$X_\omega \otimes v(\mu) = \sum_{\gamma\in\Sigma_{\mathfrak{v}}} P_{\mu+\gamma}(X_\omega \otimes v(\mu)) .$$

This implies that

$$|P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 = 1 - \sum_{\gamma\in\Sigma_{\mathfrak{v}}, \gamma\neq\omega} |P_{\mu+\gamma}(X_\omega \otimes v(\mu))|^2 .$$

Moreover, since $P_{\mu+\gamma}(\mathfrak{v} \otimes V_\mu) \neq \{0\}$ for all $\gamma \in \Sigma_{\mathfrak{v}}$, Lemma 3.8 implies that $P_{\mu+\gamma}(X_\omega \otimes v(\mu)) \neq 0$ iff $\Pi(\omega; \gamma) \neq \phi$. When $\omega = \gamma_0$ we have $\Pi(\gamma_0; \gamma) = \phi$ for all $\gamma \neq \gamma_0, \gamma \in \Sigma_{\mathfrak{v}}$. Therefore $|P_{\mu+\gamma_0}(X_{\gamma_0} \otimes v(\mu))|^2 = 1$. On the other hand, since $\gamma_0 + \alpha \notin \Sigma_{\mathfrak{v}}$ for all $\alpha \in P_K$, we have $a_{\gamma_0}(I) = 0$ for all $I \neq \tilde{\phi}$, $I \in \Pi$. Thus by (4.1)

$$f(\lambda + \gamma_0; \gamma_0) = 1 = |P_{\mu+\gamma_0}(X_{\gamma_0} \otimes v(\mu))|^2 .$$

Let us now assume that the formula is true for all roots γ in $\Sigma_{\mathfrak{v}}$ satisfying $0 \leq n(\gamma; \gamma_0) < n(\omega; \gamma_0)$. To apply our inductive hypothesis we shall prove that if $\Pi(\omega; \gamma) \neq \phi$ and $\gamma \neq \omega$, then $n(\gamma; \gamma_0) < n(\omega; \gamma_0)$. Let I be an elemen in $\Pi(\omega; \gamma)$. Then $Q(I)^* X_\omega \in \mathfrak{g}_\gamma \backslash \{0\}$. Since $Q(J)^* X_\gamma \in \mathfrak{g}_{\gamma_0} \backslash \{0\}$ for $J \in \Pi(\gamma; \gamma_0)$, we have $Q(I \star J)^* X_\omega = Q(J)^* Q(I)^* X_\omega \in \mathfrak{g}_{\gamma_0} \backslash \{0\}$.

This implies that $I \star J \in \Pi(\omega; \gamma_0)$ for all $J \in \Pi(\gamma; \gamma_0)$. Since $n(\omega; \gamma_0) \geq \sharp(I \star J) = \sharp I + \sharp J$ and $\sharp I \geq 1$, we have $n(\gamma; \gamma_0) < n(\omega; \gamma_0)$. Applying Lemma 4.3 to $P_{\mu+\gamma}(X_\omega \otimes v(\mu))$ for $\gamma \neq \omega$, $\gamma \in \Sigma_{\mathfrak{v}}$ satisfying $\Pi(\omega; \gamma) \neq \phi$ we have

$$\begin{aligned}
&|P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 \\
&\qquad = 1 - \sum_{\gamma \in \Sigma_{\mathfrak{v}}, \gamma \neq \omega} \sum_{J \in \Pi(\omega;\gamma)} a_\omega(J) T(\lambda + \omega; J) |P_{\mu+\gamma}(X_\gamma \otimes v(\mu))|^2 ,
\end{aligned}$$

hence by the inductive hypothesis,

$$\begin{aligned}
&= 1 - \sum_{\gamma \neq \omega, J \in \Pi(\omega;\gamma)} \sum_{L \in \Pi} (-1)^{\sharp L} \times a_\omega(J) a_\gamma(L) T(\lambda + \omega; J) S(\lambda + \gamma; L) \\
&= 1 - \sum_{\gamma \neq \omega, J \in \Pi(\omega;\gamma)} \sum_{L \in \Pi} (-1)^{\sharp L} \times a_\omega(J) a_\gamma(L) T(\lambda + \omega; J) S(\lambda + \omega + \langle J \rangle; L) .
\end{aligned}$$

Since $a_\omega(J) a_\gamma(L) = a_\omega(J \star L)$ and $\cup_{\gamma \in \Sigma_{\mathfrak{v}}, \gamma \neq \omega} \Pi(\omega; \gamma) = \{J \in \Pi : J \neq \tilde{\phi}, a_\omega(J) \neq 0\}$, we have from Lemma 4.4

$$\begin{aligned}
|P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 &= 1 - \sum_{\sharp I \geq 1} \sum_{J \star L = I, J \neq \tilde{\phi}} (-1)^{\sharp L} a_\omega(I) T(\lambda + \omega; J) S(\lambda + \omega + \langle J \rangle; L) \\
&= \sum_{I \in \Pi} \sum_{J \star L = I, J \neq \tilde{\phi}} (-1)^{\sharp I} a_\omega(I) S(\lambda + \omega; I) \\
&= f(\lambda + \omega; \omega) .
\end{aligned}$$

REMARK. The assumption of this lemma is crucial to apply our induction. For example, it is not trivial that $P_{\mu+\omega}(\mathfrak{v} \otimes v(\mu)) = 0$ for $\omega \in \Sigma_{\mathfrak{v}}$, $\mu + \omega \in \Gamma_K$ implies $f(\lambda + \omega; \omega) = 0$.

The following two lemmas will be applied to prove Theorem 5.5.

LEMMA 4.8. *Let $\omega \in \Sigma_n$ and $\mu \in \Gamma_K$. Assume that $\mu + \omega \in \Gamma_K$ and $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$. Then we have*

$$\sum_{\gamma \in \Sigma_n} |P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 = f(-\lambda - \omega; -\omega) |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 .$$

PROOF. We can assume that $\omega \in \Sigma_{\mathfrak{v}}$. Since $P_{\mu+\omega}(\tau(\mathfrak{v}) \otimes V_\mu) = \{0\}$, it is sufficient to prove that

$$\sum_{\gamma \in \Sigma_{\mathfrak{v}}} |P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 = f(-\lambda - \omega; -\omega) |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 .$$

Let γ be an element in $\Sigma_{\mathfrak{v}}$ satisfying $n(\gamma; \omega) \geq 1$. First we define a mapping ψ of $\Pi(\gamma; \omega)$ to $\Pi(-\omega; -\gamma)$ by

$$\psi(I) = (\alpha_p, \alpha_{p-1}, \cdots, \alpha_1) \quad \text{for } I = (\alpha_1, \alpha_2, \cdots, \alpha_p) \in \Pi(\gamma; \omega) .$$

Actually, since

$$\begin{aligned}\phi(ad(Q(I)^*)X_\gamma, X_{-\omega}) &= \phi(ad(X_{\alpha_p}X_{\alpha_{p-1}}\cdots X_{\alpha_1})X_\gamma, X_{-\omega})\\ &= (-1)^p\phi(X_\gamma, ad(X_{\alpha_1}X_{\alpha_2}\cdots X_{\alpha_p})X_{-\omega})\\ &= (-1)^p\phi(Q(\psi(I))^*X_{-\omega}, X_\gamma)\,,\end{aligned}$$

Definition 3.7 implies that $\psi(I) \in \Pi(-\omega, -\gamma)$. Furthermore, we have (see Definition 4.1)

$$a_\gamma(I) = a_{-\omega}(\psi(I)) \quad \text{for} \quad I \in \Pi(\gamma;\omega)\,, \tag{4.12}$$

and ψ is bijective, because ψ^2 is the identity on $\Pi(\gamma,\omega)$. Next we shall prove that

$$\begin{aligned}&T(\lambda+\gamma; I) = (-1)^{\sharp I}S(-\lambda-\omega;\psi(I)) \quad \text{for all } \gamma \in \Sigma_{\mathfrak{v}} \text{ and}\\ &I \in \Pi(\gamma;\omega) \text{ satisfying } n(\gamma;\omega) \geq 1\,,\end{aligned} \tag{4.13}$$

by an induction on $n(\gamma;\omega) \geq 1$. Suppose that $n(\gamma;\omega) = 1$. Then we have immediately $T(\lambda+\gamma; I) = -S(-\lambda-\omega; I)$. Let γ be an elemaent in $\Sigma_{\mathfrak{v}}$ satisfying $1 < n(\gamma,\omega)$. Let us assume that the identity (4.13) is true for all δ in $\Sigma_{\mathfrak{v}}$ satisfying $1 \leq n(\delta;\omega) < n(\gamma,\omega)$. Let I be an element in $\Pi(\gamma;\omega)$. We can assume that $I = \alpha \star I'$ for $\alpha \in P_K$ and $I' \in \Pi(\gamma+\alpha;\omega)$. Then by (4.5)

$$T(\lambda+\gamma; I) = R(\lambda+\gamma; I)T(\lambda+\gamma+\alpha; I')\,. \tag{4.14}$$

Since $n(\gamma+\alpha;\omega) < n(\gamma;\omega)$, the inductive hypothesis implies that

$$T(\lambda+\gamma+\alpha; I') = (-1)^{\sharp I'}S(-\lambda-\omega;\psi(I'))\,.$$

Since $\gamma + \langle I\rangle = \omega$, we have $R(\lambda+\gamma; I) = -R(-\lambda-\omega;\psi(I))$. Consequently, by (4.14) and (4.4) we conclude that

$$\begin{aligned}T(\lambda+\gamma; I) &= (-1)^{\sharp I}R(-\lambda-\omega;\psi(I))S(-\lambda-\omega;\psi(I'))\\ &= (-1)^{\sharp I}S(-\lambda-\omega;\psi(I))\,.\end{aligned}$$

Hence we have (4.13). Let us now prove this lemma. By using Lemma 4.3, we have

$$\begin{aligned}\sum_{\gamma\in\Sigma_{\mathfrak{v}}} |P_{\mu+\omega}(X_\gamma\otimes v(\mu))|^2 &= \sum_{\gamma\in\Sigma_{\mathfrak{v}}}\sum_{I\in\Pi(\gamma;\omega)} a_\gamma(I)T(\lambda+\gamma; I)|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2\,,\\ &= \sum_{\gamma\in\Sigma_n}\sum_{I\in\Pi(\gamma;\omega)} a_\gamma(I)T(\lambda+\gamma; I)|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2\\ &= \sum_{I\in\Pi}(-1)^{\sharp I}a_{-\omega}(I)S(-\lambda-\omega; I)|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2\\ &= f(-\lambda-\omega;-\omega)|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2\,.\end{aligned}$$

Here we used (4.12) and (4.13). By Lemma 4.7 and Lemma 4.8 we have immediately the following lemma.

