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Asymptotic values of meromorphic functions of smooth growth

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1. Introduction

In the following, the standard notation of Nevanlinna theory (e.g., see Hayman [7]) will be used.

Hayman [8] gave a striking example of a meromorphic function $f(z)$ in the whole plane such that $\delta(\infty, f)=1$ but ∞ is not an asymptotic value of $f(z)$. To point out that the singular behaviour of this $f(z)$ is essentially associated with the irregular growth of Nevanlinna characteristic $T(r, f)$, he picked up several sorts of smoothly growing conditions of $T(r, f)$, under which certain deficient values are asymptotic values.

In [8, Corollary 2], Hayman proved that, if a meromorphic function $f(z)$ satisfies the smoothness condition

$$(1) \qquad T(2r, f) \sim T(r, f) \qquad (r \to \infty),$$

then any deficient value of $f(z)$ is an asymptotic value of $f(z)$. Further, extending the result [3, Theorem 4] and answering to the question [2, 2.57], Anderson [1] proved that for $f(z)$ satisfying (1), if w is a deficient value of $f(z)$, we can find a path Γ going to ∞ and satisfying

$$(2) \qquad L(r, \Gamma) = r(1 + \mathrm{o}(1)) \qquad (r \to \infty)$$

along which

$$\liminf_{|z|\to\infty} (\log 1/|f(z) - w|)/T(|z|, f) \geqq \delta(w, f) \qquad (w \neq \infty)$$

$$\liminf_{|z|\to\infty} (\log |f(z)|)/T(|z|, f) \geqq \delta(w, f) \qquad (w = \infty)$$

where $L(r, \Gamma)$ is the length of the arc $\Gamma \cap \{z: |z| \leqq r\}$.

The aim of this paper is mainly to extend this Anderson's result to meromorphic functions of positive order ρ ($\rho < 1/2$) satisfying the smoothness condition

$$(3) \qquad \limsup_{r\to\infty} x^{-\rho} T(r, f)^{-1} T(xr, f) \leqq 1$$

for any x ($x > 1$), because meromorphic functions satisfying (1) have order 0 (see Hayman [8, p. 130]). But, we could not get any result corresponding to (2), since we did not use the method depending essentially on Boutroux-Cartan's lemma which provides an estimate from below for the modulus of a polynomial. So, Hayman conjectures that it would be possible to take an asymptotic path Γ in our results which has also the property

$$L(r, \Gamma) = \mathrm{O}(r) \qquad (r \to \infty).$$

It seems that this (3) is a natural generalization of (1) to higher order ρ $(0 \leqq \rho < 1/2)$ of $T(r, f)$. In fact, we have Hayman's result [8, Corollary 2] by putting $\rho = 0$ in Theorem 3 which says that if (3) and

$$\delta(w, f) > 2\rho$$

are satisfied, w is an asymptotic value of $f(z)$.

We introduce another smoothness condition which generalizes the concepts of 'very regular growth' and 'perfectly regular growth' in the sense that $T(r, f)$ is compared not only with r^{ρ} $(0 \leqq \rho < 1/2)$ but also with $r^{\rho(r)}$: there exist a proximate order $\rho(r)$ $(\rho(r) \to \rho)$ and two constants c_1, c_2 such that

$$(4) \qquad 0 < c_1 \leqq \liminf_{r \to \infty} r^{-\rho(r)} T(r, f) \leqq \limsup_{r \to \infty} r^{-\rho(r)} T(r, f) \leqq c_2 < +\infty.$$

We shall also consider an analogous problem for the functions satisfying (4) instead of (3) to obtain sharper results. As one of them Corollary 5 is a result sharper than Hayman's [8, Corollary 3].

Our results are deeply based on problems of finding a path on which an entire function $g(z)$ having the smooth growth of

$$B(r, g) = \max_{|z| = r} \log |g(z)|$$

grows quickly, and the problems also depend on Denjoy integral inequality (Lemma 2) whose proof is completely elementary and which is far-reaching. It should be remarked that we need the value of the constant K as accurate as possible, in obtaining the following type of results: There is a path along which

$$\liminf_{|z| \to \infty} B(|z|, g)^{-1} \log |g(z)| \geqq K.$$

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2. Definitions and a lemma

Let $h(r)$ be a positive non-decreasing function defined on the interval (R, ∞), where R is a positive constant. For $\rho \geqq 0$, we put

$$C(x, r) = x^{-\rho} h(xr)/h(r).$$

We say that $h(r)$ satisfies the *smoothness condition* (A) *of type* (ρ, c), if $h(r)$ satisfies the condition:

$$\limsup_{r\to\infty} C(x, r) \leqq c$$

for any x $(x>1)$. It is easy to see $c \geqq 1$.

REMARK 1. Let $h(r)$ satisfy the smoothness condition (A) of type (ρ, c). For any μ $(\mu>\rho)$, put

$$x_0 = (c+1)^{1/(\mu-\rho)}$$

and then take r_0 such that

$$C(x_0, r) \leqq c+1 \qquad (r \geqq r_0).$$

Then,

$$h(x_0 r)/h(r) \leqq x_0^\mu \qquad (r \geqq r_0).$$

Now, for any r $(r \geqq r_0)$, choose an integer p such that

$$x_0^p \leqq r/r_0 < x_0^{p+1}.$$

We have

$$h(r) \leqq h(x_0^{p+1} r_0) \leqq (x_0^{p+1})^\mu h(r_0) \leqq (x_0/r_0)^\mu h(r_0) r^\mu \qquad (r \geqq r_0).$$

This shows that

$$\limsup_{r\to\infty} \log h(r)/\log r \leqq \rho.$$

In the following, we shall consider only the smoothness condition (A) of type (ρ, c), where

$$\rho = \limsup_{r\to\infty} \log h(r)/\log r.$$

A differentiable function $\rho(r)$ that satisfies the conditions

$$\lim_{r\to\infty} \rho(r) = \rho, \quad \text{where} \quad \rho \geqq 0 \quad \text{is a constant,}$$

and

$$\lim_{r\to\infty} r\rho'(r)\log r = 0$$

is called a *proximate order* (see Cartwright [4, p. 54] and Levin [10, p. 32]). In the following, $\rho(r)$ always denotes a proximate order. We remark (see Cartwright [4, p. 55 and p. 58] and Levin [10, pp. 32–35]) that $\rho(r)$ has the following properties:

$$r^{-\rho(r)}(xr)^{\rho(xr)} \to x^\rho \qquad (r\to\infty) \tag{5}$$

for any fixed $x>1$, and

$$\int_r^\infty t^{-(1+\alpha)} t^{\rho(t)} dt \sim (\alpha-\rho)^{-1} r^{-\alpha} r^{\rho(r)} \qquad (r \to \infty) \tag{6}$$

for any constant α ($\alpha > \rho$).

