Further results on the exponent of convergence of zeros of solutions of certain higher order linear differential equations *

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Abstract

In this paper, we further investigate the exponent of convergence of the zero-sequence of solutions of the differential equation

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + \psi(z)f = 0,$$

where $\psi(z) = \sum_{j=1}^{\iota} Q_j(z)e^{P_j(z)} (\iota \geq 3, \iota \in N_+)$, $P_j(z)$ are polynomials of degree $n \geq 1$, $Q_j(z), a_\Lambda(z)(\Lambda = 1, 2, \cdots, k-1; j = 1, 2, \ldots, \iota)$ are entire functions of order less than n, and $k \geq 2$.

1 Introduction and Results

Complex oscillation theory of solutions of linear differential equations in the complex plane $\mathbb{C}$ was started by Bank and Laine [1, 2]. After their well-known work, many important results have been obtained see [3, 12, 13].

We will use the notation $\sigma(f)$ to denote the order of growth of a meromorphic function $f(z), \lambda(f)$ to denote the exponent of convergence of the zero-sequence

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of $f(z)$(see [9, 13]). Throughout our paper, we are always interested in non-trivial solutions f only, that is, $f \not\equiv 0$.

In 1987, Bank and Langley investigated the oscillation of solutions of certain linear differential equations and obtained

Theorem A (see [4]) *Suppose that $k \geq 2$ and that $A(z) = \Pi(z)e^{P(z)} \not\equiv 0$ where the entire function $\Pi(z)$ and the polynomial $P(z) = a_n z^n + \cdots + a_0$ satisfy:*

(i) $\sigma(\Pi) < n$;

(ii) there exists $\theta_0 \in \mathbb{R}$ with $\delta(P, \theta_0) = Re(a_n e^{in\theta_0}) = 0$ and a positive ε such that $\Pi(z)$ has only finitely many zeros in $|\arg z - \theta_0| < \varepsilon$.

Then if $n \geq 2$ and Q is a polynomial whose degree d_Q satisfies $d_Q + k < kn$, all non-trivial solutions f of

$$y^{(k)} + (A(z) + Q(z))y = 0$$

satisfy $\lambda(f) = \infty$. The same conclusion holds if $n = 1$ and Q is identically zero.

In 1997, Ishizaki and Tohge [10, 11] have studied the exponent of convergence of the zero-sequence of solutions of the equation

$$f'' + (e^{P_1(z)} + e^{P_2(z)} + Q_0(z))f = 0, \tag{1}$$

where $P_1(z), P_2(z)$ are non-constant polynomials

$$P_1(z) = \zeta_1 z^n + \cdots, \quad P_2(z) = \zeta_2 z^m + \cdots, \quad \zeta_1\zeta_2 \neq 0 \quad (n, m \in N).$$

and $Q_0(z)$ is an entire function of order less than $\max\{n, m\}$, and $e^{P_1(z)}$ and $e^{P_2(z)}$ are linearly independent. They have obtained the following results:

Theorem B (see [11]). *Suppose that $n = m$, and that $\zeta_1 \neq \zeta_2$ in (1). If $\frac{\zeta_1}{\zeta_2}$ is non-real, then for any non-trivial solution f of (1), we have $\lambda(f) = \infty$.*

Theorem C (see [10]). *Suppose that $n = m$, and that $\frac{\zeta_1}{\zeta_2} = \rho > 0$ in (1). If $0 < \rho < \frac{1}{2}$ or $Q_0(z) \equiv 0$, $\frac{3}{4} < \rho < 1$, then for any non-trivial solution f of (1), we have $\lambda(f) \geq n$.*

In 2007, Tu and Chen [15] studied the exponent of convergence of the zero-sequence of solutions of

$$f'' + \left(Q_1(z)e^{P_1(z)} + Q_2(z)e^{P_2(z)} + Q_3(z)e^{P_3(z)}\right) f = 0, \tag{2}$$

and obtain the following results.

Theorem D (see [15]). *Let $Q_1(z), Q_2(z), Q_3(z)$ be entire functions of order less than n, and $P_1(z), P_2(z), P_3(z)$ be polynomials of degree $n \geq 1$,*

$$P_1(z) = \zeta_1 z^n + \cdots, \; P_2(z) = \zeta_2 z^n + \cdots, \; P_3(z) = \zeta_3 z^n + \cdots,$$

where $\zeta_1, \zeta_2, \zeta_3$ are complex numbers.

(i) If $\frac{\zeta_1}{\zeta_2}$ is non-real, $0 < \lambda = \frac{\zeta_3}{\zeta_2} < \frac{1}{2}$, then for any non-trivial solution f of (2), we have $\lambda(f) = \infty$.

(ii) If $0 < \frac{\zeta_2}{\zeta_1} < \frac{1}{4}$, $0 < \lambda = \frac{\zeta_3}{\zeta_2} < 1$, then for any non-trivial solution f of (2), we have $\lambda(f) \geq n$.

Recently, Tu and Yang [16] investigated the exponent of convergence of the zero-sequence of solutions of the differential equation

$$(2') \qquad f'' + \left(Q_1(z)e^{P_1(z)} + Q_2(z)e^{P_2(z)} + \cdots + Q_\iota(z)e^{P_\iota(z)}\right) f = 0,$$

and obtained the following result which extended Theorem D:

Theorem E (see [16]). *Let $Q_1(z)(\not\equiv 0), Q_2(z), \cdots, Q_l(z)(l \geq 3)$ be entire functions of order less than n, and $P_1(z), P_2(z), \cdots, P_l(z)(l \geq 3)$ be polynomials of degree $n \geq 1$,*

$$P_1(z) = \zeta_1 z^n + \cdots, \quad P_2(z) = \zeta_2 z^n + \cdots, \ \cdots, \ P_l(z) = \zeta_l z^n + \cdots,$$

where $\zeta_1, \zeta_2, \cdots, \zeta_l$ are complex numbers.

(i) If $\frac{\zeta_1}{\zeta_2}$ is non-real, $0 < \lambda_j = \frac{\zeta_j}{\zeta_2} < \frac{1}{2}$ $(j = 3, \cdots, l)$, then any non-trivial solution f of $(2')$ satisfies $\lambda(f) = \infty$.

(ii) If $0 < \rho = \frac{\zeta_2}{\zeta_1} < \frac{1}{4}$, $\lambda_j = \frac{\zeta_j}{\zeta_2} > 0$ and $\sum_{j=3}^{l} \lambda_j < 1$, then any non-trivial solution f of $(2')$ satisfies $\lambda(f) \geq n$.

It is natural to ask: what results can we get when we investigate the exponent of convergence of the zero-sequence of solutions of the higher order linear differential equation

$$(3) \qquad f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + \psi(z)f = 0,$$

where $\psi(z) = \sum_{j=1}^{\iota} Q_j(z)e^{P_j(z)}(\iota \geq 3, \iota \in N_+)$, $P_j(z)$ are polynomials of degree $n \geq 1$, $Q_j(z), a_\Lambda(z)(\Lambda = 1, 2, \cdots, k-1; j = 1, 2, \ldots, \iota)$ are entire functions of order less than n, and $k \geq 2$.