COROLLARY 4.9. *Let $\mu \in \Gamma_K$, and assume that $\mu + \delta \in \Gamma_K$ for all $\delta \in \Sigma_n$. Then we have*

$$\sum_{\gamma \in \Sigma_n} |P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 = f(-\lambda - \omega; -\omega) f(\lambda + \omega; \omega)$$

for each ω in Σ_n.

5. Functional equations of $f(\eta; \omega)$

For each ω in Σ_n, we shall consider the rational function $f(\eta; \omega)$ in η (see (4.1)). Our purpose of this section is to prove Theorem 5.4 and Theorem 5.5. We note that Theorem 5.5 is a refinement of Lemma 4.7.

LEMMA 5.1. *Let $\mu \in \Gamma_K$, and assume that $\mu + \omega \in \Gamma_K$, $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$ for a noncompact root $\omega \in \Sigma_n$. Then we have*

$$\prod_{\alpha \in P_K} (\lambda, \alpha) |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 = \prod_{\alpha \in P_K} (\lambda + \omega, \alpha) |P_\mu(X_{-\omega} \otimes v(\mu + \omega))|^2 , \tag{5.1}$$

$$\sum_{\gamma \in \Sigma_n} |P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 = \prod_{\alpha \in P_K} \frac{(\lambda + \omega, \alpha)}{(\lambda, \alpha)} , \tag{5.2}$$

where $v(\mu)$ (resp. $v(\mu + \omega)$) is a highest weight vector in V_μ (resp. $V_{\mu+\omega}$) normalized as $|v(\mu)| = |v(\mu + \omega)| = 1$, and $\lambda = \mu + \rho_K$.

REMARK. The identity (5.1) is due to N. Tatsuuma (cf. [9]).

PROOF OF LEMMA 5.1. By Schur orthogonality relation we have

$$\begin{aligned} C &= \int_K (k(X_\omega \otimes v(\mu)), X_\omega \otimes v(\mu)) \overline{(kv(\mu + \omega), v(\mu + \omega))} dk \\ &= \int_K (kP_{\mu+\omega}(X_\omega \otimes v(\mu)), P_{\mu+\omega}(X_\omega \otimes v(\mu))) \overline{(kv(\mu + \omega), v(\mu + \omega))} dk \\ &= (\deg \pi_{\mu+\omega})^{-1} |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 . \end{aligned}$$

On the other hand, we have

$$\begin{aligned} C &= \int_K (kX_\omega, X_\omega)(kv(\mu), v(\mu)) \overline{(kv(\mu + \omega), v(\mu + \omega))} dk \\ &= \int_K (kv(\mu), v(\mu)) \overline{(k(X_{-\omega} \otimes v(\mu + \omega)), X_{-\omega} \otimes v(\mu + \omega))} dk \\ &= \int_K (kv(\mu), v(\mu)) \overline{(kP_\mu(X_{-\omega} \otimes v(\mu + \omega)), P_\mu(X_{-\omega} \otimes v(\mu + \omega)))} dk \\ &= (\deg \pi_\mu)^{-1} |P_\mu(X_{-\omega} \otimes v(\mu + \omega))|^2 . \end{aligned}$$

Hence we have

$$\deg \pi_\mu |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 = \deg \pi_{\mu+\omega} |P_\mu (X_{-\omega} \otimes v(\mu + \omega))|^2 . \tag{5.3}$$

Bearing in mind $\deg \pi_\mu = \prod_{\alpha \in P_K} \frac{(\lambda,\alpha)}{(\rho_K,\alpha)}$ and the corresponding formula for $\deg \pi_{\mu+\omega}$, (5.3) implies the identity (5.1). Let us prove the identity (5.2). Let $\{u_i : 1 \le i \le N\}, N = \deg \pi_{\mu+\omega}$, be an orthonormal basis of $V_{\mu+\omega}$. By using Schur orthogonality relation, we have

$$\begin{aligned} E &= \sum_{\gamma \in \Sigma_n} \sum_{i=1}^{N} \int_K (k(X_\gamma \otimes v(\mu)), X_\gamma \otimes v(\mu))\overline{(ku_i, u_i)} dk \\ &= \sum_{\gamma \in \Sigma_n} \sum_{i=1}^{N} N^{-1} (P_{\mu+\omega}(X_\gamma \otimes v(\mu)), u_i)\overline{(P_{\mu+\omega}(X_\gamma \otimes v(\mu)), u_i)} \\ &= N^{-1} \sum_{\gamma \in \Sigma_n} |P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 . \end{aligned} \tag{5.4}$$

On the other hand, we have

$$\begin{aligned} E &= \int_K (kv(\mu), v(\mu)) \sum_{\gamma \in \Sigma_n} \sum_{i=1}^{N} \overline{(kX_{-\gamma}, X_{-\gamma})(ku_i, u_i)} dk \\ &= \int_K (kv(\mu), v(\mu)) \overline{\mathrm{trace}(\mathrm{Ad} \otimes \pi_{\mu+\omega})(k)} dk \\ &= m(\mu) \int_K (kv(\mu), v(\mu)) \overline{\mathrm{trace}\, \pi_\mu(k)} dk \\ &= m(\mu)(\deg \pi_\mu)^{-1} . \end{aligned}$$

Here we used

$$\mathrm{trace}(\mathrm{Ad} \otimes \pi_{\mu+\omega})(k) = \sum_{\gamma \in \Sigma_n, \mu+\omega+\gamma \in \Gamma_K} m(\mu + \omega + \gamma) \mathrm{trace}(\pi_{\mu+\omega+\gamma}(k)) ,$$

where $m(\mu + \omega + \gamma) = 1$ or $= 0$ (see Lemma 3.4). Hence by (5.4) we have the identity (5.2).

We define a subset D of Γ by

$$D = \{\mu \in \Gamma : 2(\mu, \beta_i)|\beta_i|^{-2} \ge 9 \text{ for all } i = 1, 2, \cdots, \ell\} , \tag{5.5}$$

where $\Psi = \{\beta_1, \beta_2, \cdots, \beta_\ell\}$ is the same as in (4.8).

LEMMA 5.2. *Let μ be an element in D. Then we have $\mu + \omega \in \Gamma$ for all $\omega \in \Sigma_n$. Furthermore, we have $6n\lambda_i + \mu \in D$ for all positive integers n and $i = 1, 2, \cdots, \ell$.*

PROOF. Let ω be a noncompact root in Σ. Since μ and ω are integral, $\mu + \omega$ is also an integral form on $\mathfrak{b}_{\mathbf{C}}$. We shall prove $\mu + \omega$ is P-dominant. Let α be a root in P. By (2.5) we have

$$2(\mu + \omega, \alpha)|\alpha|^{-2} \ge 2(\mu, \alpha)|\alpha|^{-2} - 3 . \tag{5.6}$$

Let $\alpha = \sum_{i=1}^{\ell} m_i\beta_i$ be the expression of α by the simple roots in Ψ. Then all m_i's are nonnegative integers. Furthermore, we can assume that $m_k > 0$ for k, $1 \leq k \leq \ell$. Since $|\beta_k|^2|\alpha|^{-2} \geq 1/3$ and $2(\mu, \beta_k)|\beta_k|^{-2} \geq 9$, we have

$$2(\mu, \alpha)|\alpha|^{-2} \geq m_k(2(\mu, \beta_k)|\beta_k|^{-2})(|\beta_k|^2|\alpha|^{-2}) \geq 3 .$$

Hence by (5.6), we have $2(\mu + \omega, \alpha)|\alpha|^{-2} \geq 0$. Thus $\mu + \omega \in \Gamma$ as claimed. Let us prove the second assertion of this lemma. It is sufficient to prove that $6n\lambda_i \in \Gamma$. Let α be as above. If $m_i = 0$, we have $2(6n\lambda_i, \alpha) = 0$. Assume that $m_i > 0$. Since

$$2(6n\lambda_i, \alpha)|\alpha|^{-2} = 6m_in(|\beta_i|^2|\alpha|^{-2}) ,$$

(2.5) implies that $6n\lambda_i$ is a P-dominant integral form on $\mathfrak{b}_{\mathbf{C}}$.