Let ρ ($\rho \geqq 0$) be a constant. If there exist a proximate order $\rho(r)$, $\rho(r) \to \rho$ ($r \to \infty$), and a constant c ($c \geqq 1$), such that

$$1 \leqq \liminf_{r\to\infty} r^{-\rho(r)} h(r) \leqq \limsup_{r\to\infty} r^{-\rho(r)} h(r) \leqq c < +\infty,$$

we say that $h(r)$ satisfies the *smoothness condition* (B) *of type* (ρ, c).

It is easily seen from (5) that if $h(r)$ satisfies the smoothness condition (B) of type (ρ, c), then $h(r)$ also satisfies the smoothness condition (A) of type (ρ, c).

REMARK 2. The case that there exist a $\rho(r)$, $\rho(r) \to \rho$ ($r \to \infty$), and two constants c_1, c_2 satisfying (4) can be reduced to the case that $h(r)$ satisfies the smoothness condition (B) of type (ρ, c_2/c_1) by considering a new proximate order $\rho(r) + \log c_1/\log r$.

We give a lemma which will be used in the next section.

LEMMA 1. *Let c, ρ and α be three constants satisfying $c \geqq 1$, $\rho \geqq 0$ and $\alpha > \rho$. Let x ($x > 1$) be a number satisfying*

$$\alpha > \log c/\log x + \rho.$$

If $h(r)$ satisfies

$$h(xr)/h(r) \leqq cx^\rho \qquad (r \geqq r_0)$$

for some r_0, then

$$\int_r^\infty t^{-(1+\alpha)} h(t) dt \leqq S(x\colon \rho, \alpha, c) r^{-\alpha} h(r) \qquad (r \geqq r_0)$$

where

$$S(x\colon \rho, \alpha, c) = \alpha^{-1} c(x^\alpha - 1)/(x^{\alpha-\rho} - c).$$

PROOF. Put

$$\mu = \log c/\log x + \rho.$$

Then, we have

$$h(xr)/h(r) \leqq x^\mu \qquad (r \geqq r_0).$$

Since

$$h(x^{i+1}r) \leqq (x^\mu)^{i+1} h(r) \qquad (r \geqq r_0) \quad (i = 0, 1, 2, 3, \ldots),$$

we get

$$\int_r^\infty t^{-(1+\alpha)}h(t)dt \leqq \sum_{i=0}^\infty h(x^{i+1}r)\int_{x^i r}^{x^{i+1}r} t^{-(1+\alpha)}dt$$
$$\leqq \alpha^{-1}x^\mu(1-x^{-\alpha})r^{-\alpha}h(r)\sum_{i=0}^\infty (x^{\mu-\alpha})^i$$
$$= S(x\colon \rho, \alpha, c)r^{-\alpha}h(r) \qquad (r \geqq r_0).$$

Now, consider the function $S(x\colon \rho, \alpha, c)$ of x for a triple (ρ, α, c), $0 \leqq \rho < 1/2$, $c \geqq 1$, $\alpha > \rho$, and denote the greatest lower bound of $S(x\colon \rho, \alpha, c)$ on the open interval $(c^{1/(\alpha-\rho)}, \infty)$ by $m(\rho, \alpha, c)$. When $c>1$ and $\rho>0$, $m(\rho, \alpha, c)$ is attained at a unique value $x = X(c) = X(\rho, \alpha, c)$ on $(c^{1/(\alpha-\rho)}, \infty)$. When $c=1$ or $\rho=0$, $m(\rho, \alpha, c) = c/(\alpha-\rho)$. Further, put

$$d(\rho, \alpha, c) = \{c\alpha m(\rho, \alpha, c)\}^{1/(\alpha-\rho)}.$$

Since

$$S(x\colon \rho, \alpha, c) \geqq \alpha^{-1}(x^\alpha - 1)/(x^{\alpha-\rho} - 1) \geqq (\alpha-\rho)^{-1},$$

it is seen that $d(\rho, \alpha, c) \geqq 1$.

3. Integral functions of order less than 1/2

Let $g(z)$ be an integral function. We denote

$$A(r, g) = \min_{|z|=r} \log |g(x)|.$$

Throughout this section, we shall take $B(r, g)$ as $h(r)$ in section 2.

LEMMA 2 (Denjoy [5] and Kjellberg [9, pp. 17–18]). *Let $g(z)$ be an integral function of order ρ $(0 \leqq \rho < 1/2)$ for which $g(0)=1$. Then, for any α $(\rho<\alpha<1/2)$,*

$$r^\alpha \int_r^\infty \{A(t, g) - B(t, g)\cos \pi\alpha\}t^{-(1+\alpha)}dt > \alpha^{-1}(1-\cos \pi\alpha)B(r, g) \quad (0<r<\infty).$$

LEMMA 3. *Let $g(z)$ be an integral function of order ρ $(0 \leqq \rho < 1/2)$ for which $g(0)=1$ and $B(r, g)$ satisfies the smoothness condition* (A) *of type (ρ, c) $(c \geqq 1)$. If α is any constant satisfying $\rho<\alpha<1/2$, then for any k,*

$$k > d(\rho, \alpha, c), \tag{7}$$

we can find $r_0>0$ such that

$$A(t, g) > B(t, g)\cos \pi\alpha$$

for some t in any interval (t, kr) $(r \geqq r_0)$.