In the present paper we shall investigate the above problem and obtain the following result which improve all the previous theorems mentioned earlier.

Theorem 1.1. *Let $P_j(z), Q_j(z)(j = 1, 2, \ldots, \iota(\geq 3))$ be defined in Theorem D and $a_\Lambda(z)$ $(\Lambda = 1, 2, \cdots, k-1)$ be entire functions of order less than n,$k \geq 2$.*

(i) If $\frac{\zeta_1}{\zeta_2}$ is non-real, $0 < \lambda_j = \frac{\zeta_j}{\zeta_2} < \frac{1}{k}(j = 3, 4, \ldots, \iota)$, then for any non-trivial solution f of (3), we have $\lambda(f) = \infty$.

(ii) If $0 < \frac{\zeta_1}{\zeta_2} < \frac{1}{2k}$, $0 < \lambda_j = \frac{\zeta_j}{\zeta_2}$ and $\sum_{j=3}^{\iota} \lambda_j < 1$, then for any non-trivial solution f of (3), we have $\lambda(f) \geq n$.

2 Notation and Some Lemmas

To prove the theorem, we need some notations and a series of lemmas. Let $P_j(z)(j = 1, 2, \ldots, \iota)$ be polynomials of degree $n \geq 1$, $P_j(z) = (\alpha_j + i\beta_j)z^n + \cdots, \alpha_j, \beta_j \in \mathbb{R}$. Define

$$\delta(P_j, \theta) = \delta_j(\theta) = \alpha_j \cos n\theta - \beta_j \sin n\theta, \qquad \theta \in [0, 2\pi)(j = 1, 2, \ldots, \iota),$$

$$S_j^+ = \{\theta | \delta_j(\theta) > 0\}, \qquad S_j^- = \{\theta | \delta_j(\theta) < 0\} \qquad (j = 1, 2, \ldots, \iota).$$

Let $f(z), a(z)$ be meromorphic functions in the complex plane $\mathbb{C}$ and satisfy

$$T(r,a) = o\{T(r,f)\},$$

except possibly for a set of r having finite linear measure, we say that $a(z)$ is a small function with respect to $f(z)$.

Lemma 2.1. (see [8]). *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$, k, j be two integers which satisfy $k > j \geq 0$. And let $\varepsilon > 0$ be a given constant, then there exists a set $E \subset [0, 2\pi)$ which has linear measure zero, such that if $\varphi \in [0, 2\pi) \backslash E$, there is a constant $R_1 = R_1(\varphi) > 1$, such that for all z satisfying $\arg z = \varphi$ and $|z| = r > R_1$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Lemma 2.2. (see [5, 14]). *Suppose that $P(z) = (\alpha + \beta i)z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geq 1$, that $A(z)(\not\equiv 0)$ is an entire function with $\sigma(A) < n$. Set $g(z) = A(z)e^{P(z)}, z = re^{i\theta}, \delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$ that has the linear measure zero, such that for any $\theta \in [0, 2\pi) \backslash (H_1 \cup H_2)$, there is $R > 0$ such that for $|z| = r > R$, we have:*

(i) If $\delta(P, \theta) > 0$, then

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1+\varepsilon)\delta(P,\theta)r^n\};$$

(ii) If $\delta(P, \theta) < 0$, then

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1-\varepsilon)\delta(P,\theta)r^n\},$$

where $H_2 = \{\theta \in [0, 2\pi); \delta(P, \theta) = 0\}$ is a finite set.

Lemma 2.3. (see [5]). *Supposes $\pi(z)$ is the canonical product formed with the zeros $\{z_n : n = 1, 2, \ldots, \}(z_n \neq 0)$ of an entire function $f(z)$. Set $O_n = \{z : |z - z_n| < |z_n|^{-\alpha}\}(\alpha(> \lambda(f))$ is a constant). Then for any given $\varepsilon > 0$,*

$$|\pi(z)| \geq \exp\{-|z|^{\lambda(f)+\varepsilon}\}$$

holds for $z \notin \bigcup_{n=1}^{\infty} O_n$.

Lemma 2.4. (see [7]). *Let $f(z)$ be an entire function of order $\sigma(f) = \alpha < +\infty$. Then for any given $\varepsilon > 0$, there is a set $E \subset [1, \infty)$ that has finite linear measure and finite logarithmic measure such that for all z satisfying $|z| \notin [0, 1] \cup E$, we have*

$$\exp\{-r^{\alpha+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\alpha+\varepsilon}\}.$$

Lemma 2.5. (see [16]). *Let $P_j(z)(j = 1, \cdots, \iota)$ be polynomials of degree $n \geq 1$,*

$$P_1(z) = \zeta z^n + B_1(z), \quad P_2(z) = \rho_2 \zeta z^n + B_2(z), \quad \cdots, \quad P_\iota(z) = \rho_\iota \zeta z^n + B_\iota(z),$$

where $\zeta = \alpha + \beta i$, $\alpha, \beta \in \mathbb{R}$, $|\alpha| + |\beta| \neq 0$, $0 < \rho_j < 1, j = 2, \cdots, \iota$, $B_1(z), \cdots, B_\iota(z)$ are polynomials of degree at most $n - 1$. Let $Q_1(z) \not\equiv 0, Q_2(z), \cdots, Q_\iota(z)$ be entire functions of order less than n, then for any given $\varepsilon > 0$, there exist a set E with finite linear measure and a constant $\xi(n - 1 < \xi < n)$ such that

$$m(r, Q_1 e^{P_1} + Q_2 e^{P_2} + \cdots + Q_\iota e^{P_\iota}) \geq (1-\varepsilon)m(r, e^{P_1}) + O(r^\xi), \quad r \to \infty, \quad (r \notin E).$$

Lemma 2.6. (see [9, 17]). *Let $f(z)$ be an entire function and write $f(z) = \pi e^h$. Then we have*

(i)

$$\frac{f^{(k)}}{f} = (h')^k + k\frac{\pi'}{\pi}(h')^{k-1} + \frac{k(k-1)}{2}(h')^{k-2}h'' + H_{k-2}(h'), \quad (k \geq 2),$$

where $H_{k-2}(h')$ is a differential polynomial of degree no more than $k-2$ in h', its coefficients are terms of the type $c(\frac{\pi'}{\pi})^{s_1}(\frac{\pi''}{\pi})^{s_2}\cdots(\frac{\pi^{(k)}}{\pi})^{s_k}$, where c is a constant, $s_1, s_2, \cdots, s_k$ are non-negative integers.