LEMMA 5.3. *Let F be an element in $\mathbf{R}[\eta]$. Suppose that $F(\lambda) = 0$ for all $\lambda \in D + \rho_K$. Then we have $F \equiv 0$.*

PROOF. F is written by

$$F(\eta) = \sum_{i=0}^{m} (\eta_1)^i F_i(\eta_2, \cdots, \eta_\ell) . \tag{5.7}$$

Let $\lambda = \mu + \rho_K$ be an element in $D + \rho_K$. We put $\lambda = \sum_{i=1}^{\ell} p_i\lambda_i$. Then p_i is a rational number. We shall prove the assertion by using an induction on ℓ. We first assume that $F(\eta) = F(\eta_1)$. By Lemma 5.2 we have $6n\lambda_1 + \lambda \in D + \rho_K$ for all positive integers n. Hence by our assumption for F, we have $F((6n + p_1)\lambda_1) = 0$. Since the polynomial $F(\eta_1)$ has the infinitely many zeros, we have $F \equiv 0$. Let F be the same as in (5.7). Since $\sum_{i=0}^{m}(6n + p_1)^i F_i(p_2, \cdots, p_\ell) = 0$ for all positive integers n,

$$F_i(p_2, \cdots, p_\ell) = 0 \quad \text{for all } \ i = 0\,,\ \ 1, \cdots, m .$$

We have $F_i(\lambda) = 0$ for all i and $\lambda \in D + \rho_K$. Thus by the hypothesis of our induction we conclude that $F \equiv 0$.

Let η be an element in $(\sqrt{-1}\mathfrak{b})^*$ and α an element in P_K. We put $\alpha = \sum_{i=1}^{\ell} m_i\beta_i$, where m_i is a nonnegative integer. Then

$$(\eta, \alpha) = \sum_{i=1}^{\lambda} \frac{m_i}{2}|\beta_i|^2\eta_i .$$

Especially, $(\eta, \alpha) \in \mathbf{R}[\eta]$ for all $\alpha \in P_K$.

THEOREM 5.4. *Let ω be an element in Σ_n. Then we have the following functional equations in $\mathbf{R}(\eta)$.*

$$\prod_{\alpha\in P_K}(\eta,\alpha)f(\eta+\omega;\omega)=\prod_{\alpha\in P_K}(\eta+\omega,\alpha)f(\eta;-\omega)\,, \tag{5.8}$$

$$f(\eta+\omega;\omega)f(-\eta-\omega;-\omega)=\prod_{\alpha\in P_K}(\eta+\omega,\alpha)(\eta,\alpha)^{-1}\,. \tag{5.9}$$

PROOF. We put

$$f(\eta+\omega;\omega)=\frac{q(\eta)}{p(\eta)}\ \text{and}\ f(\eta;-\omega)=\frac{s(\eta)}{r(\eta)}$$

where $p,q,r,s\in\mathbf{R}[\eta]$. By Lemma 4.7 and Lemma 5.1 we have

$$\prod_{\alpha\in P_K}(\lambda,\alpha)f(\lambda+\omega;\omega)=\prod_{\alpha\in P_K}(\lambda+\omega,\alpha)f(\lambda;-\omega)\quad\text{for all }\ \lambda\in D+\rho_K\,.$$

This implies that

$$\prod_{\alpha}(\lambda,\alpha)r(\lambda)q(\lambda)-\prod_{\alpha}(\lambda+\omega,\alpha)s(\lambda)p(\lambda)=0\quad\text{for all }\ \lambda\in D+\rho_K\,.$$

By Lemma 5.3 this identity holds for all $\eta\in(\sqrt{-1}\mathfrak{b})^*$, and therefore, we have the identity (5.8). The identity (5.9) is also proved by using the same arguments as above.

THEOREM 5.5. *Let $\mu\in\Gamma_K$ and $\omega\in\Sigma_n$. Suppose that $\mu+\omega\in\Gamma_K$ and $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu)\neq\{0\}$ for a noncompact root ω. Then we have*

$$|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2=f(\lambda+\omega;\omega)\,.$$

PROOF. Combining Lemma 4.8 with Lemma 5.1 we have

$$f(-\lambda-\omega;-\omega)|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2=\prod_{\alpha\in P_K}(\lambda+\omega,\alpha)(\lambda,\alpha)^{-1}\,.$$

By the second identity in Theorem 5.4, we conclude that

$$|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2=f(\lambda+\omega;\omega)\,.$$

The following lemma will be used to calculate the explicit formula of $f(\eta;\omega)$.

LEMMA 5.6. *Let f_i,h_i $(i=1,2)$ be four polynomials in $\mathbf{R}[\eta]$. We assume that $\deg h_1=\deg h_2=1$ and $f_1h_1=f_2h_2$. If h_2 is distinct to a non-zero constant multiple of h_1, then f_2 is divisible by h_1.*

PROOF. Since $\deg h_1=1$, there exists a number i, $1\le i\le\ell$, such that the first partial derivative $\frac{\partial h_1}{\partial\eta_i}$ is a non-zero constant. We can assume that $i=1$, and put $\zeta_1=h_1$, $\zeta_i=\eta_i$ for

$2 \le i \le \ell$. Then we have $\mathbf{R}[\zeta] = \mathbf{R}[\zeta_1, \zeta_2, \cdots, \zeta_\ell] = \mathbf{R}[\eta]$. Let g_i $(i = 1, 2)$ and h be three polynomials in $\mathbf{R}[\zeta]$ satisfying $g_i(\zeta) = f_i(\eta)$ and $h(\zeta) = h_2(\eta)$. We put

$$h(\zeta) = \sum_{i=1}^{\ell} c_i \zeta_i + c_0, \; g_1(\zeta) = \sum_{j=0}^{m} \zeta_1^j r_j(\zeta) \quad \text{and} \quad g_2(\zeta) = \sum_{j=0}^{m} \zeta_1^j s_j(\zeta)\,,$$

where $r_j, s_j \in \mathbf{R}[\zeta_2, \cdots, \zeta_\ell]$ and $c_i \in \mathbf{R}$. Since

$$\begin{aligned} 0 &= f_1(\eta)h_1(\eta) - f_2(\eta)h_2(\eta) \\ &= \sum_{j=0}^{m} \zeta_1^{j+1} r_j(\zeta) - \sum_{j=0}^{m} \zeta_1^j s_j(\zeta)(h(\zeta) - c_1\zeta_1) - \sum_{j=0}^{m} c_1 \zeta_1^{j+1} s_j(\zeta)\,, \end{aligned}$$

we have

$$\sum_{j=0}^{m} \zeta_1^{j+1}(r_j(\zeta) - c_1 s_j(\zeta)) = \sum_{j=0}^{m} \zeta_1^j s_j(\zeta)(h(\zeta) - c_1\zeta_1)\,.$$

Bearing in mind $h(\zeta) - c_1\zeta_1, s_j, r_j \in \mathbf{R}[\zeta_2, \cdots, \zeta_\ell]$, it follows that $s_0(\zeta)(h(\zeta) - c_1\zeta_1) = 0$. On the other hand, since $h(\zeta) = h_2(\eta)$ is not a constant multiple of $\zeta_1 = h_1(\eta)$, we have $h(\zeta) - c_1\zeta_1 \neq 0$. Since $\mathbf{R}[\zeta]$ is an integral domain, we conclude that $s_0(\zeta) = 0$. Thus $f_2(\eta) = \sum_{j=1}^{m} \zeta_1^j s_j(\zeta)$ is divisible by $\zeta_1 = h_1(\eta)$. This completes our proof.

6. Product formula for $f(\eta + \omega; \omega)$

For each $\omega \in \Sigma_n$ we define a rational function $f(\eta; \omega)$ and a real number $a_\omega(I)$, $I \in \Pi$, by Definition 4.1. In this section we shall prove that $f(\eta; \omega)$ has a product formula. First we define a subset $\hat{\Delta}(\omega)$ in $(\sqrt{-1}\mathfrak{b})^*$ by

$$\hat{\Delta}(\omega) = \{\langle I \rangle : a_\omega(I) \neq 0, I \in \Pi \backslash \Pi_0\}\,. \tag{6.1}$$

We define the polynomials $p_\xi(\eta)$ $(\xi \in \hat{\Delta}(\omega))$ and $p(\eta; \omega)$ in $\mathbf{R}[\eta]$ by

$$p_\xi(\eta) = 2(\eta, \xi) + |\xi|^2\,, \quad p(\eta; \omega) = \prod_{\xi \in \hat{\Delta}(\omega)} p_\xi(\eta)\,. \tag{6.2}$$

Since $p(\eta; \omega)$ is the least common multiple of the denominators of fractional terms $S(\eta; I)$ $(I \in \Pi)$ in $f(\eta; \omega)$, there exists a polynomial $g(\eta; \omega)$ such that

$$p(\eta; \omega) f(\eta; \omega) = g(\eta; \omega)\,. \tag{6.3}$$

We put $\Delta_\pm(\omega) = \{\alpha \in P_K : \pm(\omega, \alpha) > 0\}$. By Theorem 5.4 we have

$$\begin{aligned} &p(\eta; -\omega) \prod_{\alpha \in \Delta_-(\omega) \cup \Delta_+(\omega)} (\eta, \alpha) g(\eta + \omega; \omega) \\ &= p(\eta + \omega; \omega) \prod_{\alpha \in \Delta_-(\omega) \cup \Delta_+(\omega)} (\eta + \omega, \alpha) g(\eta; -\omega)\,. \end{aligned} \tag{6.4}$$

We now define the subsets $\Delta(\omega)$, $\Delta_m(\omega)$ and $\Delta_m(\omega)^*$ of P_K, where m is an integer, by

$$\Delta(\omega) = \{\alpha \in P_K : \omega + \alpha \in \Sigma\},$$
$$\Delta_m(\omega) = \{\alpha \in P_K : 2(\omega, \alpha)|\alpha|^{-2} = m, \omega + \alpha \in \Sigma\},$$
$$\Delta_m(\omega)^* = \{\alpha \in \Delta_m(\omega) : \omega - \alpha \in \Sigma\}.$$

We note that $\Delta(\omega) \subset \hat{\Delta}(\omega)$.