PROOF. Suppose that ρ is positive. From (7), we can choose c_1 $(c_1>c)$, sufficiently close to c, such that

$$k > c\alpha S(X(c_1)\colon \rho, \alpha, c_1)^{1/(\alpha-\rho)}.$$

Since

$$\limsup_{r\to\infty} C(k, r) \leqq c$$

we can also choose r_1 such that

$$\alpha^{-1} > C(k, r)S(X(c_1)\colon \rho, \alpha, c_1)k^{\rho-\alpha} \qquad (r \geqq r_1). \tag{8}$$

Further, choose r_0 ($r_0 \geqq r_1$) such that

$$B(r, g)^{-1}B(X(c_1)r, g) \leqq c_1\{X(c_1)\}^\rho \qquad (r \geqq r_0)$$

from the fact

$$\limsup_{r\to\infty} C(X(c_1), r) \leqq c.$$

Then, since we have

$$\alpha > \log c_1/\log X(c_1) + \rho$$

from the fact $X(c_1) > c_1^{1/(\alpha-\rho)}$, we obtain

$$\int_r^\infty t^{-(1+\alpha)}B(t, g)dt \leqq S(X(c_1)\colon \rho, \alpha, c_1)r^{-\alpha}B(r, g) \qquad (r \geqq r_0)$$

by the aid of Lemma 1. Thus, we get

$$\begin{aligned} &r^\alpha \int_{kr}^\infty \{A(t, g) - B(t, g)\cos \pi\alpha\}t^{-(1+\alpha)}dt \\ &\leqq (1 - \cos \pi\alpha)r^\alpha S(X(c_1)\colon \rho, \alpha, c_1)(kr)^{-\alpha}B(kr, g) \\ &= (1 - \cos \pi\alpha)S(X(c_1)\colon \rho, \alpha, c_1)C(k, r)k^{\rho-\alpha}B(r, g) \qquad (r \geqq r_0). \end{aligned}$$

Since $g(z)$ has order ρ, we finally have from Lemma 2 that

$$\begin{aligned} &r^\alpha \int_r^{kr} \{A(t, g) - B(t, g)\cos \pi\alpha\}t^{-(1+\alpha)}dt \\ &\geqq (1 - \cos \pi\alpha)\{\alpha^{-1} - S(X(c_1)\colon \rho, \alpha, c_1)C(k, r)k^{\rho-\alpha}\}B(r, g) \qquad (r \geqq r_0) \end{aligned}$$

in which the right-hand side is positive from (8) and the left-hand side is also positive. This fact gives the conclusion in the case $0 < \rho < 1/2$.

In the case $\rho = 0$, choose c_1 ($c_1 > c$), sufficiently close to c, and c_2 satisfying

$$\alpha > \log c_1/\log c_2.$$

If we replace $X(c_1)$ with c_2 and put $\rho = 0$ in all the previous expressions, we also obtain our conclusion in this case.

LEMMA 4. *Let $g(z)$ be an integral function of order ρ $(0 \leqq \rho < 1/2)$ for which*

$B(r, g)$ satisfies the smoothness condition (A) *of type* (ρ, c) $(c \geqq 1)$. *Then, for any constant* α, $\rho < \alpha < 1/2$, *we can find a polygonal path going to* ∞ *along which*

$$\liminf_{|z|\to\infty} B(|z|, g)^{-1} \log |g(z)| \geqq c^{-2}\{d(\rho, \alpha, c)\}^{-2\rho} \cos \pi\alpha.$$

PROOF. Since we evidently have the conclusion with $d(0, \alpha, 1) = 1$ in the case that $g(z)$ is a polynomial, we can assume that $z = \infty$ is an essential singularity of $g(z)$. Then we may assume $g(0) = 1$ from the fact

$$\lim_{r\to\infty} B(r, g)^{-1} \log r = 0.$$

Now, for each

$$k_n = d(\rho, \alpha, c) + n^{-1} \qquad (n = 1, 2, 3, \ldots),$$

take a constant $r_0^{(n)}$ and a sequence $\{t_j^{(n)}\}$ such that

$$k_n^j r_0^{(n)} < t_j^{(n)} < k_n^{j+1} r_0^{(n)} \qquad (j = 0, 1, 2, 3, \ldots)$$

and

$$\log |g(z)| \geqq A(t_j^{(n)}, g) > B(t_j^{(n)}, g) \cos \pi\alpha \qquad (|z| = t_j^{(n)})$$

by Lemma 3. Then, the set

$$\{z : \log |g(z)| > B(t_j^{(n)}, g) \cos \pi\alpha\}$$

which includes $\{z : |z| = t_j^{(n)}\}$, contains $\{z : |z| = t_{j+1}^{(n)}\}$. Hence, we can connect both points $z = t_j^{(n)}$ and $z = t_{j+1}^{(n)}$ with a polygonal path $\Gamma_j^{(n)}$ in $\{z : t_j^{(n)} \leqq |z| \leqq t_{j+1}^{(n)}\}$ on which

$$\log |g(z)| > B(t_j^{(n)}, g) \cos \pi\alpha.$$

Here, if we choose $r_1^{(n)}$ $(r_1^{(n)} \geqq r_0^{(n)})$ such that

$$k_n^{-\rho} B(|z|, g)^{-1} B(k_n|z|, g) \leqq c + n^{-1} \qquad (|z| \geqq r_1^{(n)}),$$

we have

$$\begin{aligned}\log |g(z)| > B(t_j^{(n)}, g) \cos \pi\alpha &\geqq B(k_n^{-2}|z|, g) \cos \pi\alpha \\ &\geqq k_n^{-2\rho}(c + 1/n)^{-2} B(|z|, g) \cos \pi\alpha\end{aligned}$$

for $z \in \Gamma_j^{(n)}$, $|z| \geqq r_1^{(n)} k_n^2$. Thus, we get the polygonal path

$$\Gamma_n = \cup_{j=0}^{\infty} \Gamma_j^{(n)} \qquad (n = 1, 2, 3, \ldots)$$

going to ∞ on which

$$\log |g(z)| > (c + 1/n)^{-2}\{d(\rho, \alpha, c) + 1/n\}^{-2\rho} B(|z|, g) \cos \pi\alpha \qquad (|z| \geqq r_1^{(n)} k_n^2).$$

Now, choose a sequence $\{j_n\}$ of integers such that

$$t_{j_n}^{(n)} > r_1^{(n)} k_n^2, \quad t_{j_{n+1}}^{(n+1)} > t_{j_n}^{(n)} \qquad (n = 1, 2, 3, \ldots)$$

and make a new path Γ in the following way: As soon as we reach the circle $\{z: |z| = t_{j_n}^{(n)}\}$ along Γ_{n-1}, we move along the circular arc C_n until we reach $z = t_{j_n}^{(n)}$ and then move along Γ_n $(n = 2, 3, 4, \ldots)$. It is also possible to replace C_n with a polygonal path in

$$\{z: k_n^{j_n} r_0^{(n)} < |z| \leqq t_{j_n}^{(n)}\}$$

on which

$$\log |g(z)| > B(t_{j_n}^{(n)}, g) \cos \pi\alpha \geqq B(|z|, g) \cos \pi\alpha.$$

Then, we finally get

$$\log |g(z)| > [c^{-2}\{d(\rho, \alpha, c)\}^{-2\rho} \cos \pi\alpha - \mathrm{o}(1)] B(|z|, g) \qquad (|z| \to \infty)$$

on the path Γ going to ∞.