(ii)

$$\frac{f^{(k+1)}}{f} - \frac{f^{(k)}}{f}\frac{f'}{f} = k(h')^{k-1}h'' + H_{k-1}(h') \quad (k \geq 1),$$

where $H_{k-1}(h')$ is a differential polynomial of degree no more than $k-1$ in h', its coefficients are terms of the type $c(\frac{\pi'}{\pi})^{s_1}(\frac{\pi''}{\pi})^{s_2}\cdots(\frac{\pi^{(k)}}{\pi})^{s_k}(\frac{\pi^{(k+1)}}{\pi})^{s_{k+1}}$, where c is a constant, $s_1, s_2, \cdots, s_{k+1}$ are non-negative integers.

Lemma 2.7. (see [17]). *Let $U_1(z), h(z), Q_1(z), P_1(z)$ be entire functions and satisfy $U_1 = Q_1 h'' - \frac{1}{k}(Q_1' + Q_1 P_1')h'$. Then*

$$Q_1^{n-1}h^{(n)} = A_{1,n-2}(U_1, Q_1) + B_{n-1}(Q_1)h', \quad (n \geq 2),$$

where $A_{1,n-2}(U_1, Q_1)$ is an algebraic expression in the terms $U_1^{(j)}, Q_1^{(j)}$, $P_1^{(j)}$ $(j = 0, 1, \ldots, l)$, such as addition,subtraction and multiplication, where the degree of $U_1^{(j)}$ is no more than 1 and the degree of $Q_1^{(j)}$ is no more than l; $B_d(Q_1)$ is a differential polynomial of degree no more than d in Q_1, its coefficients are algebraic expressions in terms $P_1^{(i)}(i = 1, 2, \ldots, d)$ and $\frac{1}{k}$, such as addition,subtraction and multiplication.

Lemma 2.8. *Let $h(z), c_j(z)(j = 0, 1, \ldots, k-1)$ be meromorphic functions and satisfy*

$$c_{k-1}(z)(h')^{k-1} + c_{k-2}(z)(h')^{k-2} + \cdots + c_1(z)h' + c_0(z) = 0.$$

Then we have

$$m(r, h') \leq \sum_{j=0}^{k-1} T(r, c_j(z)) + O(1).$$

Lemma 2.9. *Let h is a meromorphic function of finite order, $E_{k-1}(h')$ is a differential polynomial of degree no more than $k-1$, its coefficients are meromorphic functions $a_j(z)(j = 0, 1, \ldots, k-1)$ satisfying $\sigma(a_j) < n$. Then for sufficiently large r,*

$$m(r, (h')^k + E_{k-1}(h')) \leq km(r, h') + O(r^{\xi}),$$

where $0 < max\{\sigma(a_j) | j = 0, 1, \ldots, k-1\} < \xi < n$.

Remark 2.1. *Lemma 2.8 and 2.9 are immediate consequences of the Valiron-Mohon'ko theorem (see [11]) and/or Clunie technique.*

3 Proof of Theorem 1.1(i)

Since $\zeta_j = \lambda_j \zeta_2, \lambda_j > 0 (j = 3, 4, \ldots, \iota)$, we have $S_2^+ = S_3^+ = \cdots = S_\iota^+, S_2^- = S_3^- = \cdots = S_\iota^-$. We see that S_j^+ and S_j^- have n components $S_{j\ell}^+$ and $S_{j\ell}^-$ respectively $(j = 1, 2, \ldots, \iota; \ell = 1, 2, \ldots, n)$. Hence we write

$$S_j^+ = \bigcup_{\ell=1}^{n} S_{j\ell}^+, \quad S_j^- = \bigcup_{\ell=1}^{n} S_{j\ell}^- \quad (j = 1, 2, \ldots, \iota).$$

(i) Let $f \not\equiv 0$ be a solution of (3). Suppose that $\lambda(f) < \infty$. Write $f = \pi e^h$, where π is the canonical product from the zeros of f, and h is an entire function. From our hypothesis, we have $\sigma(\pi) = \lambda(\pi) < \infty$. From (3), we get

$$\frac{f^{(k)}}{f} + a_{k-1}\frac{f^{(k-1)}}{f} + \cdots + a_1\frac{f'}{f} + \psi(z) = 0, \tag{4}$$

By Lemma 2.6(i), we get

$$(h')^k = E_{k-1}(h') - Q_1(z)e^{P_1(z)} - Q_2(z)e^{P_2(z)} - \cdots - Q_\iota(z)e^{P_\iota(z)}, \tag{5}$$

where $E_{k-1}(h')$ is a differential polynomial of degree no more than $k-1$ in h', its coefficients are terms of type $ca_j^p(z)(\frac{\pi'}{\pi})^{s_1}(\frac{\pi''}{\pi})^{s_2}\cdots(\frac{\pi^{(k)}}{\pi})^{s_k}(j = 1, 2, \ldots, k-1)$, where c is a constant, $s_1, s_2, \cdots, s_k$ are non-negative integers and p is 0 or 1.

Eliminating e^{P_1} from (4), we have

$$Q_1\left(\frac{f^{(k+1)}}{f} - \frac{f^{(k)}}{f}\frac{f'}{f}\right) + a_{k-1}Q_1\left(\frac{f^{(k)}}{f} - \frac{f^{(k-1)}}{f}\frac{f'}{f}\right) + \cdots + a_1Q_1\left(\frac{f''}{f} - \frac{f'}{f}\frac{f'}{f}\right)$$
$$-(Q_1' + Q_1P_1')\left(\frac{f^{(k)}}{f} + a_{k-1}\frac{f^{(k-1)}}{f} + \cdots + a_1\frac{f'}{f} + \sum_{j=2}^{\iota} Q_je^{P_j}\right)$$
$$+Q_1\left[a_{k-1}'\frac{f^{(k-1)}}{f} + \cdots + a_1'\frac{f'}{f}\right] + Q_1\sum_{j=2}^{\iota}\left(Q_j' + Q_jP_j'\right)e^{P_j} = 0.$$

By Lemma 2.6(ii), we can write this as

$$kU_1(h')^{k-1} = F_{k-1}^1(h') + \sum_{j=2}^{\iota}\left[Q_j(Q_1' + Q_1P_1') - Q_1(Q_j' + Q_jP_j')\right]e^{P_j}, \tag{6}$$

where

$$U_1 = Q_1h'' - \frac{1}{k}(Q_1' + Q_1P_1')h', \tag{7}$$

and $F_{k-1}^1(h')$ is a differential polynomial of degree no more than $k-1$ in h', its coefficients are terms of the type $c(a_j(z))^p(a_j'(z))^q(Q_1)^l(Q_1')^t(P_1')^u(\frac{\pi'}{\pi})^{s_1}(\frac{\pi''}{\pi})^{s_2}\cdots$

$(\frac{\pi^{(k)}}{\pi})^{s_k}$, where c is a constant, $s_1, s_2, \cdots, s_k$ are non-negative integers and each of p, q, l, t, u is 0 or 1. Similarly, eliminating e^{P_2} from (4), we obtain

$$(8) \quad kU_2(h')^{k-1} = F^2_{k-1}(h') + \sum_{j=1, j\neq 2}^{\iota} \left[Q_j(Q_2' + Q_2P_2') - Q_2(Q_j' + Q_jP_j')\right] e^{P_j},$$

where

$$(9) \qquad U_2 = Q_2h'' - \frac{1}{k}(Q_2' + Q_2P_2')h',$$

and $F^2_{k-1}(h')$ is a differential polynomial of degree no more than $k-1$ in h', its coefficients are terms of the type $c(a_j(z))^p(a_j'(z))^q(Q_2)^l(Q_2')^t(P_2')^u(\frac{\pi'}{\pi})^{s_1}(\frac{\pi''}{\pi})^{s_2}\cdots(\frac{\pi^{(k)}}{\pi})^{s_k}$, where c is a constant, $s_1, s_2, \cdots, s_k$ are non-negative integers and each of p, q, l, t, u is 0 or 1.

