Functional Differential Equations of Second Order

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Abstract

In this paper we study a boundary value problem for functional differential equations of second order. Applying a quasilinearization technique we obtain two monotone sequences showing that they converge to the unique solution and this convergence is superlinear.

1 Introduction

Let $C_0 = C(J_0, \mathbb{R})$ with $J_0 = [-\tau, 0]$ for $\tau > 0$ and put $J = [0, 1]$. Suppose that $f \in C(J \times C_0, \mathbb{R})$, $\Phi_0 \in C_0$ and let us consider the functional differential problem of the form

$$\begin{cases} -x''(t) = f(t, x_t), & t \in J, \\ \quad x_0 = \Phi_0, & x(1) = k_1. \end{cases} \tag{1}$$

Here, for any $t \in J$, $x_t \in C_0$ is defined by $x_t(s) = x(t+s)$ for $s \in J_0$. According to the above notation, $x_0 \in C_0$, and $x_0(s) = x(s)$, $s \in J_0$. It means that in this case the condition $x_0 = \Phi_0$ implies that $x(s) = \Phi(s)$ on J_0, where the function Φ is given and continuous on J_0. Note that the differential equation from problem (1) includes, for example as special cases, ordinary differential equations, differential equations with delayed arguments and integro–differential equations too. There are some books devoted to functional differential equations (see for example [3],[4]; see also [2]). Second order nonlinear boundary problems arises in many physical

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phenomena. Many examples with some discussion about solutions (and sometimes also about lower and upper solutions) are in [1], [2].

The method of quasilinearization has been widely applied in the study of nonlinear differential problems with initial and boundary conditions (see, for example [6]–[9]). In this paper we extend this method to boundary value problems for functional differential equations of second order. Two monotone sequences are constructed and sufficient conditions which imply the convergence of these sequences to the unique solution of problem (1) are given. This convergence is superlinear. We must point out that the treatment of our problem leads us to prove some results of existence and uniqueness of solutions to linear problems of second order that are of independent interest. Some examples which satisfy the assumptions are presented.

Finally, we note that the main result of this paper is new and problem (1) is quite general containing others considered, for example, in [2], [7], [8].

2 Assumptions

Put $C^* = C(\bar{J}, \mathbb{R}) \cap C^2(J, \mathbb{R})$ with $\bar{J} = [-\tau, 1]$. A function $u \in C^*$ is said to be a lower solution of problem (1) if

$$\begin{cases} -u''(t) \le f(t, u_t), & t \in J, \\ \quad u_0 \le \Phi_0, & u(1) \le k_1, \end{cases}$$

and an upper solution of (1) if the above inequalities are reversed.

We introduce the following assumptions:

(H_1) $f \in C(J \times C_0, \mathbb{R})$,

(H_2) $y_0, z_0 \in C^*$ are lower and upper solutions of (1) and $y_0(t) \le z_0(t)$ on J,

(H_3) the Frechet derivative f_Φ exists, is a continuous linear operator satisfying:

(a) $|f_\Phi(t, \Phi)v| \le L|v|_0$ for $t \in J$, $\Phi, v \in C_0$ with $L \in [0, 8)$, and $|v|_0 = \max_{s \in [-\tau, 0]} |v(s)|$,

(b) if $u, v \in C_0$, and $y_{0,t} \le u \le v \le z_{0,t}$, then

$$f(t, v) \ge f(t, u) + f_\Phi(t, u)(v - u), \quad t \in J,$$

(c) if $v \le w$, and $u, v, w \in C_0$, then

$$f_\Phi(t, u)v \le f_\Phi(t, u)w, \quad y_{0,t} \le u \le z_{0,t},$$

(d) if $u, v, w \in C_0$, $u \ge 0$ and $y_{0,t} \le v \le w \le z_{0,t}$, then

$$f_\Phi(t, v)u \le f_\Phi(t, w)u,$$

(H_4) there exist constants $L_1 \ge 0$ and $\alpha \in [0, 1]$ such that the condition

$$|f_\Phi(t, u) - f_\Phi(t, v)| \le L_1 |u - v|_0^\alpha$$

holds for $t \in J$, $u, v \in C_0$.

3 Lemmas

Lemma 1 gives sufficient conditions under which problem (1) has at most one solution.