LEMMA 6.1. *Let G be an inner type noncompact real simple Lie group and ω a noncompact root in Σ. Then we have the followings.*

(1) $\Delta(\omega) = \Delta_-(\omega) \cup \Delta_0(\omega) \cup \Delta_1(\omega)$, $\Delta_0(\omega) = \Delta_0(\omega)^*$ *and* $\Delta_1(\omega) = \Delta_1(\omega)^*$.

(2) *If* $\Delta_0(\omega)^* \neq \phi$, *then G is one of* $S_p(n, \mathbf{R})$ *and* $SO(2m, 2n+1)$, *and* $\Delta(\omega) = \Delta_0(\omega)^* \cup \Delta_{-1}(\omega)$.

(3) *If* $\Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^* \neq \phi$, *then G is of the type G_2.*

PROOF. Let α be an element in P_K and $\omega + j\alpha$ $(-q \leq j \leq p)$ the α-series containing ω. We put $A = \Delta_-(\omega) \cup \Delta_0(\omega) \cup \Delta_1(\omega)$. We first prove that $\Delta(\omega) = A$. Let α be an element in A. Since $\Delta_0(\omega) \cup \Delta_1(\omega) \subset \Delta(\omega)$, we can assume $\alpha \in \Delta_-(\omega)$. Then by (2.3) we have $\alpha \in \Delta(\omega)$, and hence, $A \subset \Delta(\omega)$. Let us now assume that $\alpha \in \Delta(\omega)$. Since $p \geq 1$ and $p + q \leq 3$, (2.3) implies that $\alpha \in A$. Thus $A = \Delta(\omega)$. Moreover, we have $\Delta_0(\omega) = \Delta_0(\omega)^*$ and $\Delta_1(\omega)^* = \Delta_1(\omega)$. Let us prove (2) and (3). Suppose that $\alpha \in \Delta_0(\omega)^*$. Then $\omega + \alpha \in \Sigma$ and $2(\omega + \alpha, \alpha)|\alpha|^{-2} = 2$. In view of (2.8), (2.9), (2.10) and (2.11) we have G is one of $SO(2m, 2n+1)$ and $S_p(n, \mathbf{R})$. It remains to prove that if $\Delta_0(\omega)^* \neq \phi$, then $\Delta(\omega) = \Delta_0(\omega) \cup \Delta_{-1}(\omega)$. By (1) it is sufficient to prove $\Delta_-(\omega) = \Delta_{-1}(\omega)$ and $\Delta_1(\omega) = \phi$. If $\Delta_0(\omega) \neq \phi$, then ω is a short root. By (2.9) and (2.10) we have $|2(\omega, \alpha)|\alpha|^{-2}| \leq 1$ for all $\alpha \in P_K$. This implies $\Delta_-(\omega) = \Delta_{-1}(\omega)$. Suppose that $\alpha \in \Delta_1(\omega)$. Since $\omega + \alpha \in \Sigma_n$ and $2(\omega + \alpha, \alpha)|\alpha|^{-2} = 3$, G is of the type G_2. This implies that $\Delta_1(\omega) = \phi$ for the case $\Delta_0(\omega)^* = \phi$. Consider the case $\Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^* \neq \phi$. When $\Delta_1(\omega)^* = \Delta_1(\omega) \neq \phi$, the above argument implies G is of type G_2. If $\alpha \in \Delta_{-1}(\omega)^*$, then we have $2(\omega - \alpha, \alpha)|\alpha|^{-2} = -3$. This implies also the same conclusion.

LEMMA 6.2. *Consider a noncompact root ω and a compact root α in P_K. Then for each $\xi \in \hat{\Delta}(\omega)$ and $\zeta \in \hat{\Delta}(-\omega)$ we have the followings.*

(1) $p_\xi(\eta + \omega)$ *is divisible by* (η, α) *iff* $\alpha \in \Delta_-(\omega)$ *and* $\xi = -\frac{2(\omega,\alpha)}{|\alpha|^2}\alpha$.

(2) $p_\zeta(\eta)$ *is divisible by* $(\eta + \omega, \alpha)$ *iff* $\alpha \in \Delta_+(\omega)$ *and* $\zeta = \frac{2(\omega,\alpha)}{|\alpha|^2}\alpha$.

(3) $p_\xi(\eta + \omega)$ *is divisible by* $p_\zeta(\eta)$ *iff one of the following three cases*;

(i) $\xi = \zeta \in \Delta_0(\omega)^*$, (ii) $\xi \in \Delta_1(\omega)^*$ *and* $\zeta = 2\xi$,

(iii) $\zeta \in \Delta_{-1}(\omega)^*$ *and* $\xi = 2\zeta$.

PROOF. Let us prove (1). Assume that there exists a nonzero real number k such that $p_\xi(\eta+\omega) = 2k(\eta,\alpha)$. Then we have $\xi = k\alpha$ and $2(\omega,\alpha) + k|\alpha|^2 = 0$. On the other hand, since $\xi \in \hat{\Delta}(\omega)$, there is $I \in \Pi\backslash\Pi_0$ such that $\xi = \langle I\rangle$. This implies that k is a positive real number. Consequently, we have $\alpha \in \Delta_-(\omega)$ and $\xi = -\frac{2(\omega,\alpha)}{|\alpha|^2}\alpha$. Conversely if α and ξ satisfy these conditions, then we can prove that $p_\xi(\eta+\omega)$ is a non-zero constant multiple of (η,α). Similarly by using the same arguments we can prove (2). We shall prove (3). Suppose that there exists a nonzero real number k such that $kp_\xi(\eta+\omega) = p_\zeta(\eta)$. Then we have

$$\zeta = k\xi \text{ and } 2(\omega,\xi) = (k-1)|\xi|^2 . \tag{6.5}$$

By the first identity in (6.5), k is positive. We put $\delta = \omega + \xi$. Since $\xi \in \hat{\Delta}(\omega)$, it follows from the definition in (6.1) that $\delta \in \Sigma$. We have

$$|\delta|^2 - |\omega|^2 = 2(\omega,\xi) + |\xi|^2 = k|\xi|^2 . \tag{6.6}$$

This implies that $|\delta|^2 > |\omega|^2$. On the other hand, since $|\xi|^2 = |\delta|^2 - 2(\omega,\delta) + |\omega|^2$, we have

$$2(\omega,\delta) = \left(1+\frac{1}{k}\right)|\omega|^2 + \left(1-\frac{1}{k}\right)|\delta|^2 . \tag{6.7}$$

When $|\delta|^2 = 2|\omega|^2$, we have $k=1$ and $\frac{2(\delta,\omega)}{|\omega|^2} = 2$. Therefore $\xi = \zeta \in \Delta_0(\omega)^*$ which is the case (i). When $|\delta|^2 = 3|\omega|^2$, (6.7) implies that $2(\omega,\delta) = (4-2/k)|\omega|^2$. Since

$$\frac{2(\delta,\omega)}{|\omega|^2} \in \{0, \pm 3\} ,$$

it follows that $k=2$ or $k=1/2$ or $k=2/7$. In the first case, we have $\delta - \omega = \xi \in \Delta_1(\omega)^*$ and $\zeta = 2\xi$. Let us consider the second case: $\frac{2(\delta,\omega)}{|\omega|^2} = 0$ and $k = 1/2$. We put $\delta' = -\omega + \zeta$. Then we have $\delta' \in \Sigma$. Furthermore, by (6.5) we have

$$2(\omega,\zeta) = -\frac{1}{4}|\xi|^2 = -\frac{1}{4}|\delta-\omega|^2 = -|\omega|^2 .$$

Therefore

$$2(\delta',\omega) = 2(\zeta,\omega) - 2|\omega|^2 = -3|\omega|^2 .$$

Hence we have $\zeta = \delta' + \omega \in \Sigma$ and $2(\zeta,\omega) = -|\omega|^2 = -|\zeta|^2$. Thus $\zeta \in \Delta_{-1}(\omega)^*$ and $\xi = 2\zeta$. Suppose that $k = 2/7$. Since $\xi \in \hat{\Delta}(\omega)$, there are two nonnegative integers p, q such that $\xi = p\alpha_0 + q\alpha_2$, where $\{\alpha_0, \alpha_2\}$ is the positive root system P_K of type G_2 (see (2.11)). Since $|\delta|^2 = 3|\omega|^2$, the equation in (6.6) implies that $\xi = \alpha_0 + 2\alpha_2$. Then, $\zeta = \frac{2}{7}\xi \notin \hat{\Delta}(-\omega)$. This is contradict to the assumption $\zeta \in \hat{\Delta}(-\omega)$. Thus the final case does not occur.