From the assumptions of Theorem 1.1, there exists three positive real numbers ξ_1, ξ_2, ξ_3 such that $\max\{\sigma(Q_j), \sigma(a_\Lambda), j = 1, 2, \ldots, \iota; \Lambda = 1, 2, \ldots, k-1\} < \xi_1 < \xi_2 < \xi_3 < n$, from Lemma 2.4 we get

$$|Q_j(re^{i\theta})| \leq \exp(r^{\xi_1}), (j = 1, 2, \ldots, \iota); \quad |a_\Lambda(re^{i\theta})| \leq \exp(r^{\xi_1}), (\Lambda = 1, 2, \ldots, k-1),$$

for sufficiently large r and for any $\theta \in [0, 2\pi)$. Applying the Clunie Lemma [9, Lemma 3.3] to (5), for any given $\varepsilon > 0$,

$$\begin{aligned} T(r, h') = m(r, h') &\leq m(r, Q_1e^{P_1} + Q_2e^{P_2} + \cdots + Q_\iota e^{P_\iota}) \\ &\quad + O\left(\sum_{j=1}^{k} m(r, \frac{\pi^{(j)}}{\pi}) + \sum_{\Lambda=1}^{k-1} m(r, a_\Lambda)\right) + S(r, h') \\ &\leq O(r^{n+\varepsilon}) + S(r, h'), \end{aligned}$$

which implies $\sigma(h') \leq n$. It follows from (7) and (9) that $\sigma(U_1) \leq n$ and $\sigma(U_2) \leq n$ respectively.

In the following, we will show that there exists a set $E_0 \subset [0, 2\pi), m(E_0) = 0$ such that if $\theta \in S_2^- \backslash E_0$, then

$$(10) \qquad |U_1(re^{i\theta})| \leq O(\exp\{r^{\xi_2}\}), \qquad r \to \infty.$$

If $|h'(re^{i\theta})| \leq 1$, from Lemmas 2.1,2.2 and 2.4 and (7), we have

(11)

$$\begin{aligned} |U_1(re^{i\theta})| &\leq \frac{|h''(re^{i\theta})|}{|h'(re^{i\theta})|}|Q_1(re^{i\theta})| + \frac{1}{k}|P_1'(re^{i\theta})||Q_1(re^{i\theta})| + \frac{1}{k}\frac{|Q_1'(re^{i\theta})|}{|Q_1(re^{i\theta})|}|Q_1(re^{i\theta})| \\ &\leq O(\exp\{r^{\xi_2}\}), \qquad r \to \infty. \end{aligned}$$

If $|h'(re^{i\theta})| \geq 1$. Since $F^1_{k-1}(h')$ is the sum of a finite number of terms of the type

$$\begin{aligned} H(z) &= c(a_j(z))^p(a_j'(z))^q(Q_1)^l(Q_1')^t(P_1')^u(\frac{\pi'}{\pi})^{s_1}(\frac{\pi''}{\pi})^{s_2}\cdots(\frac{\pi^{(k)}}{\pi})^{s_k} \\ &\quad \times (h')^{l_0}(h'')^{l_1}\cdots(h^{(v)})^{l_{v-1}}, \end{aligned}$$

where $l_0, l_1, \cdots, l_{v-1}$ are non-negative integers and $l_0 + l_1 + \cdots + l_{v-1} \le k-1$, from Lemma 2.1 we can get

$$(12)\quad \begin{aligned} \frac{|H(re^{i\theta})|}{|h'(re^{i\theta})|^{k-1}} &\le |c||a_j(re^{i\theta})|^p |a_j'(re^{i\theta})|^q |Q_1(re^{i\theta})|^l |Q_1'(re^{i\theta})|^t |P_1'(re^{i\theta})|^u \\ &\quad \times |\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}|^{s_1} \cdots |\frac{\pi^{(k)}(re^{i\theta})}{\pi(re^{i\theta})}|^{s_k} \frac{|h''(re^{i\theta})|^{l_1}}{|h'(re^{i\theta})|} \cdots \frac{|h^{(v)}(re^{i\theta})|^{l_{v-1}}}{|h'(re^{i\theta})|} \\ &\le O(\exp\{r^{\xi_2}\}). \end{aligned}$$

Thus

$$(13)\qquad \frac{|F_{k-1}^1(re^{i\theta})|}{|h'(re^{i\theta})|^{k-1}} \le O(\exp\{r^{\xi_2}\}).$$

From (6),(13) and Lemma 2.2, we get

$$(14)\quad \begin{aligned} k|U_1(re^{i\theta})| &\le \frac{|F_{k-1}^1(re^{i\theta})|}{|h'(re^{i\theta})|^{k-1}} + \sum_{j=2}^{\iota} |e^{P_j(re^{i\theta})}| \Big| \Big(Q_1'(re^{i\theta}) + Q_1(re^{i\theta}) P_1'(re^{i\theta}) \Big) \\ &\quad \times Q_j(re^{i\theta}) - Q_1(re^{i\theta})((Q_j'(re^{i\theta}) + Q_j(re^{i\theta}) P_j'(re^{i\theta})) \Big| \\ &\le O(\exp\{r^{\xi_2}\}), \qquad r \to \infty. \end{aligned}$$

From (11) and (14), we obtain (10).