Lemma 1. *Let the assumptions H_1 and $H_3(a)$ hold. Then problem (1) has at most one solution.*

Proof. Assume that problem (1) has two distinct solutions x and y. Put $p = x - y$. Then

$$\begin{cases} -p''(t) = f(t, x_t) - f(t, y_t), \quad t \in J, \\ \quad p(s) = 0, \quad s \in J_0, \quad p(1) = 0. \end{cases}$$

Note that this is equivalent to the following integral equation

$$\begin{cases} p(t) = \int_0^1 G(t,s)[f(s,x_s) - f(s,y_s)]ds, \quad t \in J, \\ p(s) = 0, \quad s \in J_0 \end{cases}$$

with G as the Green function defined by

$$G(t,s) = \begin{cases} s(1-t) & \text{if} \quad 0 \le s \le t, \\ t(1-s) & \text{if} \quad t < s \le 1. \end{cases}$$

Let $|p|_* = \max_{t \in J} |p(t)|$. Then, a mean value theorem and assumption $H_3(a)$ yield

$$|p|_* = \max_{t \in J} \left| \int_0^1 G(t,s) \int_0^1 f_\Phi(s, rx_s + (1-r)y_s) dr p_s ds \right| \le \frac{L}{8} |p|_*.$$

Hence $|p|_* = 0$ since $L < 8$. This proves that problem (1) has at most one solution.
The lemma is proved.

We shall now prove the basic comparison result.

Lemma 2. *Let the assumptions $H_1, H_3(a,b,c)$ hold. Let $u, v \in C^*$ be lower and upper solutions of problem (1), respectively, and $[u,v] \subset [y_0, z_0]$. Then the problems*

$$\text{(2)} \qquad \begin{cases} -p''(t) = f(t,u_t) + f_\Phi(t,u_t)[p_t - u_t], \quad t \in J, \\ \qquad p_0 = \Phi_0, \quad p(1) = k_1, \end{cases}$$

and

$$\text{(3)} \qquad \begin{cases} -q''(t) = f(t,v_t) + f_\Phi(t,u_t)[q_t - v_t], \quad t \in J, \\ \qquad q_0 = \Phi_0, \quad q(1) = k_1 \end{cases}$$

have, in the segment $[u,v]$, their unique solutions $p, q \in C^*$, and moreover $p \le q$.

Proof. Note that problems (2) and (3) are equivalent to the following integral equations

$$\text{(4)} \qquad \begin{cases} p(t) = \int_0^1 G(t,s) U_1(s,p) ds + \Phi(0) + [k_1 - \Phi(0)]t \equiv A_1 p(t), \quad t \in J, \\ \quad p_0 = \Phi_0, \end{cases}$$

and

$$(5)\qquad \begin{cases} q(t) = \int_0^1 G(t,s)U_2(s,q)ds + \Phi(0) + [k_1 - \Phi(0)]t \equiv A_2p(t), & t \in J, \\ q_0 = \Phi_0, \end{cases}$$

where

$$\begin{aligned} U_1(t,p) &= f(t,u_t) + f_\Phi(t,u_t)[p_t - u_t], \\ U_2(t,p) &= f(t,v_t) + f_\Phi(t,u_t)[p_t - v_t]. \end{aligned}$$

Knowing that u, v are lower and upper solutions of problem (1), respectively, and using assumption $H_3(b)$, we get

$$\begin{aligned} U_1(t,u) &= f(t,u_t) \geq -u''(t), \\ U_1(t,v) &= f(t,u_t) + f_\Phi(t,u_t)[v_t - u_t] - f(t,v_t) + f(t,v_t) \\ &\leq f(t,v_t) \leq -v''(t), \end{aligned}$$

and

$$\begin{aligned} U_2(t,u) &= f(t,v_t) + f_\Phi(t,u_t)[u_t - v_t] - f(t,u_t) + f(t,u_t) \\ &\geq f(t,u_t) \geq -u''(t), \\ U_2(t,v) &= f(t,v_t) \leq -v''(t). \end{aligned}$$