Let ω be a fixed noncompact root in Σ_n. In order to calculate a product formula for $f(\eta;\omega)$ we shall consider two cases: $\Delta_1(\omega)^* \cup \Delta_{-1}(\omega)^*$ is empty or not. For the first case we define two polynomials $p(\eta;\omega)$ and $g(\eta;\omega)$ as in (6.3). We now put

$$p'(\eta+\omega;\omega) = p(\eta+\omega;\omega)\prod_{\alpha\in\Delta_-(\omega)\cup\Delta_+(\omega)}(\eta+\omega,\alpha)\,,$$
$$p'(\eta;-\omega) = p(\eta;-\omega)\prod_{\alpha\in\Delta_-(\omega)\cup\Delta_+(\omega)}(\eta,\alpha)\,.$$

Then by (6.4) we have

$$p'(\eta;-\omega)g(\eta+\omega;\omega) = p'(\eta+\omega;\omega)g(\eta;-\omega)\,. \tag{6.8}$$

We also define two polynomials $s(\eta;\omega)$ and $q(\eta;\omega)$ in $\mathbf{R}[\eta]$ by

$$\begin{aligned} s(\eta;\omega) &= \prod_{\alpha\in\Delta_0(\omega)^*}(2(\eta,\alpha)+|\alpha|^2),\\ q(\eta;\omega) &= s(\eta;\omega)\prod_{\alpha\in\Delta_-(\omega)}(\eta-\omega,\alpha)\prod_{\alpha\in\Delta_+(\omega)}(\eta,\alpha)\,. \end{aligned} \tag{6.9}$$

Since $q(\eta;\omega)$ and $s(\eta;\omega)$ are invariant under the transformation: $(\eta,\omega)\to(\eta-\omega,-\omega)$, we can define $q(\eta;-\omega)$ and $s(\eta;-\omega)$ by

$$q(\eta;-\omega) = q(\eta+\omega;\omega) \quad \text{and} \quad s(\eta;-\omega) = s(\eta+\omega;\omega)\,. \tag{6.10}$$

LEMMA 6.3. *Assume that $\Delta_1(\omega)^*\cup\Delta_{-1}(\omega)^* = \phi$. Then the greatest common divisor of $p'(\eta+\omega;\omega)$ and $p'(\eta;-\omega)$ is given by $q(\eta+\omega;\omega) = q(\eta;-\omega)$.*

PROOF. By (6.2) we have $p(\eta+\omega;\omega) = \prod_{\xi\in\hat{\Delta}(\omega)} p_\xi(\eta+\omega)$. The polynomials $p_\xi(\eta)$ and $p_\zeta(\eta)$ are mutually prime for two distinct ξ and ζ in $\hat{\Delta}(\omega)$. Actually, if $p_\xi(\eta) = cp_\zeta(\eta)$ for a non-zero real number c, then $\xi = c\zeta$ and $|\xi|^2 = c|\zeta|^2$. These imply $c = 1$ and $\xi = \zeta$. Let p be a common prime divisor of $p'(\eta+\omega;\omega)$ and $p'(\eta;-\omega)$. Since p is a divisor of $p'(\eta+\omega;\omega)$, we can assume that $p = p_\xi(\eta+\omega)$ ($\xi\in\hat{\Delta}(\omega)$) or $p = (\eta+\omega,\alpha)$ ($\alpha\in\Delta_-(\omega)\cup\Delta_+(\omega)$). If $p = p_\xi(\eta+\omega)$, then the formula of $p'(\eta;-\omega)$ implies that $cp = (\eta,\beta)$, $\beta\in\Delta_-(\omega)\cup\Delta_+(\omega)$ or $cp = p_\zeta(\eta)$, $\zeta\in\hat{\Delta}(-\omega)$, where c is a constant. In the first case, (1) in Lemma 6.2 implies $cp = (\eta,\beta)$, $\beta\in\Delta_-(\omega)$. For the second case, from (3) in Lemma 6.2 it follows that $cp = 2(\eta,\beta)+|\beta|^2$, $\beta\in\Delta_0(\omega)^*$. If $p = (\eta+\omega,\alpha)$, $\alpha\in\Delta_-(\omega)\cup\Delta_+(\omega)$, then by (2) in Lemma 6.2 we have $\alpha\in\Delta_+(\omega)$. Therefore, p is a divisor of

$$q(\eta+\omega;\omega) = \prod_{\gamma\in\Delta_-(\omega)}(\eta,\gamma)\prod_{\gamma\in\Delta_+(\omega)}(\eta+\omega,\gamma)\prod_{\gamma\in\Delta_0(\omega)^*}(2(\eta,\gamma)+|\gamma|^2)\,.$$

Thus $q(\eta+\omega;\omega)$ is divisible by all common prime divisors of $p'(\eta+\omega;\omega)$ and $p'(\eta;-\omega)$. Again by Lemma 6.2 $q(\eta+\omega;\omega)$ is a common divisor of $p'(\eta+\omega;\omega)$ and $p'(\eta;-\omega)$ which implies the conclusion of this lemma.

We keep the assumption $\Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^* = \phi$, and put $h(\eta;\omega) = p'(\eta;\omega)q(\eta;\omega)^{-1}$, $h(\eta;-\omega) = p'(\eta;-\omega)q(\eta;-\omega)^{-1}$. By (6.8) we have

$$h(\eta;-\omega)g(\eta+\omega;\omega) = h(\eta+\omega;\omega)g(\eta;-\omega)\,. \tag{6.11}$$

By Lemma 6.3, $h(\eta+\omega;\omega)$ and $h(\eta;-\omega)$ are mutually prime. Therefore, it follows from Lemma 5.6 that there are two polynomials $k(\eta;\omega)$ and $k(\eta;-\omega)$ such that

$$g(\eta+\omega;\omega) = k(\eta+\omega;\omega)h(\eta+\omega;\omega)\,, \quad g(\eta;-\omega) = k(\eta;-\omega)h(\eta;-\omega)\,. \tag{6.12}$$

Substituting the first identity for (6.3), we have

$$\begin{aligned} f(\eta+\omega;\omega) &= g(\eta+\omega;\omega)p(\eta+\omega;\omega)^{-1}\,, \\ &= k(\eta+\omega;\omega)h(\eta+\omega;\omega)p(\eta+\omega;\omega)^{-1} \\ &= k(\eta+\omega;\omega)p'(\eta+\omega;\omega)(q(\eta+\omega;\omega)p(\eta+\omega;\omega))^{-1} \\ &= k(\eta+\omega;\omega)\prod_{\alpha\in\Delta_-(\omega)}(\eta+\omega,\alpha)(\eta,\alpha)^{-1}s(\eta+\omega;\omega)^{-1}\,. \end{aligned}$$

LEMMA 6.4. *Assume that $\Delta_1(\omega)^* \cup \Delta_{-1}(\omega)^* = \phi$. Then we have $k(\eta+\omega;\omega) = k(\eta;-\omega)$ and the following identities.*

(1) $$f(\eta+\omega;\omega) = k(\eta+\omega;\omega)\prod_{\alpha\in\Delta_-(\omega)}(\eta+\omega,\alpha)(\eta,\alpha)^{-1}s(\eta+\omega;\omega)^{-1}\,,$$

$$f(\eta;-\omega) = k(\eta;-\omega)\prod_{\alpha\in\Delta_+(\omega)}(\eta,\alpha)(\eta+\omega,\alpha)^{-1}s(\eta;-\omega)^{-1}\,.$$

(2) $k(-\eta-\omega;-\omega)k(\eta+\omega;\omega) = s(-\eta-\omega;-\omega)s(\eta+\omega;\omega)\,.$

PROOF. The first identity in (1) is already shown, and the second one also follows from the same calculation. It remains to prove the identities $k(\eta+\omega;\omega) = k(\eta;-\omega)$ and (2). By using the first identity in Theorem 5.4 we have

$$\prod_{\alpha\in\Delta_-(\omega)\cup\Delta_+(\omega)}(\eta,\alpha)f(\eta+\omega;\omega) = \prod_{\alpha\in\Delta_-(\omega)\cup\Delta_+(\omega)}(\eta+\omega,\alpha)f(\eta;-\omega)\,.$$

Hence by (6.10) and the identities in (1), we have $k(\eta+\omega;\omega) = k(\eta;-\omega)$. By the second identity in (1) we have

$$f(-\eta-\omega;-\omega) = k(-\eta-\omega;-\omega)\prod_{\alpha\in\Delta_+(\omega)}(\eta+\omega,\alpha)(\eta,\alpha)^{-1}s(-\eta-\omega,-\omega)^{-1}\,,$$

and then, by the first identity in (1) we have

$$\begin{aligned} f(\eta+\omega;\omega)f(-\eta-\omega;-\omega) &= k(\eta+\omega;\omega)k(-\eta-\omega;-\omega)\{s(\eta+\omega;\omega)s(-\eta-\omega;-\omega)\}^{-1} \\ &\quad\times \prod_{\alpha\in\Delta_-(\omega)\cup\Delta_+(\omega)}(\eta+\omega,\alpha)(\eta,\alpha)^{-1}\,. \end{aligned}$$

Therefore, the second identity in Theorem 5.4 implies the assertion of (2).