We note that there exist $\bar{\theta}_j (j = 1, 2, \ldots, \iota)$ satisfying $\delta_j(\theta) = 0$ on the rays $\arg z = \bar{\theta}_j + \frac{\gamma\pi}{n}$, where $\gamma = 0, \ldots, 2n-1$, which form $2n$ sectors of opening $\frac{\pi}{n}$ respectively, thus we may assume that $\bar{\theta}_j \in [0, \frac{\pi}{n})$. Since $\zeta_j = \lambda_j \zeta_2, \lambda_j > 0$ $(j = 3, 4, \ldots, \iota)$, we have $\bar{\theta}_j = \bar{\theta}_2 (j = 3, 4, \ldots, \iota)$. Write $\bar{\theta}_{j\gamma} = \bar{\theta}_j + \frac{\gamma\pi}{n}, j = 1, 2$, if there are some integers γ_1 and γ_2 such that $\bar{\theta}_{1\gamma_1} = \bar{\theta}_{2\gamma_2}$, then $\bar{\theta}_1 - \bar{\theta}_2 + (\gamma_1 - \gamma_2)\frac{\pi}{n} = 0$, we have that $\tan n\bar{\theta}_j = \frac{\alpha_j}{\beta_j}, j = 1, 2$. This gives

$$0 = \tan(n\bar{\theta}_1 - n\bar{\theta}_2 + (\gamma_1 - \gamma_2)\pi) = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\alpha_1\alpha_2 + \beta_1\beta_2}.$$

This contradicts the assumption that $\frac{\zeta_1}{\zeta_2}$ is non-real. Hence we see that each component of S_1^+ and S_2^+ contains a component of $S_1^+ \cap S_2^+$. The boundaries of the components of $S_1^+ \cap S_2^+$ are some of the rays $\arg z = \bar{\theta}_{j\gamma}$, we fix a component of $S_1^+ \cap S_2^+$, say S^*. We may write

$$S^* = \{\theta \in S_1^+ \cap S_2^+ : \theta_1^* < \theta < \theta_2^*, \delta_1(\theta_1^*) = \delta_2(\theta_2^*) = 0\}$$

or

$$S^* = \{\theta \in S_1^+ \cap S_2^+ : \theta_2^* < \theta < \theta_1^*, \delta_1(\theta_1^*) = \delta_2(\theta_2^*) = 0\}.$$

We define

$$D_{12} = \left\{ \theta \in S_1^+ \cap S_2^+ : \delta_1(\theta) > \frac{k(\lambda+1)}{k-1} \delta_2(\theta) \right\},$$

$$D_{21} = \left\{ \theta \in S_1^+ \cap S_2^+ : \delta_2(\theta) > \frac{\lambda+1}{\lambda}\delta_1(\theta) \right\},$$

where $\lambda = \max\{\lambda_j : j = 3, 4, \ldots, \iota\} < \frac{1}{k}$. Since every component of S_1^+ and S_2^+ is a sector of opening $\frac{\pi}{n}$, the rays $\arg z = \theta_1^*$ and $\arg z = \theta_2^*$ are contained in S_2^+ and S_1^+ respectively. We treat the first case, the proof of the second case can be obtained similarly. Hence there exist $\eta_1 > 0, \eta_2 > 0$ such that

$$\{\theta : \theta_1^* < \theta < \theta_1^* + \eta_1\} \subset D_{21}, \qquad \{\theta : \theta_2^* - \eta_2 < \theta < \theta_2^*\} \subset D_{12}.$$

Hence there exists a $\theta \in (S_{2k}^+ \cap D_{12}) \backslash E_0$ for any $k = 1, 2, \ldots, n$. Set $0 < \frac{k(\lambda+1)}{k-1}\delta_2 < \sigma_2 < \sigma_1 < \delta_1, 0 < \varepsilon_1 < 1 - \frac{\sigma_1}{\delta_1}, 0 < \varepsilon_2 < \frac{(k-1)\sigma_2}{k\delta_2} - 1, \ldots, 0 < \varepsilon_\iota < \frac{(k-1)\sigma_2}{k\lambda_\iota\delta_2} - 1$. By Lemma 2.2, we have

$$\begin{aligned}
(15)\qquad & |Q_1 e^{P_1(re^{i\theta})} + Q_2 e^{P_2(re^{i\theta})} + \cdots + Q_\iota e^{P_\iota(re^{i\theta})}| \\
& \geq \left|Q_1 e^{P_1(re^{i\theta})}\right| \left| 1 - \left|\frac{Q_2}{Q_1} e^{P_2(re^{i\theta}) - P_1(re^{i\theta})}\right| - \cdots - \left|\frac{Q_\iota}{Q_1} e^{P_\iota(re^{i\theta}) - P_1(re^{i\theta})}\right| \right| \\
& \geq \exp\{(1-\varepsilon_1)\delta_1 r^n\}(1 - o(1)) \\
& \geq \exp\{\sigma_1 r^n\}(1 - o(1)), \qquad r \to \infty.
\end{aligned}$$

We assume that there exists an unbounded sequence $\{r_\kappa\}_{\kappa=1}^\infty$ such that $0 < |h'(r_\kappa e^{i\theta})| \leq 1$. From (5) and (15) and Lemma 2.1, we get

$$\begin{aligned}
\exp\{\sigma_1 r_\kappa^n\}(1 - o(1)) \;&\leq\; |h'(r_\kappa e^{i\theta})|^k + |E_{k-1}(h'(r_\kappa e^{i\theta}))| \\
&\leq\; 1 + \sum |c||a_\Lambda(r_\kappa e^{i\theta})|^p |\frac{\pi'(r_\kappa e^{i\theta})}{\pi(r_\kappa e^{i\theta})}|^{s_1} \cdots |\frac{\pi^{(k)}(r_\kappa e^{i\theta})}{\pi(r_\kappa e^{i\theta})}|^{s_k} \\
&\quad\; \times |h'(r_\kappa e^{i\theta})|^{l_0} \cdots |h^{(v)}(r_\kappa e^{i\theta})|^{l_{v-1}} \\
&\leq\; 1 + \sum |c||a_\Lambda(r_\kappa e^{i\theta})|^p |\frac{\pi'(r_\kappa e^{i\theta})}{\pi(r_\kappa e^{i\theta})}|^{s_1} \cdots |\frac{\pi^{(k)}(r_\kappa e^{i\theta})}{\pi(r_\kappa e^{i\theta})}|^{s_k} \\
&\quad\; \times |\frac{h''(r_\kappa e^{i\theta})}{h'(r_\kappa e^{i\theta})}|^{l_1} \cdots |\frac{h^{(v)}(r_\kappa e^{i\theta})}{h'(r_\kappa e^{i\theta})}|^{l_{v-1}} \\
&\leq\; O(\exp\{r_\kappa^{\xi_2}\}), \qquad (\kappa \to \infty),
\end{aligned}$$

which is not true. Hence we may assume that $|h'(re^{i\theta})| \geq 1$ for all r sufficiently large. From (5),(15) and Lemma 2.2, we get

$$\begin{aligned}
\exp\{\sigma_1 r^n\}(1 - o(1)) \;&\leq\; |h'(re^{i\theta})|^k + |E_{k-1}(h'(re^{i\theta}))| \\
&\leq\; |h'(re^{i\theta})|^k [1 + \sum |c||a_\Lambda(re^{i\theta})|^p |\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}|^{s_1} \cdots |\frac{\pi^{(k)}(re^{i\theta})}{\pi(re^{i\theta})}|^{s_k} \\
&\quad\; \times |\frac{h''(re^{i\theta})}{h'(re^{i\theta})}|^{l_1} \cdots |\frac{h^{(v)}(re^{i\theta})}{h'(re^{i\theta})}|^{l_{v-1}}] \\
&\leq\; |h'(re^{i\theta})|^k (1 + O(\exp\{r^{\xi_2}\})), \qquad (r \to \infty),
\end{aligned}$$

i.e.