In the following, we denote by $M(\rho, c)$ the least upper bound of the function

$$M(\alpha: \rho, c) = c^{-2}\{d(\rho, \alpha, c)\}^{-2\rho} \cos \pi\alpha$$

of α on the open interval $(\rho, 1/2)$. In the case $\rho > 0$, we see from the fact

$$\lim_{\alpha \to \rho + 0} d(\rho, \alpha, c) = +\infty$$

that there is an α_0 $(\rho < \alpha_0 < 1/2)$ such that

$$M(\rho, c) = M(\alpha_0: \rho, c).$$

Also we see that $M(0, c) = 1/c^2$.

THEOREM 1. *Let $g(z)$ be an integral function of order ρ $(0 \leqq \rho < 1/2)$ for which $B(r, g)$ satisfies the smoothness condition* (A) *of type (ρ, c) $(c \geqq 1)$. Then, we can find a polygonal path going to ∞ along which*

$$\liminf_{|z| \to \infty} B(|z|, g)^{-1} \log |g(z)| \geqq M(\rho, c).$$

PROOF. In the case $\rho > 0$, this immediately follows from Lemma 4, if only we put $\alpha = \alpha_0$ there. Hence, we shall consider the case $\rho = 0$.

For each $\alpha = 1/m$ $(m = 1, 2, 3, \ldots)$, we denote the sequence and the number corresponding to $\{t_j^{(n)}\}$ and $r_0^{(n)}$ in the proof of Lemma 4 by $\{t_j^{(n,m)}\}$ and $r_0^{(n,m)}$, respectively. Now, for each $\alpha = 1/m$ $(m = 1, 2, 3, \ldots)$, make a polygonal path Γ_m corresponding to Γ in Lemma 4, on which

$$\log |g(z)| > \{c^{-2} \cos m^{-1}\pi - \mathrm{o}(1)\} B(|z|, g) \qquad (|z| \to \infty).$$

Further, choose an $r_2^{(m)}$ such that

$$\log |g(z)| > \{c^{-2} \cos m^{-1}\pi - m^{-1}\} B(|z|, g) \qquad (|z| \geqq r_2^{(m)})$$

on Γ_m, and choose a sequence $\{i_m\}$ such that

$$t_{i_m}^{(m,m)} > r_2^{(m)}, \quad t_{i_{m+1}}^{(m+1,m+1)} > t_{i_m}^{(m,m)} \qquad (m = 1, 2, 3,\ldots).$$

Now, we make a new path Γ from $\{\Gamma_m\}$ in the following way: As soon as we reach the circle $\{z\colon |z| = t_{i_m}^{(m,m)}\}$ along Γ_{m-1}, we move along the circular arc to a point on Γ_m and then move along Γ_m $(m=2, 3, 4,\ldots)$. It is also possible to replace the circular arc with a polygonal path in

$$\{z\colon t_{i_m}^{(m,m)} \geqq |z| > h_m^{i_m} r_0^{(m,m)}\}$$

where $h_m = d(0, 1/m, c) + 1/m$, on which

$$\log |g(z)| > B(t_{i_m}^{(m,m)}, g) \cos m^{-1}\pi \geqq B(|z|, g) \cos m^{-1}\pi.$$

Then, we get

$$\log |g(z)| > \{c^{-2} - o(1)\} B(|z|, g) \qquad (|z| \to \infty)$$

along Γ.

Remark 3. Suppose that there exists a δ $(\delta > 1)$ such that

$$\limsup_{r\to\infty} x^{-\rho} B(r, g)^{-1} B(xr, g) \leqq 1$$

for any x $(1 < x < \delta)$. Now, take any x $(x > 1)$ and choose an integer p satisfying

$$\delta^p \leqq x < \delta^{p+1}.$$

If we put $y = x^{1/(p+1)}$, we see that for any $\varepsilon > 0$, there is an r_0 such that

$$B(r, g)^{-1} B(yr, g) \leqq (1 + \varepsilon) y^\rho \qquad (r \geqq r_0).$$

Since

$$\begin{aligned} B(r, g)^{-1} B(xr, g) &= B(r, g)^{-1} B(y^{p+1} r, g) \\ &\leqq (1 + \varepsilon)^{p+1} (y^\rho)^{p+1} = (1 + \varepsilon)^{p+1} x^\rho \qquad (r \geqq r_0), \end{aligned}$$

we get

$$\limsup_{r\to\infty} x^{-\rho} B(r, g)^{-1} B(xr, g) \leqq 1$$

for this x. Thus, it is seen that $B(r, g)$ satisfies the smoothness condition (A) of type $(\rho, 1)$, if and only if there exists a δ $(\delta > 1)$ such that

$$\limsup_{r\to\infty} x^{-\rho} B(r, g)^{-1} B(xr, g) \leqq 1$$

for any x $(1 < x < \delta)$.

Corollary 1 (Anderson [1, Theorem 2]). *If $g(z)$ is an integral function for which*

$$B(2r, g) \sim B(r, g) \qquad (r \to \infty),$$

then we can find a polygonal path going to ∞ *along which*

$$\liminf_{|z| \to \infty} B(|z|, g)^{-1} \log |g(z)| \geqq 1.$$

Proof. Take $\delta = 2$ in Remark 3, since

$$1 \leqq B(r, g)^{-1} B(xr, g) \leqq B(r, g)^{-1} B(2r, g) \to 1 \qquad (r \to \infty)$$

for x $(1 < x < 2)$. The required conclusion follows from Theorem 1 because of $M(0, 1) = 1$.

Let ρ, α and c be the numbers satisfying $0 \leqq \rho < 1/2$, $\rho < \alpha$ and $1 \leqq c$. We denote by $M^*(\rho, c)$ the least upper bound of the function

$$M^*(\alpha\colon \rho, c) = (1/c)^{2\alpha/(\alpha-\rho)} [\alpha^{-1}(\alpha - \rho)]^{2\rho/(\alpha-\rho)} \cos \pi\alpha$$

of α on the open interval $(\rho, 1/2)$. In the case $\rho > 0$,

$$M^*(\rho, c) = M^*(\alpha_0\colon \rho, c)$$

for some α_0 $(\rho < \alpha_0 < 1/2)$ and further in the case $c > 1$

$$M^*(\rho, c) > M(\rho, c)$$

because of the fact

$$S(X(c)\colon \rho, \alpha, c) \geqq \alpha^{-1} [\{X(c)\}^{\alpha} - 1] / [\{X(c)\}^{\alpha-\rho} - 1] > (\alpha - \rho)^{-1}.$$

In the case $\rho = 0$ or $c = 1$

$$M^*(0, c) = M(0, c) = 1/c^2.$$

The following Theorem 2 shows that we can have a result sharper than Theorem 1, in the case that $B(r, g)$ satisfies the smoothness condition (B) of type (ρ, c) where $0 < \rho < 1/2$ and $c > 1$.