Then integration by parts gives

$$\begin{aligned} A_1u(t) &= \int_0^1 G(t,s)U_1(s,u)ds + \Phi(0) + [k_1 - \Phi(0)]t \\ &\geq -\int_0^1 G(t,s)u''(s)ds + \Phi(0) + [k_1 - \Phi(0)]t \\ &= \int_0^t s(t-1)u''(s)ds + \int_t^1 t(s-1)u''(s)ds + \Phi(0) + [k_1 - \Phi(0)]t \\ &= u(t) + (1-t)[\Phi(0) - u(0)] + t[k_1 - u(1)] \\ &\geq u(t), \quad t \in J, \\ A_1v(t) &= \int_0^1 G(t,s)U_1(s,v)ds + \Phi(0) + [k_1 - \Phi(0)]t \\ &\leq (t-1)\int_0^t sv''(s)ds + t\int_t^1 (s-1)v''(s)ds + \Phi(0) + [k_1 - \Phi(0)]t \\ &= v(t) + (1-t)[\Phi(0) - v(0)] + t[k_1 - v(1)] \\ &\leq v(t), \quad t \in J, \end{aligned}$$

and

$$\begin{aligned} A_2u(t) &= \int_0^1 G(t,s)U_2(s,u)ds + \Phi(0) + [k_1 - \Phi(0)]t \\ &\geq -\int_0^1 G(t,s)u''(s)ds + \Phi(0) + [k_1 - \Phi(0)]t \\ &\geq u(t), \quad t \in J, \\ A_2v(t) &= \int_0^1 G(t,s)U_2(s,v)ds + \Phi(0) + [k_1 - \Phi(0)]t \\ &\leq -\int_0^1 G(t,s)v''(s)ds + \Phi(0) + [k_1 - \Phi(0)]t \\ &\leq v(t), \quad t \in J. \end{aligned}$$

Now, let $u(t) \leq v_1(t) \leq v_2(t) \leq v(t),\ t \in J,\ v_1, v_2 \in C^*$. Assumption $H_3(c)$ yields

$$\begin{aligned} A_1 v_1(t) &= \int_0^1 G(t,s)U_1(s,v_1)ds + \Phi(0) + [k_1 - \Phi(0)]t \\ &\leq \int_0^1 G(t,s)U_1(s,v_2)ds + \Phi(0) + [k_1 - \Phi(0)]t = A_1 v_2(t), \quad t \in J, \\ A_2 v_1(t) &= \int_0^1 G(t,s)U_2(s,v_1)ds + \Phi(0) + [k_1 - \Phi(0)]t \\ &\leq \int_0^1 G(t,s)U_2(s,v_2)ds + \Phi(0) + [k_1 - \Phi(0)]t = A_2 v_2(t), \quad t \in J \end{aligned}$$

showing that operators A_1 and A_2 map the segment $[u, v]$ into itself. Since A_1 and A_2 are completely continuous operators on $[u, v]$, so the sequences $\bar{u}_{n+1} = A_1\bar{u}_n,\ \ \bar{v}_{n+1} = A_1\bar{v}_n,\ \ \bar{u}_0 = u,\ \ \bar{v}_0 = v$ and $\tilde{u}_{n+1} = A_2\tilde{u}_n,\ \ \tilde{v}_{n+1} = A_2\tilde{v}_n,\ \ \tilde{u}_0 = u,\ \ \tilde{v}_0 = v$ converge to fixed points $\bar{u}, \bar{v}, \tilde{u}, \tilde{v} \in [u, v]$ of A_1 and A_2, respectively, and $\bar{u} \leq \bar{v},\ \ \tilde{u} \leq \tilde{v}$.

Now we are going to show that problem (2) has a unique solution. Assume that it has two distinct solutions x and y. Set $m = x - y$, so $m(s) = 0$ on J_0, and $m(1) = 0$. Then m satisfies the following problem:

$$\text{(6)} \qquad \begin{cases} -m''(t) = f_\Phi(t, u_t)m_t, \quad t \in J, \\ \quad m(s) = 0, \quad s \in J_0, \quad m(1) = 0. \end{cases}$$

Moreover, $m(t) = 0,\ t \in J$ is a solution of (6). Since (6) is equivalent to the following one

$$\begin{cases} m(t) = \displaystyle\int_0^1 G(t,s)f_\Phi(s, u_s)m_s ds, \quad t \in J, \\ m(s) = 0, \quad s \in J_0, \end{cases}$$

using assumption $H_3(a)$, it is easy to show that $m(t) = 0$ on J is the unique solution of (6). It proves that problem (2) has a unique solution p, so $\bar{u}(t) = \bar{v}(t) = p(t)$ on J. Similarly, we can prove that problem (3) has a unique solution q, so $\tilde{u}(t) = \tilde{v}(t) = q(t)$ on J.