$$|h'(re^{i\theta})|^k \geq \frac{1 - o(1)}{1 + O(\exp\{r^{\xi_2}\})} \exp\{\sigma_1 r^n\}, \qquad (r \to \infty).$$

Then we obtain for all r large enough

$$|h'(re^{i\theta})| \geq \exp\left\{\frac{1}{k}\sigma_2 r^n\right\}. \tag{16}$$

From Lemma 2.1,(6) and (16), we get

(17)

$$\begin{aligned}
k|U_1(re^{i\theta})| \leq & \frac{|F^1_{k-1}(re^{i\theta})|}{|h'(re^{i\theta})|^{k-1}} + \sum_{j=2}^{\iota} \frac{|e^{P_j(re^{i\theta})}|}{|h'(re^{i\theta})|^{k-1}} \\
& \times \left[|Q_j(re^{i\theta})| \left(\frac{|Q_1'(re^{i\theta})|}{|Q_1(re^{i\theta})|}|Q_1(re^{i\theta})| + |Q_1(re^{i\theta})| \cdot |P_1'(re^{i\theta})|\right)\right. \\
& \left. + |Q_1(re^{i\theta})| \times \left(\frac{|Q_j'(re^{i\theta})|}{|Q_j(re^{i\theta})|}|Q_j(re^{i\theta})| + |Q_j(re^{i\theta})||P_j'(re^{i\theta})|\right)\right] \\
\leq & O(\exp\{r^{\xi_2}\}) + (1+o(1))\exp\left\{(\delta_2(1+\varepsilon_2) - \frac{(k-1)\sigma_2}{k})r^n\right\} \\
& + \cdots + (1+o(1))\exp\left\{(\lambda_\iota\delta_2(1+\varepsilon_\iota) - \frac{(k-1)\sigma_2}{k})r^n\right\}, \quad (r \to \infty).
\end{aligned}$$

Since $\delta_2(1+\varepsilon_2) - \frac{(k-1)\sigma_2}{k} < 0, \ldots, \lambda_\iota\delta_2(1+\varepsilon_\iota) - \frac{(k-1)\sigma_2}{k} < 0$, it gives that for all sufficiently large r,

$$|U_1(re^{i\theta})| \leq O(\exp\{r^{\xi_2}\}). \tag{18}$$

Now we fix a $\Phi(=\Phi_{2k}) \in (S^+_{2k} \cap D_{12})\backslash E_0, k = 1, 2, \ldots, n$. Then we find $\Phi_1, \Phi_2 \in S^-_2\backslash E_0, \Phi_1 < \Phi < \Phi_2$ such that $\Phi - \Phi_1 < \frac{\pi}{n}, \Phi_2 - \Phi < \frac{\pi}{n}$. We first prove that for any $\theta, \Phi_1 \leq \theta \leq \Phi$, we have

$$|U_1(re^{i\theta})| \leq O(\exp\{r^{\xi_3}\}), \qquad (r \to \infty). \tag{19}$$

Write $\Phi - \Phi_1 = \frac{\pi}{n+\tau_1}, \tau_1 > 0$, since $\sigma(U_1) \leq n$, we have that $|U_1(re^{i\theta})| \leq e^{r^{n+\tau_2}}, 0 < \tau_2 < \tau_1$ for sufficiently large r. Set $g(z) = U_1(z)/\exp((ze^{-\frac{\Phi+\Phi_1}{2}})^{\xi_3})$, then $g(z)$ is regular in the region $\{z : \Phi_1 \leq \arg z \leq \Phi\}$. Since $\Phi_1 \leq \arg z = \theta \leq \Phi, \Phi - \Phi_1 < \frac{\pi}{n}$, we infer that $\cos(\arg((ze^{-\frac{\Phi+\Phi_1}{2}})^{\xi_3}) \geq K$ for some $K > 0$. In fact,

$$-\frac{\pi}{2} < -\frac{\pi\xi_3}{2n} \leq -\xi_3\frac{\Phi-\Phi_1}{2} \leq \arg\left((ze^{-\frac{\Phi+\Phi_1}{2}})^{\xi_3}\right) \leq \xi_3\frac{\Phi-\Phi_1}{2} \leq \frac{\pi\xi_3}{2n} < \frac{\pi}{2}.$$

Hence for $\Phi_1 < \theta < \Phi$,

$$|g(re^{i\theta})| \leq \left|\frac{U_1(re^{i\theta})}{\exp\{Kr^{\xi_3}\}}\right| \leq O(\exp\{r^{n+\tau_2}\}), \qquad (r \to \infty).$$

It follows from (10) and (18) that for some $M > 0$, as $r \to \infty$

$$|g(re^{i\Phi_1})| \leq \frac{O(e^{r^{\xi_2}})}{\exp\{Kr^{\xi_3}\}} \leq M$$

and

$$|g(re^{i\Phi})| \le \frac{O(e^{r^{\xi_2}})}{\exp\{Kr^{\xi_3}\}} \le M.$$

By the Phragmen-Lindelöf theorem, we obtain (19). Similarly we see that (19) holds for $\Phi < \theta < \Phi_2$. Hence we conclude that (19) holds for any $\theta \in [0, 2\pi)$.

By a similar proof as before we can prove that for any $\theta \in [0, 2\pi)$

$$(20) \qquad |U_2(re^{i\theta})| \le O(\exp\{r^{\xi_3}\}), \qquad (r \to \infty).$$

By (7) and (9), we have

$$(21) \qquad Q_2U_1 - Q_1U_2 = \frac{1}{k}h'[Q_1(Q_2' + Q_2P_2') - Q_2(Q_1' + Q_1P_1')].$$

Since $\sigma(Q_j) < \xi_2 < \xi_3 (j = 1, 2, 3)$, by (5),(10),(20), (21) and Lemma 2.9, we have

$$\begin{aligned}(22) \qquad & m(r, Q_1e^{P_1(z)} + Q_2e^{P_2(z)} + \cdots + Q_\iota e^{P_\iota(z)}) \\ & \le km(r, h') + O(\log r) \le km(r, Q_2U_1 - Q_1U_2) + O(r^{\xi_2}) \\ & \le O(r^{\xi_3}), \quad (r \to \infty).\end{aligned}$$