THEOREM 6.5. *Let G be an inner type noncompact real simple Lie group and ω a noncompact root. We define $f(\eta;\omega)$ by* (4.1). *Then $f(\eta;\omega)$ has one of the following product formulae.*

(1) *If $\Delta_0(\omega)^* \cup \Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^* = \phi$, then*

$$f(\eta+\omega;\omega) = \prod_{\alpha\in\Delta_-(\omega)} (\eta+\omega,\alpha)(\eta,\alpha)^{-1}\,.$$

(2) *If $\Delta_0(\omega)^* \neq \phi$, then $\Delta_1(\omega)^* \cup \Delta_{-1}(\omega)^* = \phi$ and*

$$\begin{aligned} f(\eta+\omega;\omega) &= \prod_{\alpha\in\Delta_0(\omega)^*} (2(\eta,\alpha)-|\alpha|^2)(2(\eta,\alpha)+|\alpha|^2)^{-1} \\ &\times \prod_{\alpha\in\Delta_{-1}(\omega)} (\eta+\omega,\alpha)(\eta,\alpha)^{-1}\,. \end{aligned}$$

(3) *If $\Delta_1(\omega)^* \cup \Delta_{-1}(\omega)^* \neq \phi$, then $\Delta_0(\omega)^* = \phi$ and*

$$\begin{aligned} f(\eta+\omega;\omega) = &\prod_{\alpha\in\Delta_-(\omega)} (\eta+\omega,\alpha)(\eta,\alpha)^{-1} \prod_{\alpha\in\Delta_1(\omega)^*} (2(\eta,\alpha)-|\alpha|^2)\{2((\eta,\alpha)+|\alpha|^2)\}^{-1} \\ &\times \prod_{\alpha\in\Delta_{-1}(\omega)^*} 2((\eta,\alpha)-|\alpha|^2)(2(\eta,\alpha)+|\alpha|^2)^{-1}\,. \end{aligned}$$

PROOF. If $\Delta_0(\omega)^* \cup \Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^* = \phi$, then by (6.9) we have $s(\eta+\omega;\omega) = 1$. (2) in Lemma 6.4 implies that $k(\eta+\omega;\omega) = c$, where c is a real constant. Hence by (1) in Lemma 6.4

$$f(\eta+\omega;\omega) = c\prod_{\alpha\in\Delta_-(\omega)} (\eta+\omega,\alpha)(\eta,\alpha)^{-1}\,.$$

We shall prove $c=1$. Let λ_0 be a P_K-regular dominant integral form on $\mathfrak{b}_{\mathbf{C}}$. By the above identity, we have

$$\lim_{a\to+\infty} f(a\lambda_0+\omega;\omega) = c\,.$$

Let $S(\eta;I)$, $I\in\Pi$, be the rational function as in Definition 4.1. Since $\lim_{a\to+\infty} S(a\lambda_0;I) = 0$ for $I\neq\tilde{\phi}$, we have $\lim_{a\to+\infty} f(a\lambda_0+\omega) = 1$. Hence we can prove (1). Let us assume that $\Delta_0(\omega)^*\neq\phi$. By (2), (3) in Lemma 6.1 we have $\Delta_{-1}(\omega)^*\cup\Delta_1(\omega)^* = \phi$ and $\Delta_-(\omega) = \Delta_{-1}(\omega)$. We put

$$\begin{aligned} s(\eta) &= s(\eta+\omega;\omega),\quad k(\eta) = k(\eta+\omega;\omega)\,, \\ u(\eta) &= \sum_{\alpha\in\Pi_1} a_\omega(\alpha)p_\alpha(\eta+\omega)^{-1}\,, \\ v(\eta) &= \prod_{\alpha\in\Delta_{-1}(\omega)} (\eta+\omega,\alpha)\,, \end{aligned}$$

$$w(\eta) = \prod_{\alpha \in \Delta_{-1}(\omega)} (\eta, \alpha)\,,$$

$$r(\eta) = f(\eta + \omega; \omega) - 1 + u(\eta)\,.$$

By (4.1) there exist two polynomials r_1, r_2 in $\mathbf{R}[\eta]$ such that $r = r_1 r_2^{-1}$ and $\deg r_1 \leq \deg r_2 - 2$. Since each prime divisor of r_2 is of degree one, it follows from Lemma 5.6 that we can assume r_1 and r_2 are mutually prime. By (1) in Lemma 6.4 we have

$$s(\eta)\{1 - u(\eta) + r(\eta)\}w(\eta) = k(\eta)v(\eta)\,. \tag{6.13}$$

Let N be the degree of the polynomial $k(\eta)v(\eta)$. We shall prove that srw is a plolynomial and $\deg(srw) \leq N - 2$. By (2) in Lemma 6.1 and (2.4) we have

$$\begin{aligned} u(\eta) &= \sum_{\alpha \in \Delta(\omega)} a_\omega(\alpha) p_\alpha(\eta + \omega)^{-1} \\ &= \sum_{\alpha \in \Delta_0(\omega)^*} 2|\alpha|^2 (2(\eta, \alpha) + |\alpha|^2)^{-1} + \sum_{\alpha \in \Delta_{-1}(\omega)} \frac{1}{2}|\alpha|^2 (\eta, \alpha)^{-1}\,. \end{aligned} \tag{6.14}$$

By this formula and (6.9) suw is a plynomial in η, and hence by (6.13), srw is also a polynomial and $\deg(srw) \leq N - 2$. Then it follows from (6.14) that

$$\begin{aligned} &s(\eta + \omega; \omega)(1 - u(\eta) + r(\eta))w(\eta) \\ &= \prod_{\alpha \in \Delta_0(\omega)^* \cup \Delta_{-1}(\omega)} (\eta, \alpha) - \sum_{\alpha \in \Delta_0(\omega)^*} |\alpha|^2 \prod_{\beta \in (\Delta_0(\omega)^* \setminus \{\alpha\}) \cup \Delta_{-1}(\omega)} (\eta, \beta) \\ &\quad - \sum_{\alpha \in \Delta_{-1}(\omega)^*} \frac{1}{2}|\alpha|^2 \prod_{\beta \in \Delta_0(\omega)^* \cup \Delta_{-1}(\omega) \setminus \{\alpha\}} (\eta, \beta) + \text{ the lower terms}\,. \end{aligned} \tag{6.15}$$

Let us now determine the polynomial $k(\eta)$. By the functional equation of (2) in Lemma 6.4, we have

$$k(\eta) = c \prod_{\alpha \in \Delta_0(\omega)^*} (2(\eta, \alpha) + \varepsilon_\alpha |\alpha|^2)\,,$$

where c is a constant and $\varepsilon_\alpha = \pm 1$. Comparing the highest and the second highest terms in (6.15) with $k(\eta)v(\eta)$, we have $c = 1$ and $\varepsilon_\alpha = -1$ for all $\alpha \in \Delta_0(\omega)^*$. Hence by (1) in Lemma 6.4 we have (2) in this theorem. Finally, let us consider the case $\Delta_1(\omega)^* \cup \Delta_{-1}(\omega)^* \neq \phi$. Since G is of the type G_2 (see (3) in Lemma 6.1). P_K and Σ_n are respectively given by

$$P_K = \{\beta, 2\gamma - 3\beta\},\ \Sigma_n = \{\pm\gamma, \pm(\gamma - \beta), \pm(\gamma - 2\beta), \pm(\gamma - 3\beta)\},\ \frac{2(\gamma, \beta)}{|\beta|^2} = 3\,.$$

Let α be an element in $\Delta_1(\omega)^* \cup \Delta_{-1}(\omega)^*$. Since $\omega \pm \alpha \in \Sigma$, we have $|\alpha| = |\omega|$ and ω is a short root. Therefore

$$\Delta_1(\omega)^* \cup \Delta_{-1}(\omega)^* = \{\beta\} \quad \text{and} \quad \omega \in \{\pm(\gamma - \beta), \pm(\gamma - 2\beta)\}\,.$$

Furthermore, we have

$$\hat{\Delta}(\gamma-\beta)=\Delta_1(\gamma-\beta)^*=\Delta_{-1}(\gamma-2\beta)^*=\{\beta\},\ \hat{\Delta}(\gamma-2\beta)=\{\beta,\ 2\beta\}\,.$$

A direct calculation shows that

$$\begin{aligned} f(\eta+\gamma-\beta;\gamma-\beta)&=1-\frac{3|\beta|^2}{2((\eta,\beta)+|\beta|^2)}=\frac{2(\eta,\beta)-|\beta|^2}{2((\eta,\beta)+|\beta|^2)}\,,\\ f(\eta+\gamma-2\beta;\gamma-2\beta)&=1-\frac{4|\beta|^2}{2(\eta,\beta)}+\frac{4|\beta|^2}{2(\eta,\beta)}\frac{3|\beta|^2}{2(2(\eta,\beta)+|\beta|^2)}\\ &=\frac{(\eta+\gamma-2\beta,\beta)}{(\eta,\beta)}\frac{2((\eta,\beta)-|\beta|^2)}{2(\eta,\beta)+|\beta|^2}\,. \end{aligned}$$

Hence, for $\omega=\gamma-\beta$ or $=\gamma-2\beta$, we have (3) of this theorem. For the case $-\omega=-\gamma+\beta,-\gamma+2\beta$, we have also (3) by using the identity (5.8) in Theorem 5.4.

7. Main theorem

Let $\mu\in\Gamma_K$ and V_μ a simple K-module with the highest weight μ. By Lemma 3.4

$$\mathfrak{p}_{\mathbf{C}}\otimes V_\mu=\oplus_{\omega\in\Sigma_n,\mu+\omega\in\Gamma_K}P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu)\,,$$

where $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu)=\{0\}$ or is a simple K-module. In this section we shall prove that the K-module $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu)$ is nontrivial if and only if $f(\lambda+\omega;\omega)>0$, where $\lambda=\mu+\rho_K$.