Theorem 2. *If $g(z)$ is an integral function of order ρ $(0 \leqq \rho < 1/2)$ for which $B(r, g)$ satisfies the smoothness condition* (B) *of type (ρ, c) $(c \geqq 1)$, then we can find a polygonal path going to ∞ along which*

$$\liminf_{|z| \to \infty} B(|z|, g)^{-1} \log |g(z)| \geqq M^*(\rho, c).$$

Proof. We can assume $g(0) = 1$. Now, let α be any number satisfying $\rho < \alpha < 1/2$. Since for $x > 1$

$$\begin{aligned} r^{\alpha} \int_{xr}^{\infty} \{A(t, g) - B(t, g) \cos \pi\alpha\} t^{-(1+\alpha)} dt &\leqq (1 - \cos \pi\alpha) r^{\alpha} \int_{xr}^{\infty} t^{-(1+\alpha)} B(t, g) dt \\ &\leqq (1 - \cos \pi\alpha)(c + \mathrm{o}(1))(\alpha - \rho)^{-1} r^{\rho(r)} x^{\rho-\alpha} \qquad (r \to \infty) \end{aligned}$$

from (5) and (6), we get

$$r^{\alpha}\int_{r}^{xr}\{A(t, g) - B(t, g)\cos\pi\alpha\}t^{-(1+\alpha)}dt$$
$$> (1-\cos\pi\alpha)[\alpha^{-1}\{1-\mathrm{o}(1)\} - (\alpha-\rho)^{-1}\{c+\mathrm{o}(1)\}x^{\rho-\alpha}]r^{\rho(r)} \qquad (r\to\infty)$$

by Lemma 2. Thus, if we take any x satisfying

$$x > \{(\alpha-\rho)^{-1}c\alpha\}^{1/(\alpha-\rho)}$$

we can make the right-hand side positive for sufficiently large r. On the other hand, we have for any $x>1$

$$B(r, g)^{-1}B(xr, g) \leqq \{c+\mathrm{o}(1)\}x^{\rho} \qquad (r\to\infty)$$

from (5), since

$$\{1-\mathrm{o}(1)\}r^{\rho(r)} \leqq B(r, g) \leqq \{c+\mathrm{o}(1)\}r^{\rho(r)} \qquad (r\to\infty).$$

Hence, if we use both these facts, we obtain the conclusion in the same way as in Theorem 1.

4. Meromorphic functions of order less than 1/2

First of all, we remark that the smoothness of the growth of $T(r,f)$ is compatible with the largeness of the deficiency i.e., for any ρ $(0<\rho<1/2)$ and any v $(0\leqq v\leqq 1)$, there exists a meromorphic function of order ρ for which

$$\delta(\infty, f) = v \quad \text{and} \quad T(r,f)\sim Kr^{\rho} \qquad (r\to\infty),$$

where K is a constant (see Edrei and Fuchs [6, pp. 247–248] and Hayman [7, pp. 117–118]).

Throughout this section, we shall mainly take $T(r,f)$ as $h(r)$ in section 2. Now, we shall give Theorem 3 which generalizes Hayman [8, Corollary 2].

LEMMA 5. *Let $g(z)$ be an integral function for which $N(r, 1/g)$ satisfies the smoothness condition* (A) *of type* $(\rho, 1)$. *Then,*

$$\limsup_{r\to\infty} N(r, 1/g)^{-1}n(r, 1/g) \leqq \rho.$$

PROOF. Put

$$N(r, 1/g)^{-1}N(xr, 1/g) = C(x, r)x^{\rho}.$$

Then, for any x $(x>1)$, we have

$$n(r, 1/g)\log x \leqq \int_{r}^{xr} t^{-1}n(t, 1/g)dt$$
$$= N(xr, 1/g) - N(r, 1/g) = \{C(x, r)x^{\rho} - 1\}N(r, 1/g)$$

which gives

$$N(r, 1/g)^{-1}n(r, 1/g) \leqq \{C(x, r)x^{\rho} - 1\}/\log x \qquad (x > 1).$$

This immediately gives the conclusion, since

$$\lim_{x\to 1} (x^{\rho} - 1)/\log x = \rho \quad \text{and} \quad \limsup_{r\to\infty} C(x, r) \leqq 1 \qquad (x > 1).$$

THEOREM 3. *Let $f(z)$ be a meromorphic function in the whole plane of order ρ $(0 \leqq \rho < 1/2)$, for which $T(r, f)$ satisfies the smoothness condition* (A) *of type* $(\rho, 1)$. *Then, if*

$$\delta(w, f) > 2\rho,$$

w is an asymptotic value of $f(z)$.

PROOF. We can assume $w = \infty$ without loss of generality. Since

$$N(r, 1/(f - w)) \sim T(r, f) \qquad (r \to \infty)$$

for all w outside a set of capacity zero (see Nevanlinna [11, p. 264]), we can choose w_0 $(w_0 \neq \infty)$ such that

$$N(r, 1/(f - w_0)) \sim T(r, f) \qquad (r \to \infty). \tag{9}$$

Further, we can write

$$f(z) - w_0 = g_1(z)/g_2(z),$$

where $g_1(z)$ and $g_2(z)$ are both integral functions of order at most ρ, having no zeros in common. (In fact, $g_1(z)$ has order ρ by (9).) Since

$$T(t, f) \sim N(t, 1/g_1) \qquad (t \to \infty)$$

from (9) and

$$\begin{aligned} \log\{N(r, 1/g_1)^{-1}N(t, 1/g_1)\} &= \int_r^t u^{-1}N(u, 1/g_1)^{-1}n(u, 1/g_1)du \\ &\leqq \{\rho + o(1)\} \log r^{-1}t \qquad (r \to \infty) \end{aligned} \tag{10}$$

from Lemma 5, we obtain

$$T(t, f) \leqq (1 + \varepsilon)(r^{-1}t)^{\rho+\varepsilon}T(r, f) \qquad (t \geqq r \geqq r_0(\varepsilon))$$

for any $\varepsilon > 0$. This yields

$$\limsup_{r\to\infty} 2^{-1}T(r, f)^{-1}r^{1/2}\int_r^{\infty} t^{-3/2}T(t, f)dt \leqq (1 - 2\rho)^{-1},$$

since ε is any positive number. Hence, Hayman [8, Corollary 1] gives the conclusion.