Since $\frac{\zeta_1}{\zeta_2}$ is non-real, $S_1^+ \cap S_2^-$ contains an interval $I = [\varphi_1, \varphi_2]$ satisfying $\min_{\theta \in I} \delta_1(\theta) = \chi > 0$. By Lemma 2.2, there exists an $R(I)(> 0)$ such that for any $\theta \in I$ and $r \ge R(I)$,

$$|Q_1e^{P_1(re^{i\theta})}| \ge \exp((1-\varepsilon)\delta_1 r^n), \quad |Q_2e^{P_2(re^{i\theta})}| \le \exp((1-\varepsilon)\delta_2 r^n), \quad \ldots,$$

and

$$|Q_\iota e^{P_\iota(re^{i\theta})}| \le \exp((1-\varepsilon)\lambda_\iota\delta_2 r^n).$$

Hence, we have

$$\begin{aligned}(23) \qquad & m\left(r, Q_1e^{P_1(z)} + Q_2e^{P_2(z)} + \cdots + Q_\iota e^{P_\iota(z)}\right) \\ & \ge \int_{\varphi_1}^{\varphi_2} \log^+ |Q_1e^{P_1(z)} + Q_2e^{P_2(z)} + \cdots + Q_\iota e^{P_\iota(z)}|d\theta \\ & \ge \int_{\varphi_1}^{\varphi_2} (1 - o(1)) \log^+ |Q_1e^{P_1(z)}|d\theta \\ & \ge \int_{\varphi_1}^{\varphi_2} (1 - o(1))(1-\varepsilon)sr^n d\theta \\ & \ge (1 - o(1))(1-\varepsilon)sr^n(\varphi_2 - \varphi_1), \quad (r \to \infty).\end{aligned}$$

Combining (22) and (23) and recalling that $\xi_3 < n$, we get a contradiction. Hence, $\lambda(f) = \infty$.

4 Proof of Theorem 1.1(ii)

Let $f \not\equiv 0$ be a solution of (3). Write $f = \pi e^h$, suppose that $\lambda(f) < n$. From our hypothesis, we have $\sigma(\pi) = \lambda(\pi) < n$. Eliminating e^{P_1} from (5), we have

$$kU(h')^{k-1} = F_{k-1}(h') + \sum_{j=2}^{\iota} e^{P_j}[Q_j(Q_1' + Q_1P_1') - Q_1(Q_j' + Q_jP_j')], \tag{24}$$

where

$$U = Q_1h'' - \frac{1}{k}(Q_1' + Q_1P_1')h'. \tag{25}$$

From (24), (25) and Lemma 2.7, we have

$$\begin{aligned} c_{k-1}(z)(h')^{k-1} &+ c_{k-2}(h')^{k-2} + \cdots + c_1(z)h' \\ &= c_0(z) + \sum_{j=2}^{\iota} e^{P_j}[Q_j(Q_1' + Q_1P_1') - Q_1(Q_j' + Q_jP_j')], \end{aligned} \tag{26}$$

where $c_j(z)(j = 0, 1, 2, \ldots, k-1)$ is an algebraic expression in the terms $U^{(l)}$ $(l = 0, 1, \ldots, k-2), Q_1^{(i)}(i = 0, 1 \ldots, k-1), P_1^{(s)}(s = 0, 1, \ldots, l-1), \frac{1}{k}, \frac{1}{Q_1}, \frac{\pi^{(t)}}{\pi}$ $(t = 1, 2, \ldots, k)$ and $a_j, a_j'(j = 1, 2, \ldots, k-1)$, such as addition, subtraction and multiplication.

Now we suppose that at least one of $c_j(z)(j = 1, 2, \ldots, k-1)$ is not identically vanishing and $c_0(z) + \sum_{j=2}^{\iota} e^{P_j}[Q_j(Q_1' + Q_1P_1') - Q_1(Q_j' + Q_jP_j')] \not\equiv 0$. Without loss of generality, suppose $c_{k-1}(z) \not\equiv 0$, from (26) and Lemma 2.8, we have

$$\begin{aligned} T(r, h') = m(r, h') \le \sum_{i=0}^{k-1} T(r, c_i(z)) &+ m\Big(r, \sum_{j=2}^{\iota} e^{P_j}[Q_j(Q_1' + Q_1P_1') \\ &- Q_1(Q_j' + Q_jP_j')]\Big) + O(1). \end{aligned} \tag{27}$$

Set $\max\{\lambda(f), \sigma(Q_j) : (j = 1, 2, \ldots, \iota)\} < \xi_2 < \xi_3 < n$. From (5), we obtain

$$T\left(r, Q_1e^{P_1(z)} + Q_2e^{P_2(z)} + \cdots + Q_\iota e^{P_\iota(z)}\right) \le kT(r, h') + O(\log r). \tag{28}$$

By Lemma 2.5, we have

$$\begin{aligned} m\Big(r, Q_1e^{P_1(z)} &+ Q_2e^{P_2(z)} + \cdots + Q_\iota e^{P_\iota(z)}\Big) \\ &\ge (1-\varepsilon)m(r, e^{P_1}) + O(r^{\xi_3}), \quad (r \to \infty, r \notin E), \end{aligned} \tag{29}$$

where E has finite linear measure. From (28) and (29), we obtain

$$T(r, h') \ge \frac{1-\varepsilon}{k}T(r, e^{P_1}) + O(r^{\xi_3}), \qquad (r \to \infty, r \notin E). \tag{30}$$

Since $0 < \rho = \frac{\zeta_2}{\zeta_1} < \frac{1}{2k}$, $\zeta_j = \lambda_j\zeta_2$, $\lambda_j > 0$ and $0 < \sum_{j=3}^{\iota} \lambda < 1$, we get $\delta(P_2, \theta) = \rho\delta(P_1, \theta)$, and

$$S_{1m}^+ = S_{2m}^+ = \cdots = S_{\iota m}^+, \quad S_{1m}^- = S_{2m}^- = \cdots = S_{\iota m}^-, \quad (m = 1, \ldots, n).$$

By the same reasoning as in (11) and (14), we have

$$|U(re^{i\theta})| \leq O(\exp\{r^{\xi_2}\}), \qquad (r \to \infty) \tag{31}$$

for any $\theta \in S_1^- \backslash E_0, m(E_0) = 0$. Also by the same reasoning as in (15)-(18), we have

$$|U(re^{i\theta})| \leq O(\exp\{r^{\xi_2}\}), \qquad (r \to \infty) \tag{32}$$

for any $\theta \in S_1^+ \backslash E_0, m(E_0) = 0$. Since $\sigma(U) \leq n$, by the Phragmen-Lindelöf theorem, we have

$$|U(re^{i\theta})| \leq O(\exp\{r^{\xi_3}\}), \qquad (r \to \infty) \tag{33}$$

for any $\theta \in [0, 2\pi)$.

We will estimate $T(r, c_j)$ as follows.