DEFINITION 7.1. Let $\mu\in\Gamma_K$, and define the following six sets for $\lambda=\mu+\rho_K$.

$$\begin{aligned} w(\lambda)&=\{\lambda+\omega:\omega\in\Sigma_n\},\\ sw(\lambda)&=\left\{\xi\in w(\lambda):\prod_{\alpha\in P_K}(\xi,\alpha)=0\right\},\\ rw(\lambda)&=\left\{\xi\in w(\lambda):\prod_{\alpha\in P_K}(\xi,\alpha)\neq 0\right\},\\ rw_0(\lambda)&=\{\lambda+\omega\in rw(\lambda):f(\lambda+\omega;\omega)=0\}\,,\\ rw_+(\lambda)&=\{\lambda+\omega:\mu+\omega\in\Gamma_K,\ f(\lambda+\omega;\omega)>0\}\,,\\ rw_-(\lambda)&=rw(\lambda)\backslash(rw_0(\lambda)\cup rw_+(\lambda))\,. \end{aligned}$$

LEMMA 7.2. *Assume that all noncompact roots in Σ have the same length. Then we have $w(\lambda)=sw(\lambda)\cup rw_+(\lambda)$.*

PROOF. First we shall prove $\Delta_0(\omega)^*\cup\Delta_{-1}(\omega)^*\cup\Delta_1(\omega)^*=\phi$. Let α be an element in $\Delta(\omega)$. Since $|\omega+\alpha|=|\omega|$, we have $2(\omega,\alpha)|\alpha|^{-2}=-1$. This implies that $\Delta(\omega)=\Delta_{-1}(\omega)$. By the proof of (3) in Lemma 6.1 if $\alpha\in\Delta_{-1}(\omega)^*$, then $\omega-2\alpha\in\Sigma_n$. Since $|\omega-2\alpha|^2>|\omega|^2$, our assumption implies $\Delta_{-1}(\omega)^*=\phi$, and hence, $\Delta_0(\omega)^*\cup\Delta_{-1}(\omega)^*\cup\Delta_1(\omega)^*=\phi$. Let

$\lambda+\omega$ be an element in $w(\lambda)$. Then for $\alpha \in P_K$, we have $(\omega, \alpha) \geq 0$ or $(\omega, \alpha) < 0$. In the first case we have $(\lambda+\omega, \alpha) > 0$. For the later case we have $2(\omega, \alpha)|\alpha|^{-2} = -1$. Consequently we have $(\lambda+\omega, \alpha) = 0$ or $(\lambda+\omega, \alpha) > 0$. Suppose that $\lambda+\omega \notin sw(\lambda)$. Since $(\lambda+\omega, \alpha) > 0$ for all $\alpha \in P_K$, (1) in Theorem 6.5 implies $\lambda+\omega \in rw_+(\lambda)$.

For α in P_K we define a linear transformation s_α on $(\sqrt{-1}\mathfrak{b})^*$ by

$$s_\alpha(\eta) = \eta - 2(\eta, \alpha)|\alpha|^{-2}\alpha,\ \eta \in (\sqrt{-1}\mathfrak{b})^* .$$

The Weyl group W_K of $(\mathfrak{k}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$ is generated by the set $\{s_\alpha; \alpha \in P_K\}$ (cf. Theorem 4.41 in [7]).

LEMMA 7.3. *Assume that G is one of $S_p(n, \mathbf{R})$ and $SO(2m, 2n+1)$. Suppose that $\lambda+\omega \in rw_0(\lambda)$. Then there exists a unique compact simple short root α in P such that $(\mu, \alpha) = 0, \alpha \in \Delta_0(\omega)^*$ and $s_\alpha(\lambda+\omega) \in rw_-(\lambda)$.*

PROOF. By the assumption for $\lambda+\omega$, $\lambda+\omega$ is P_K-regular and $f(\lambda+\omega; \omega) = 0$. We first prove that $\Delta_0(\omega)^* \neq \phi$. Suppose that $\Delta_0(\omega)^* = \phi$. By (2), (3) in Lemma 6.1 we can assume $\Delta_0(\omega)^* \cup \Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^* = \phi$. Since $\lambda+\omega$ is P_K-regular, (1) in Theorem 6.5 implies a contradiction : $f(\lambda+\omega; \omega) \neq 0$. Thus $\Delta_0(\omega)^* \neq \phi$. By (2) in Theorem 6.5 there exists $\alpha \in P_K$ such that $(\omega, \alpha) = 0$ and $2(\lambda, \alpha) = |\alpha|^2$. Therefore ω and α are short roots, and

$$(\mu, \alpha) = 0,\ 2(\rho_K, \alpha)|\alpha|^{-2} = 1 . \tag{7.1}$$

Let us prove that α is a simple root in P. Let $\Psi_K = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ be the simple root system of P_K. Then α is written by $\alpha = \sum_{i=1}^{\ell} m_i\alpha_i$, where m_i is a nonnegative integer. Consequently we have

$$1 = 2(\rho_K, \alpha)|\alpha|^{-2} = \sum_{i=1}^{\ell} m_i 2(\rho_K, \alpha_i)|\alpha_i|^{-2}(|\alpha_i|^2|\alpha|^{-2}) .$$

Since α is a short root, all $|\alpha_i|^2|\alpha|^{-2}$'s are positive integers. This implies that $\alpha = \alpha_i$ for a suitable i. Hence α is a simple short root. Here we shall use the Dynkin diagrams (2.9) and (2.10). Consider the case $G = SO(2m, 2n+1)$. In view of the Dynkin diagram of P_K the simle short root α is unique. Furthermore, α is also a unique short root of the Dynkin diagram of P. Let us prove that $s_\alpha(\lambda+\omega) \in rw_-(\lambda)$. Since $2(\lambda+\omega, \alpha)|\alpha|^{-2} = 1$ and $\lambda+\omega$ is P_K-regular, we have $s_\alpha(\lambda+\omega) = \lambda+\omega-\alpha \in rw(\lambda)$. Since $\omega-\alpha$ is a noncompact long root, we have

$$\Delta_0(\omega-\alpha)^* \cup \Delta_{-1}(\omega-\alpha)^* \cup \Delta_1(\omega-\alpha)^* = \phi .$$

The formula (1) in Theorem 6.5 implies that $f(\lambda+\omega-\alpha; \omega-\alpha) \neq 0$. Moreover, since $2(\mu+\omega-\alpha, \alpha)|\alpha|^{-2} < 0$, we have $s_\alpha(\lambda+\omega) \in rw_-(\lambda)$. Consider the case $G = S_p(n, \mathbf{R})$. Let $\Psi = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be the simple root system of the Dynkin diagram (2.9). Then all

α_i's ($1 \leq i \leq n-1$) are compact simple short roots in P_K. The set of all noncompact positive short roots is given by

$$\{\alpha_k + \cdots + \alpha_{s-1} + 2\alpha_s + \cdots + 2\alpha_{n-1} + \alpha_n : 1 \leq k < s < n\}.$$

Let γ be an element in this set. Then $\Delta_0(\gamma)$ is nonempty iff γ is of the form

$$\gamma_k = \alpha_k + 2\alpha_{k+1} + \cdots + 2\alpha_{n-1} + \alpha_n \ (1 \leq k \leq n).$$

If $(\gamma_k, \alpha) = 0$ for a compact root α, then $\alpha = \alpha_k$. Especially, $\Delta_0(\gamma_k)^* = \{\alpha_k\}$. Moreover, we can prove that $s_\alpha(\lambda+\omega) \in rw_-(\lambda)$ by using the same argument as in the case of $SO(2m, 2n+1)$.

LEMMA 7.4. *Let G be the same as in the previous lemma. Suppose that $\lambda + \omega \in rw_-(\lambda)$. Then there exists a unique compact simple short root $\alpha \in P$ such that $(\mu, \alpha) = 0$, $\alpha \in \Delta_0(\omega+\alpha)^*$ and $s_\alpha(\lambda+\omega) = \lambda + \omega + \alpha \in rw_0(\lambda)$.*

PROOF. Let $\lambda + \omega$ be an element in $rw_-(\lambda)$. First we shall prove that there exists a simple root α in P_K such that $(\mu, \alpha) = 0$ and $2(\omega, \alpha)|\alpha|^{-2} = -2$. Since $\lambda + \omega \notin rw_0(\lambda) \cup rw_+(\lambda)$, we have either $\mu + \omega \notin \Gamma_K$ and $f(\lambda+\omega; \omega) \neq 0$ or $f(\lambda+\omega; \omega) < 0$. If $\mu+\omega \notin \Gamma_K$, then there exists a simple root $\alpha \in P_K$ such that $(\mu+\omega, \alpha) < 0$. Since μ is P_K-dominant, the pair $(2(\mu, \alpha)|\alpha|^{-2}, 2(\omega, \alpha)|\alpha|^{-2})$ is one of the followings: $(0, -1)$, $(0, -2)$ and $(1, -2)$. For the cases $(0, -1)$ and $(1, -2)$ we have $(\lambda+\omega, \alpha) = 0$. If $\Delta_0(\omega)^* = \phi$, then (1) in Theorem 6.5 implies $f(\lambda+\omega; \omega) = 0$. When $\Delta_0(\omega)^* \neq \phi$, Lemma 6.1 implies that the case $(1, -2)$ is impossible. Consider the case $\Delta_0(\omega)^* \neq \phi$ and $(0, -1)$. Since $\alpha \in \Delta_-(\omega)$ and $(\lambda+\omega, \alpha) = 0$, (2) in Theorem 6.5 implies $f(\lambda+\omega; \omega) = 0$. Consequently if $\mu + \omega \notin \Gamma_K$ and $f(\lambda+\omega; \omega) \neq 0$, then $(\mu, \alpha) = 0$ and $2(\omega, \alpha)|\alpha|^{-2} = -2$. Let us consider the case $f(\lambda+\omega; \omega) < 0$. Since $\lambda+\omega$ is P_K-regular, it follows from (1) and (2) in Theorem 6.5 there exists a simple root α in P_K such that $(\lambda+\omega, \alpha) < 0$. Then we have also $(\mu, \alpha) = 0$ and $2(\omega, \alpha)|\alpha|^{-2} = -2$. Let us prove $s_\alpha(\lambda+\omega) \in rw_0(\lambda)$. Since $s_\alpha(\lambda+\omega) = \lambda+\omega+\alpha$, $\alpha \in \Delta_0(\omega+\alpha)^*$ and $2(\lambda+\omega+\alpha, \alpha)|\alpha|^{-2} = 1$, (2) in Theorem 6.5 implies that $f(\lambda+\omega+\alpha; \omega+\alpha) = 0$, and therefore, $s_\alpha(\lambda+\omega) \in rw_0(\lambda)$. It remains to prove that α is simple in P and is unique. In view of the proof of the previous lemma it is enough to consider the case $G = S_p(n, \mathbf{R})$. Then the set of all noncompact positive long roots is given by

$$\{\gamma_i = 2\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-1} + \alpha_n : 1 \leq i \leq n\}.$$