Now, we need to show that $p(t) \leq q(t)$ on J. Note that for all $w \in C^*$, assumption $H_3(b)$ yields $U_1(t, w) \leq U_2(t, w)$ which proves that $A_1 w \leq A_2 w$. Since $\bar{u}_0 = u \leq v = \tilde{v}_0$, then $\bar{u}_1 = A_1\bar{u}_0 \leq A_1\tilde{v}_0 \leq A_2\tilde{v}_0 = \tilde{v}_1$. Assume that $\bar{u}_k \leq \tilde{v}_k$ for some fixed $k > 1$. Then,

$$\bar{u}_{k+1} = A_1\bar{u}_k \leq A_1\tilde{v}_k \leq A_2\tilde{v}_k = \tilde{v}_{k+1}.$$

By induction, it proves that $\bar{u}_n \leq \tilde{v}_n$ for all $n \geq 0$. Now, if $n \to \infty$, then $p(t) \leq q(t)$ on J showing that $u(t) \leq p(t) \leq q(t) \leq v(t)$ on J.

This completes the proof of the lemma.

4 Main result

We are now in a position to prove the following main result of this paper.

Theorem 1. *Let assumptions $H_1, H_2, H_3(a,c,d)$ and H_4 hold. Then there exist monotone sequences $\{y_n\}$, $\{z_n\}$ which converge uniformly to the unique solution x of problem (1) on J and that convergence is superlinear.*

Proof. Note that assumption $H_3(d)$ and the mean value theorem prove that $H_3(b)$ is satisfied. Let us define the sequences $\{y_n\}, \{z_n\}$ by formulas

$$\begin{cases} -y''_{n+1}(t) = f(t, y_{n,t}) + f_\Phi(t, y_{n,t})[y_{n+1,t} - y_{n,t}], \;\; t \in J, \\ \quad y_{n+1,0} = \Phi_0, \;\; y_{n+1}(1) = k_1, \end{cases}$$
$$\begin{cases} -z''_{n+1}(t) = f(t, z_{n,t}) + f_\Phi(t, y_{n,t})[z_{n+1,t} - z_{n,t}], \;\; t \in J, \\ \quad z_{n+1,0} = \Phi_0, \;\; z_{n+1}(1) = k_1 \end{cases}$$

for $n = 0, 1, \cdots$. Since $y_0, z_0 \in C^*$ are lower and upper solutions of problem (1), respectively, by Lemma 2, the elements y_1, z_1 are well defined and moreover

$$y_0(t) \le y_1(t) \le z_1(t) \le z_0(t), \;\; t \in J.$$

Next, using assumption $H_3(b, d)$, we obtain

$$\begin{aligned} -y''_1(t) &= f(t, y_{0,t}) + f_\Phi(t, y_{0,t})[y_{1,t} - y_{0,t}] - f(t, y_{1,t}) + f(t, y_{1,t}) \\ &\le f(t, y_{1,t}), \;\; t \in J, \\ -z''_1(t) &= f(t, z_{0,t}) + f_\Phi(t, y_{0,t})[z_{1,t} - z_{0,t}] - f(t, z_{1,t}) + f(t, z_{1,t}) \\ &\ge f(t, z_{1,t}), \;\; t \in J, \end{aligned}$$

showing that y_1, z_1 are lower and upper solutions of problem (1), respectively.

Let us assume that

$$y_0(t) \le y_1(t) \le \cdots \le y_{k-1}(t) \le y_k(t) \le z_k(t) \le z_{k-1}(t) \le \cdots \le z_1(t) \le z_0(t), \; t \in J$$

and let y_k, z_k be lower and upper solutions of problem (1) for some $k \ge 1$.

Note that, by Lemma 2, y_{k+1}, z_{k+1} are well defined and

$$y_k(t) \le y_{k+1}(t) \le z_{k+1}(t) \le z_k(t), \;\; t \in J.$$

Hence, by induction, we have

$$y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t), \;\; t \in J$$

for all n. Employing standard techniques, it can be shown that the sequences $\{y_n\}$, $\{z_n\}$ converge uniformly and monotonically to the unique solution x of problem (1). The uniqueness of solutions of problem (1) is guaranteed by Lemma 1.