By our hypothesis $f = \pi e^h, \lambda(f) < \xi_3 < n$, from Lemma 2.3 we have $\overline{N}\left(r, \frac{1}{\pi}\right) \leq O(r^{\xi_3})$. Thus, from (33), the assumptions of Theorem 1.1, the forms of $c_j(z)$ and the theorem on the logarithmic derivatives, we have

$$\begin{aligned} T(r, c_j) \leq O\Bigg(& \sum_{i=0}^{k-1} T(r, Q_1^{(i)}) + \sum_{\Lambda=0}^{k-1} m(r, a_\Lambda) + \sum_{\Lambda=0}^{k-1} m(r, a'_\Lambda) + \sum_{s=0}^{k-1} m(r, P_1^{(s)}) \\ & + \sum_{t=1}^{k-2} m\left(r, \frac{U^{(t)}}{U}\right) + m(r, U) + \overline{N}\left(r, \frac{1}{\pi}\right) + O(\log r)\Bigg) \\ \leq {} & O(r^{\xi_3}), \qquad r \to \infty, \quad j = 0, 1, \ldots, k-1, \end{aligned} \tag{34}$$

and

$$\begin{aligned} & T(r, \sum_{j=2}^{\iota} e^{P_j}[Q_j(Q_1' + Q_1P_1') - Q_1(Q_j' + Q_jP_j')]\Big) \\ & \leq O(r^{\xi_3}) + T(r, e^{P_2}) + T(r, e^{P_3}) + \cdots + T(r, e^{P_\iota}) \\ & = (1 + \sum_{j=3}^{\iota} \lambda_j) T(r, e^{P_2}) + O(r^{\xi_3}) \\ & \leq (1 + \sum_{j=3}^{\iota} \lambda_j) \rho T(r, e^{P_1}) + O(r^{\xi_3}), \quad r \to \infty. \end{aligned} \tag{35}$$

From (27),(30),(34) and (35), we get

$$\begin{aligned} \frac{1-\varepsilon}{k} T(r, e^{P_1}) + O(r^{\xi_3}) & \leq T(r, h') \\ & \leq (1 + \sum_{j=3}^{\iota} \lambda_j) \rho T(r, e^{P_1}) + O(r^{\xi_3}), \quad r \to \infty, r \notin E. \end{aligned} \tag{36}$$

Thus, (36) implies

$$\left(\frac{1-\varepsilon}{k} - (1 + \sum_{j=3}^{\iota} \lambda_j)\rho - o(1) \right) T(r, e^{P_1}) \leq 0, \qquad r \to \infty, r \notin E. \tag{37}$$

From $0<\rho=\frac{\zeta_2}{\zeta_1}<\frac{1}{2k}, 0<\sum_{j=3}^{\iota}\lambda_j<1$ and (37), we get a contradiction. Hence $c_{k-1}=\cdots=c_1=c_0+\sum_{j=2}^{\iota}e^{P_j}[Q_j(Q_1'+Q_1P_1')-Q_1(Q_j'+Q_jP_j')]\equiv 0$, that is,

$$-c_0(z)=\sum_{j=2}^{\iota}e^{P_j}[Q_j(Q_1'+Q_1P_1')-Q_1(Q_j'+Q_jP_j')]. \tag{38}$$

First, we show that $Q_2(Q_1'+Q_1P_1')-Q_1(Q_2'+Q_2P_2')\not\equiv 0$ as follows. If $Q_2(Q_1'+Q_1P_1')-Q_1(Q_2'+Q_2P_2')\equiv 0$, that is, $P_1'-P_2'=\frac{Q_1'}{Q_1}-\frac{Q_2'}{Q_2}$. By solving this differential equation, we get $Q_1=\varsigma Q_2e^{P_1-P_2}$, where ς is a non-zero constant. Thus, we can get $\sigma(Q_1)=n$ which contradicts with $\sigma(Q_1)<n$. Therefore, we have $Q_2(Q_1'+Q_1P_1')-Q_1(Q_2'+Q_2P_2')\not\equiv 0$. Since $\sigma(Q_j)<n (j=1,2,\ldots,\iota)$ and $P_j(z)$ are polynomials of degree n, we have $\sigma(Q_j(Q_1'+Q_1P_1')-Q_1(Q_j'+Q_jP_j'))<n (j=2,3,\ldots,\iota)$.

Next, we assume that $\sum_{j=2}^{\iota}e^{P_j}[Q_j(Q_1'+Q_1P_1')-Q_1(Q_j'+Q_jP_j')]\not\equiv 0$. If $\sum_{j=2}^{\iota}e^{P_j}[Q_j\,(Q_1'+Q_1P_1')-Q_1(Q_j'+Q_jP_j')]\equiv 0$, that is,

$$\begin{aligned}-e^{P_2}&[Q_2(Q_1'+Q_1P_1')-Q_1(Q_2'+Q_2P_2')]\\&=\sum_{j=3}^{\iota}e^{P_j}[Q_j(Q_1'+Q_1P_1')-Q_1(Q_j'+Q_jP_j')].\end{aligned} \tag{39}$$

If $\delta(P_2,\theta)=\delta_2(\theta)>0, \theta\in[0,2\pi)$. Since $\zeta_j=\lambda_j\zeta_2, 0<\lambda_j, (j=3,4,\ldots,\iota)$, we have $\delta(P_j,\theta)=\delta_j(\theta)>0, (j=3,4,\ldots,\iota)$. Set $\lambda_0=\max\{\lambda_j: j=3,4,\ldots,\iota\}$, from (39), Lemma 2.5 and the assumptions of Theorem 1.1, for any $\varepsilon_0 (0<\varepsilon_0<\frac{1-\lambda_0}{1+\lambda_0})$, we have

$$\begin{aligned}\exp\{(1-\varepsilon_0)\delta_2r^n\}&\le\left|e^{P_2}[Q_2(Q_1'+Q_1P_1')-Q_1(Q_2'+Q_2P_2')]\right|\\&\le\left|\sum_{j=3}^{\iota}e^{P_j}[Q_j(Q_1'+Q_1P_1')-Q_1(Q_j'+Q_jP_j')]\right|\\&\le(\iota-2)\exp\{(1+\varepsilon_0)\lambda_0\delta_2r^n\}.\end{aligned} \tag{40}$$

Since $\delta_2>0, \lambda_0>0$ and $0<\varepsilon_0<\frac{1-\lambda_0}{1+\lambda_0}$, we can get a contradiction.

If $\delta(P_2,\theta)=\delta_2(\theta)<0, \theta\in[0,2\pi)$, similar to the above argument, we can also get a contradiction.

From (38), $Q_2(Q_1'+Q_1P_1')-Q_1(Q_2'+Q_2P_2')\not\equiv 0$ and $\sigma(Q_j(Q_1'+Q_1P_1')-Q_1(Q_j'+Q_jP_j'))<n (j=2,3,\ldots,\iota)$, by (34) and Lemma 2.5, we get

$$(1-\varepsilon)T(r,e^{P_2})+O(r^{\xi})\le O(r^{\xi_3}),\quad r\to\infty. \tag{41}$$

From (41), we have $\sigma(e^{P_2})\le\max\{\xi,\xi_3\}<n$, we get a contradiction. Hence $\lambda(f)\ge n$.

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