QUESTIONS. We can also prove from Hayman [8, Corollary 1]: If $f(z)$ is a meromorphic function of order ρ $(0 \leqq \rho < 1/2)$ for which $T(r, f)$ satisfies the smoothness condition (B) of type (ρ, c) $(c \geqq 1)$ and w is a value such that

$$\delta(w, f) > 1 - (1 - 2\rho)/c,$$

w is an asymptotic value of $f(z)$.

This result in the case $c = 1$ is the same one as Theorem 3. Hence, we can ask according to Hayman [8, p. 144]: Is the constant 2ρ sharp for the functions satisfying the smoothness condition (A) of type $(\rho, 1)$? We also ask whether all deficient values are necessarily asymptotic values for the functions satisfying the smoothness condition (B) of type $(\rho, 1)$. Is the constant $1 - 1/c$ also sharp for the functions satisfying the smoothness condition (B) of type $(0, c)$ $(c > 1)$?

Let Γ be a polygonal path going to ∞. We put

$$G(w, f) = \begin{cases} \liminf_{|z| \to \infty, z \in \Gamma} (\log 1/|f(z) - w|)/T(|z|, f) & (w \neq \infty) \\ \liminf_{|z| \to \infty, z \in \Gamma} (\log |f(z)|)/T(|z|, f) & (w = \infty) \end{cases}$$

THEOREM 4. *Let $f(z)$ be a meromorphic function in the whole plane of order ρ $(0 \leqq \rho < 1/2)$, for which $T(r, f)$ satisfies the smoothness condition* (A) *of type* $(\rho, 1)$. *If*

$$\delta(w, f) > 1 - P(\rho)$$

where

$$P(\rho) = (1 - \rho) M(\rho, (1 - \rho)^{-1}),$$

we can find a polygonal path going to ∞ along which

$$G(w, f) \geqq (1 - \rho)^{-1}\{\delta(w, f) - (1 - P(\rho)\}.$$

Hence, if $0 < \rho < 1/2$ and

$$\delta(w, f) > 1 - S(\rho)$$

where

$$S(\rho) = (1 - t)(1 - \rho)^{(3-t)/(1-t)}[\{2/(1 - \rho)\}^{1/(1-t)} - 1]^{2t/(t-1)} \quad (t = (2\rho)^{1/2})$$

we can find a polygonal path going to ∞ on which

$$G(w, f) \geqq (1 - \rho)^{-1}\{P(\rho) - S(\rho)\}.$$

PROOF. We can assume $w = \infty$ without loss of generality. In the same way as in Theorem 3, we can choose w_0 $(w_0 \neq \infty)$ such that $f(0) \neq w_0$ and

$$N(r, 1/(f - w_0)) \sim T(r, f) \qquad (r \to \infty). \tag{11}$$

Further, we can write

$$f(z) - w_0 = g_1(z)/g_2(z),$$

where

$$g_2(z) = z^{\lambda} + \cdots \qquad \text{at} \quad z = 0.$$

Then, we obtain

$$\begin{aligned}(12) \quad T(r,f) &\sim N(r, 1/g_1) \leqq B(r, g_1) + \mathrm{O}(1) \\ &= \int_0^{\infty} \log(1 + t^{-1}r)dn(t, 1/g_1) + \mathrm{O}(1) \leqq r\int_r^{\infty} t^{-2}N(t, 1/g_1)dt + \mathrm{O}(1) \\ & \qquad (r \to \infty)\end{aligned}$$

from (11) and

$$\begin{aligned}(13) \qquad r\int_r^{\infty} t^{-2}N(t, 1/g_1)dt &\leqq N(r, 1/g_1)((1-\rho)^{-1} + \mathrm{o}(1)) \\ &= T(r,f)((1-\rho)^{-1} + \mathrm{o}(1)) \qquad (r \to \infty)\end{aligned}$$

from (10). These (12) and (13) show that $B(r, g_1)$ also satisfies the smoothness condition (A) of type $(\rho, 1/(1-\rho))$. Hence, from Theorem 1 and (12), we can find a polygonal path Γ going to ∞ on which

$$\begin{aligned}(14) \qquad \log|g_1(z)| &> \{M(\rho, (1-\rho)^{-1}) - \mathrm{o}(1)\}B(|z|, g_1) \\ &\geqq \{M(\rho, (1-\rho)^{-1}) - \mathrm{o}(1)\}T(|z|, f) \qquad (|z| \to \infty).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\log|g_2(z)| &\leqq \int_0^{\infty} \log(1 + t^{-1}r)dn(t, 1/g_2) + n(0, 1/g_2)\log r \\ &\leqq r\int_r^{\infty} t^{-2}N(t, 1/g_2)dt \leqq \{1 - \delta(\infty, f) + \mathrm{o}(1)\}r\int_r^{\infty} t^{-2}T(t,f)dt \quad (|z| = r \to \infty)\end{aligned}$$

from the fact

$$N(r, 1/g_2) = N(r, f - w_0) = N(r, f) < \{1 - \delta(\infty, f) + \mathrm{o}(1)\}T(r, f) \quad (r \to \infty).$$

Hence, we get

$$(15) \quad \log|g_2(z)| \leqq \{1 - \delta(\infty, f) + \mathrm{o}(1)\}T(|z|, f)((1-\rho)^{-1} + \mathrm{o}(1)) \quad (|z| \to \infty),$$

since

$$\begin{aligned}r\int_r^{\infty} t^{-2}T(t,f)dt &\leqq \{1 + \mathrm{o}(1)\}r\int_r^{\infty} t^{-2}N(t, 1/g_1)dt \\ &\leqq T(r,f)((1-\rho)^{-1} + \mathrm{o}(1)) \qquad (r \to \infty)\end{aligned}$$

from (11).

Thus, from (14) and (15), we finally have

$$\log|f(z)| \sim \log|f(z) - w_0| = \log|g_1(z)| - \log|g_2(z)|$$
$$> (1-\rho)^{-1}\{\delta(\infty, f) - (1 - P(\rho)) - o(1)\}T(|z|, f)$$

along Γ as $|z|\to\infty$.

To get the latter part, we have only to put

$$\alpha = (2^{-1}\rho)^{1/2} \quad \text{and} \quad x = \{2/(1-\rho)\}^{1/(\alpha-\rho)}$$

in $S(x\colon \rho, \alpha, (1-\rho)^{-1})$, to estimate $P(\rho)$.