If $2(\gamma_i, \alpha)|\alpha|^{-2} = -2$ for a compact root α, then $2 \leq i \leq n$ and $\alpha = \alpha_{i-1}$. Especially, $\Delta_0(\gamma_i + \alpha_{i-1})^* = \{\alpha_{i-1}\}$.

Let G be of the type G_2. Then $P_K = \{\alpha, \beta\}$, α (resp. β) is short (resp. long), and α is simple and $(\alpha, \beta) = 0$ (see (2.11)).

LEMMA 7.5. *Let G be of the type G_2. Suppose that $\lambda+\omega \in rw_0(\lambda)$. Then we have $s_\alpha(\lambda+\omega) \in rw_-(\lambda)$. Conversely, suppose that $\lambda+\omega \in rw_-(\lambda)$. Then we have $s_\alpha(\lambda+\omega) \in rw_0(\lambda)$.*

PROOF. Assume that $\lambda+\omega \in rw_0(\lambda)$. Since $\lambda+\omega$ is P_K-regular and $f(\lambda+\omega;\omega)=0$, (1) in Theorem 6.5 and (2) in Lemma 6.1 imply that $\Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^* \neq \phi$, and hence, ω is a short root. By (3) in Theorem 6.5 we have

$$\prod_{\delta\in\Delta_{-1}(\omega)^*}((\lambda,\delta)-|\delta|^2)\prod_{\delta\in\Delta_1(\omega)^*}(2(\lambda,\delta)-|\delta|^2)=0\,.$$

This implies that

$$(\lambda,\delta)=|\delta|^2 \quad \text{for } \delta\in\Delta_{-1}(\omega)^* \quad \text{or} \quad 2(\lambda,\delta)=|\delta|^2 \ \text{ for } \delta\in\Delta_1(\omega)^*\,.$$

In both cases δ is a short root, and therefore $\delta=\alpha$. Consider the first case. Since $2(\lambda+\omega,\alpha)|\alpha|^{-2}=1$, we have $s_\alpha(\lambda+\omega)=\lambda+\omega-\alpha$ and $2(\mu+\omega-\alpha,\alpha)|\alpha|^{-2}=-1$. Therefore $\mu+\omega-\alpha\notin\Gamma_K$. Since $\omega-\alpha$ is a long root, we have $\Delta_0(\omega-\alpha)^*\cup\Delta_{-1}(\omega-\alpha)^*\cup\Delta_1(\omega-\alpha)^*=\phi$. This implies that $f(\lambda+\omega-\alpha;\omega-\alpha)\neq 0$. Thus $s_\alpha(\lambda+\omega)\in rw_-(\lambda)$. Consider the second case. Since $\alpha\in\Delta_1(\omega)^*$, we have $s_\alpha(\lambda+\omega)=\lambda+\omega-2\alpha$ and $\omega-2\alpha\in\Sigma_n$. Since $\omega-2\alpha$ is a long root and $2(\lambda+\omega-2\alpha,\alpha)|\alpha|^{-2}=-2$, we have also $s_\alpha(\lambda+\omega)\in rw_-(\lambda)$. The converse follows from the same arguments.

THEOREM 7.6. *Let $\omega\in\Sigma_n$ and $\mu\in\Gamma_K$. Assume that $\mu+\omega\in\Gamma_K$. Then the K-module $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu)$ is nontrivial if and only if $f(\lambda+\omega;\omega)>0$, where $\lambda=\mu+\rho_K$.*

PROOF. Assume that $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu)\neq\{0\}$. By Corollary 3.5 we have $P_{\mu+\omega}(X_\omega\otimes v(\mu))\neq 0$. Hence by Theorem 5.5 we have $f(\lambda+\omega;\omega)>0$. Let us prove the sufficiency of the theorem. Choosing a suitable covering group K^* of K, we can define the character ξ_{ρ_K} of the analytic subgroup B^* of K^* corresponding to $\mathfrak{b}$. Weyl's character formula (see (4.11)) implies that

$$(\Delta_K\operatorname{trace}(\mathrm{Ad}\otimes\pi_\mu))(\exp H)=\sum_{\lambda+\omega\in w(\lambda)}\sum_{t\in W_K}\varepsilon(t)e^{t(\lambda+\omega)(H)}$$

for all $\exp H\in B^*$, where (π_μ,V_μ) is a simple K-module with the highest weight μ and $w(\lambda)$ is the same as in Definition 7.1. We shall prove that

$$(\Delta_K\operatorname{trace}(\mathrm{Ad}\otimes\pi_\mu))(\exp H)=\sum_{\lambda+\omega\in rw_+(\lambda)}\sum_{t\in W_K}\varepsilon(t)e^{t(\lambda+\omega)(H)}\,. \tag{7.2}$$

If $\lambda+\omega\in w(\lambda)$ is P_K-singular, then

$$\sum_{t\in W_K}\varepsilon(t)e^{t(\lambda+\omega)(H)}=0\,.$$

Since $w(\lambda) = sw(\lambda) \cup rw_0(\lambda) \cup rw_-(\lambda) \cup rw_+(\lambda)$, it is enough to prove

$$\sum_{\lambda+\omega \in rw_0(\lambda)\cup rw_-(\lambda)} \sum_{t \in W_K} \varepsilon(t) e^{t(\lambda+\omega)(H)} = 0\,. \tag{7.3}$$

If G satisfies that all noncompact roots have the same length, then Lemma 7.2 implies $rw_0(\lambda) \cup rw_-(\lambda) = \phi$. Hence we can assume that G is one of $S_p(n, \mathbf{R})$ and $SO(2m, 2n+1)$, or G is of the type G_2. Consider the case G_2, and let α be the short root in P_K. By Lemma 7.5 we have $s_\alpha(rw_0(\lambda)) \subset rw_-(\lambda)$ and $s_\alpha(rw_-(\lambda)) \subset rw_0(\lambda)$. Since $(s_\alpha)^2 = 1$, we have $s_\alpha(rw_0(\lambda)) = rw_-(\lambda)$. This implies (7.3). Let G be one of $S_p(n, \mathbf{R})$ and $SO(2m, 2n+1)$. We define the mappings $\psi : rw_0(\lambda) \to rw_-(\lambda)$ and $\psi' : rw_-(\lambda) \to rw_0(\lambda)$ by the followings. Let $\lambda + \omega$ be an element in $rw_0(\lambda)$. By Lemma 7.3 there exists a unique compact simple short root α such that $(\mu, \alpha) = 0, \alpha \in \Delta_0(\omega)^*$ and $s_\alpha(\lambda+\omega) \in rw_-(\lambda)$. We note that $s_\alpha(\lambda+\omega) = \lambda+\omega-\alpha$. We now put $\psi(\lambda+\omega) = s_\alpha(\lambda+\omega)$. Similarly, by using Lemma 7.4, we define a mapping ψ'. We shall prove $\psi'\psi$, $\psi\psi'$ are the identities on $rw_0(\lambda)$ and $rw_-(\lambda)$ respectively. Let $\lambda+\omega \in rw_0(\lambda)$ and $\psi(\lambda+\omega) = s_\alpha(\lambda+\omega)$. Since $\alpha \in \Delta_0((\omega-\alpha)+\alpha)^*$ and $(\mu, \alpha) = 0$, α is the unique compact simple short root determined by $\lambda+\omega-\alpha \in rw_-(\lambda)$. This implies that $\psi'\psi(\lambda+\omega) = \lambda+\omega$. Similarly we can prove $\psi\psi'$ is the identity. Hence ψ is bijective, and thus, $rw_-(\omega) = \psi(rw_0(\omega))$. Therefore, we have (7.3) for this case. Let $\lambda+\omega \in rw_+(\lambda)$ and $\pi_{\mu+\omega}$ a simple K-module with the highest weight $\mu+\omega$. By (7.2) we have

$$\mathrm{trace}(\mathrm{Ad} \otimes \pi_\mu)(k) = \sum_{\mu+\omega \in \Gamma_K,\, f(\lambda+\omega;\omega)>0} \mathrm{trace}\, \pi_{\mu+\omega}(k)\,.$$

Thus if $\mu+\omega \in \Gamma_K$ and $f(\lambda+\omega; \omega) > 0$ then $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$ as claimed.

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