We shall next show that the convergence of y_n, z_n to the unique solution x of problem (1) is superlinear. For this purpose, we consider

$$p_{n+1} = x - y_{n+1} \ge 0, \qquad q_{n+1} = z_{n+1} - x \ge 0 \;\; t \in \bar{J}.$$

Note that $p_{n+1}(s) = q_{n+1}(s) = 0$ for $s \in J_0$, and $p_{n+1}(1) = q_{n+1}(1) = 0$. Moreover,

$$-p''_{n+1}(t) = f(t, x_t) - f(t, y_{n,t}) - f_\Phi(t, y_{n,t})[y_{n+1,t} - y_{n,t}] \equiv W_n(t), \quad t \in J,$$

so

$$p_{n+1}(t) = \int_0^1 G(t,s) W_n(s) ds, \quad t \in J.$$

Now, using the mean value theorem and assumptions H_4 and $H_3(a)$, we get

$$\begin{aligned} p_{n+1}(t) = & \int_0^1 G(t,s) \left\{ \int_0^1 f_\Phi(s, rx_s + (1-r)y_{n,s}) dr p_{n,s} \right. \\ & \left. - f_\Phi(s, y_{n,s})[p_{n,s} - p_{n+1,s}] \right\} ds \\ = & \int_0^1 G(t,s) \left\{ \int_0^1 [f_\Phi(s, rx_s + (1-r)y_{n,s}) - f_\Phi(s, y_{n,s})] \, p_{n,s} dr \right. \\ & \left. + f_\Phi(s, y_{n,s}) p_{n+1,s} \right\} ds \\ \leq & \int_0^1 G(t,s) \left\{ \int_0^1 L_1 r^\alpha |p_{n,s}|_0^\alpha p_{n,s} dr + L \max_{t \in J} |p_{n+1}(t)| \right\} ds \\ \leq & \frac{L_1}{8} \max_{t \in J} |p_{n,t}|_0^{\alpha+1} + \frac{L}{8} \max_{t \in J} |p_{n+1}(t)|. \end{aligned}$$

Hence

$$\max_{t \in J} |p_{n+1}(t)| \leq \frac{L_1}{8 - L} \max_{t \in J} |p_{n,t}|_0^{\alpha+1}.$$

Similarly, using the mean value theorem and assumptions $H_3(a), H_4$ we have an estimation for q_{n+1}, namely

$$\max_{t \in J} |q_{n+1}(t)| \leq \frac{L_1}{8 - L} \left[\max_{t \in J} |q_{n,t}|_0^{\alpha+1} + \max_{t \in J} |p_{n,t}|_0^\alpha |q_{n,t}|_0 \right].$$

The proof is complete.

Remark. If $\alpha = 1$, then the convergence of sequences $\{y_n\}, \{z_n\}$ is quadratic.

Examples **1.** Consider the following problem

$$(7) \qquad \begin{cases} -x''(t) = \left[x\left(t - \dfrac{1}{2}\right) \right]^2 - 2, & t \in J = [0,1], \\ x(s) = 0, \quad s \in \left[-\dfrac{1}{2}, 0 \right], & x(1) = 1. \end{cases}$$

If we take $y_0(t) = 0$ for $t \in [-\frac{1}{2}, 0]$, $y_0(t) = t^2$ for $t \in J$, and $z_0(t) = 1$, $t \in [-\frac{1}{2}, 1]$, then it is easy to verify that assumptions H_3, H_4 are satisfied. Moreover, y_0, and z_0 are respectively a lower and an upper solution of problem (7) and $y_0(t) \leq z_0(t)$ on J.

2. Let

$$(8) \qquad \begin{cases} -x''(t) = x\left(\dfrac{1}{2} t\right), & t \in J = [0,1], \\ x(0) = 0, \quad x(1) = 1. \end{cases}$$

Then Assumptions H_3, H_4 hold. Note that $y_0(t) = 0$, $z_0(t) = 2 - t^2$, $t \in J$ are a lower and an upper solution of (8) and $y_0(t) \leq z_0(t)$ on J.

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