REMARK 4. If there is a w such that

$$\delta(w, f) > 1 - P(\rho),$$

$f(z)$ cannot have any deficient values other than w. For, there exists a sequence $\{t_j\}$, $t_j\to\infty$ $(j\to\infty)$, such that

$$f(t_j e^{i\theta}) \to w \qquad (j\to\infty)$$

uniformly for $0\leqq\theta\leqq 2\pi$, as we see from both proofs of Theorem 1 and Theorem 4. But, this also follows from the fact

$$1 - P(\rho) \geqq 1 - \cos\pi\rho \qquad (0 \leqq \rho < 1/2)$$

(see Edrei and Fuchs [6, Corollary 1.1] and Valiron [12]).

COROLLARY 2 (Anderson [1, Theorem 1]). *Let $f(z)$ be a moromorphic function for which*

$$T(2r, f) \sim T(r, f) \qquad (r\to\infty).$$

Then if

$$\delta(w, f) > 0,$$

we can find a polygonal path going to ∞ along which

$$G(w, f) \geqq \delta(w, f).$$

PROOF. It is seen in the same way as in Remark 3 and Corollary 1 that the condition with $\rho=0$ of Theorem 4 is satisfied.

COROLLARY 3. *Let ρ be a sufficiently small positive number and $f(z)$ be a meromorphic function of order ρ for which $T(r, f)$ satisfies the smoothness condition* (A) *of type* $(\rho, 1)$. *Then, if*

$$\delta(w, f) > 10\rho,$$

we can find a polygonal path going to ∞ *along which*

$$G(w, f) \geqq 2^{-1}(1-\rho)^{-1}\rho.$$

PROOF. We shall estimate the value $P(\rho)$ for sufficiently small ρ $(\rho>0)$. Let ε be a positive number. Then, we have

$$m(\rho, \alpha, (1-\rho)^{-1}) \leqq [\{(1+\varepsilon\rho)/(1-\rho)\}^{\alpha/(\alpha-\rho)} - 1]/(\alpha\varepsilon\rho)$$

by putting

$$x = ((1+\varepsilon\rho)/(1-\rho))^{1/(\alpha-\rho)}$$

in $S(x\colon \rho, \alpha, (1-\rho)^{-1})$. Hence, if we put $\alpha = k\rho^{1/2}$ $(k=(2/\pi)^{1/2})$ and

$$\varepsilon = k2^{-1}\eta\rho^{-1/2}(1 - k2^{-1}\eta\rho^{1/2})$$

for any $\eta>0$, we get

$$1 - P(\rho) \leqq (3 + 2\pi + \eta)\rho + o(\rho) \qquad (\rho \to 0).$$

The following Theorem 5 and Theorem 6 contain better constants than the constant in Theorem 4 (see Remark 5).

THEOREM 5. *Let* $f(z)$ *be a meromorphic function in the whole plane of order* ρ $(0 \leqq \rho < 1/2)$ *for which* $T(r, f)$ *satisfies the smoothness condition* (B) *of type* (ρ, c) $(c \geqq 1)$. *Then, if*

$$\delta(w, f) > 1 - Q(\rho, c)$$

where

$$Q(\rho, c) = c^{-1}(1-\rho)M^*(\rho, c(1-\rho)^{-1}),$$

we can find a polygonal path going to ∞ *along which*

$$G(w, f) \geqq c(1-\rho)^{-1}[\delta(w, f) - \{1 - Q(\rho, c)\}].$$

Hence, if $0<\rho<1/2$ *and*

$$\delta(w, f) > 1 - U(\rho, c)$$

where

$$U(\rho, c) = \{c^{-1}(1-\rho)\}^{(3-t)/(1-t)}(1-t)^{(1+t)/(1-t)} \qquad (t = (2\rho)^{1/2})$$

we can find a polygonal path going to ∞ *along which*

$$G(w, f) \geqq c(1-\rho)^{-1}\{Q(\rho, c) - U(\rho, c)\}.$$

PROOF. We choose such a w_0 $(w_0 \neq \infty)$ that $f(0) \neq w_0$ and

$$T(r,f) \sim N(r, 1/(f-w_0)) \qquad (r \to \infty).$$

Further, write

$$f(z) - w_0 = g_1(z)/g_2(z).$$

Then, we have

$$\begin{aligned}\{1 - o(1)\}r^{\rho(r)} &\leqq T(r,f) \sim N(r, 1/g_1) \leqq B(r, g_1) + O(1) \\ &\leqq \int_0^\infty \log(1 + t^{-1}r)dn(t, 1/g_1) + O(1) \leqq r\int_r^\infty t^{-2}N(t, 1/g_1)dt + O(1) \\ &\leqq \{c + o(1)\}r\int_r^\infty t^{\rho(t)-2}dt + O(1) = \{c + o(1)\}(1-\rho)^{-1}r^{\rho(r)} + O(1) \\ &\qquad (r \to \infty)\end{aligned}$$

from (6). Hence, we get

$$1 \leqq \liminf_{r\to\infty} r^{-\rho(r)}B(r, g_1) \leqq \limsup_{r\to\infty} r^{-\rho(r)}B(r, g_1) \leqq c(1-\rho)^{-1}.$$

Thus, using Theorem 2, we obtain the conclusion by the same argument as in Theorem 4, since

$$\begin{aligned} r\int_r^\infty t^{-2}T(t,f)dt &= \{1 + o(1)\}r\int_r^\infty t^{-2}N(t, 1/g_1)dt \leqq \{c + o(1)\}(1-\rho)^{-1}r^{\rho(r)} \\ &\leqq \{c + o(1)\}(1-\rho)^{-1}T(r,f) \qquad (r \to \infty).\end{aligned}$$

To get the latter part, we have only to put $\alpha = (\rho/2)^{1/2}$ in $M^*(\alpha\colon \rho, c(1-\rho)^{-1})$.

COROLLARY 4. *Let $f(z)$ be a meromorphic function of order 0 for which $T(r,f)$ satisfies the smoothness condition* (B) *of type* $(0, c)$ $(c \geqq 1)$. *Then, if*

$$\delta(w,f) > 1 - 1/c^3,$$

we can find a polygonal path going to ∞ along which

$$G(w,f) \geqq c[\delta(w,f) - (1 - 1/c^3)].$$

The following Corollary 5 sharpens Hayman [8, Corollary 3] in the sense that there is a path along which $f(z)$ grows quickly.

COROLLARY 5. *Let $f(z)$ have very regular growth of order ρ* $(0<\rho<1/2)$, *i.e., suppose there are two positive constants c_1, c_2 such that*

$$c_1 r^\rho < T(r,f) < c_2 r^\rho$$

for sufficiently large r. Then, if

$$\delta(w,f) = 1,$$

we can find a polygonal path going to ∞ *along which*

$$\liminf_{|z|\to\infty} r^{-\rho} \log 1/|f(z) - w| \geqq C$$

where C is a positive constant dependent on c_1, c_2.

PROOF. Since $T(r, f)$ satisfies the smoothness condition (B) of type (ρ, c_2/c_1) (see Remark 2), this follows from Theorem 5.

LEMMA 6. *Let* $g(z)$ *be an integral function for which* $N(r, 1/g)$ *satisfies the smoothness condition* (B) *of type* (ρ, 1). *Then*

$$\lim_{r\to\infty} N(r, 1/g)^{-1} n(r, 1/g) = \rho.$$

PROOF. Since we have Lemma 5, we shall show that

$$\lim_{r\to\infty} N(r, 1/g)^{-1} n(r, 1/g) \geqq \rho.$$

From the fact

$$N(r, 1/g) \sim r^{\rho(r)} \qquad (r \to \infty),$$

we have

$$N(r, 1/g)^{-1} N(xr, 1/g) \geqq \{1 - \mathrm{o}(1)\} x^{\rho} \qquad (r \to \infty)$$

for any x ($x > 1$). Hence, we have

$$\begin{aligned} n(xr, 1/g) \log x &\geqq \int_r^{xr} t^{-1} n(t, 1/g) dt = N(xr, 1/g) - N(r, 1/g) \\ &\geqq [1 - \{1 - \mathrm{o}(1)\}^{-1} x^{-\rho}] N(xr, 1/g) \qquad (r \to \infty), \end{aligned}$$

which is equivalent to

$$N(r, 1/g)^{-1} n(r, 1/g) \geqq \{(x^{\rho} - 1) - \mathrm{o}(1) x^{\rho}\} \{1 - \mathrm{o}(1)\}^{-1} x^{-\rho} / \log x \qquad (r \to \infty)$$

for any x ($x > 1$). Thus, since

$$\lim_{x\to 1} (x^{\rho} - 1)/\log x = \rho,$$

we get the conclusion.

THEOREM 6. *Let* $f(z)$ *be a meromorphic function in the whole plane of order* ρ ($0 < \rho < 1/2$) *for which* $T(r, f)$ *satisfies the smoothness condition* (B) *of type* (ρ, 1). *Then, if*

$$\delta(w, f) > 1 - R(\rho)$$

where

$$R(\rho) = (1 - \rho) M^*(\rho, \pi\rho/\sin \pi\rho),$$

we can find a polygonal path going to ∞ *along which*

$$G(w,f) \geqq (1-\rho)^{-1}[\delta(w,f)-(1-R(\rho))].$$

Hence, if

$$\delta(w,f) > 1 - V(\rho)$$

where

$$V(\rho) = (1-\rho)\{(\pi\rho)^{-1}\sin\pi\rho\}^{2/(1-t)}(1-t)^{(1+t)/(1-t)} \qquad (t=(2\rho)^{1/2}),$$

we can find a polygonal path going to ∞ *along which*

$$G(w,f) \geqq (1-\rho)^{-1}\{R(\rho)-V(\rho)\}.$$

PROOF. In the usual way, we choose w_0 ($w_0 \neq \infty$) such that $f(0) \neq w_0$ and

$$T(r,f) \sim N(r, 1/(f-w_0)) \qquad (r\to\infty).$$

Further, write

$$f(z) - w_0 = g_1(z)/g_2(z).$$

Then, Lemma 6 applied to $g_1(z)$ and a result (see Cartwright [4, p. 59, Theorem 37] and Levin [10, pp. 63–64, Theorem 25]) give

$$\begin{aligned}\{1-o(1)\}r^{\rho(r)} &\sim N(r, 1/g_1) \leqq B(r, g_1)\\ &= \int_0^\infty \log(1+t^{-1}r)dn(t, 1/g_1) = \{\pi\rho/\sin\pi\rho + o(1)\}r^{\rho(r)} \qquad (r\to\infty).\end{aligned}$$

Further, we have

$$r\int_r^\infty t^{-2}T(t,f)dt = \{1+o(1)\}(1-\rho)^{-1}r^{\rho(r)} = \{1+o(1)\}(1-\rho)^{-1}T(r,f) \qquad (r\to\infty).$$

Thus, by the same argument as in Theorem 4, we get the conclusion.

To get the latter part, we have only to put $\alpha=(\rho/2)^{1/2}$ in $M^*(\alpha\colon \rho, \pi\rho/\sin\pi\rho)$.

REMARK 5. Since

$$\pi\rho/\sin\pi\rho < (1-\rho)^{-1} \quad \text{and} \quad M^*(\rho, (1-\rho)^{-1}) > M(\rho, (1-\rho)^{-1}) \quad (\rho > 0)$$

as was observed after Corollary 1, we have

$$1 - R(\rho) < 1 - Q(\rho, 1) < 1 - P(\rho).$$

COROLLARY 6. *Let* ρ *be a sufficiently small positive number and* $f(z)$ *be a meromorphic function of order* ρ *for which* $T(r,f)$ *satisfies the smoothness condition* (B) *of type* $(\rho, 1)$ (*e.g., let* $f(z)$ *have perfectly regular growth of order* ρ:

$$\lim_{r\to\infty} r^{-\rho}T(r,f) = c \qquad (0 < c < +\infty)).$$

Then, if

$$\delta(w,f) > 8\rho,$$

we can find a polygonal path going to ∞ along which

$$G(w,f) \geqq 2^{-1}\rho(1-\rho)^{-1}.$$

PROOF. Since

$$R(\rho) \geqq (1-\rho)\{(\pi\rho)^{-1}\sin\pi\rho\}^{2\alpha/(\alpha-\rho)}\{\alpha^{-1}(\alpha-\rho)\}^{2\rho/(\alpha-\rho)}\cos\pi\alpha$$

for any α $(\rho<\alpha<1/2)$, we have

$$R(\rho) \geqq 1-(1+2\pi)\rho+\mathrm{o}(\rho) \qquad (\rho\to 0)$$

by putting $\alpha=k\rho^{1/2}$ $(k=(2/\pi)^{1/2})$. This gives the conclusion.

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