A note to establish the Hyers-Ulam stability for a nonlinear integral equation with Lipschitzian kernel

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Abstract

In the stability theory, the nonlinear equations were not so much investigated. In this note we consider the stability of a nonlinear integral equation with Lipschitzian kernel. The approach is based on monotonicity properties of a nonlinear operator.

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1 Introduction and preliminaries

In 1940 S. M. Ulam raised the well-known stability problem, and in the next year, D. H. Hyers [4] gave an answer to this problem proving the stability of an additive mapping between two Banach spaces.

Over the years, many authors solved the Hyers-Ulam stability(HUS) problem for some important functional, differential or integral equations(see [1-3,5-7,10,11] and related references therein).

In this paper we study the Hyers-Ulam stability of a nonlinear integral equation, in a simple approach, much simpler than many other approaches of some linear cases.

We consider the nonlinear integral equation

$$\lambda u(x) + \int_a^b K(x, y, u(y))dy = g(x) \ \ (g \in L^2[a, b]; \lambda > 0), \tag{1}$$

where the kernel $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the Lipschitz condition

$$|K(x, y, z) - K(x, y, t)| \leq M |z - t| \ \ (M > 0). \tag{2}$$

To obtain results about the stability of some linear differential equations, Miura, Miyajima and Takahasi introduced in [10] the notion of Hyers-Ulam stability of an operator between normed spaces as follows:

Definition 1.1. Let $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *be two normed spaces and* f *be a (not necessarily linear) operator from* X *into* Y. *We say that* f *has the Hyers-Ulam stability if there exists a constant* $K > 0$ *with the following property: For any* $v \in Rg(f)$, *the range of* f, $\varepsilon > 0$ *and* $u \in X$ *with* $\|f(u) - v\| < \varepsilon$, *there exists* $u_0 \in X$ *such that* $f(u_0) = v$ *and* $\|u - u_0\| < K\varepsilon$. *We call such* $K > 0$ *a HUS constant for* f

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Thus the above mentioned authors studied the stability of some linear differential equations using the stability of the associated operator, in the sense of Definition 1.1. So, if the operator f has the Hyers-Ulam stability in the sense of Definition 1.1., then clearly the equation $f(x) = y(y \in Y)$ has the Hyers-Ulam stability.

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm denoted by $\|\cdot\|$. To better understand what follows, we recall that an operator $T : H \to H$ is said to be α-strongly monotone$(\alpha > 0)$ if

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|x - y\|^2 \text{ for all } x, y \in H.$$

(the definition of monotone operator was first given in 1960 by Kachurovski[8] and iterative methods for strongly monotone operators in Hilbert space satisfying a Lipschitz condition were first given by Vainberg [12] and Zarantonello [13])

Also, we recall that an operator $T : H \to H$ is said to be a β-Lipschitz operator$(\beta > 0)$ if

$$\|Tx - Ty\| \leq \beta \|x - y\| \text{ for all } x, y \in H.$$

2 Main results

In the following H denotes the real Hilbert space $L^2[a, b]$, $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and the corresponding norm in $L^2[a, b]$.

Theorem 2.1. *Let $F : H \to H$ be the nonlinear integral operator defined by*

$$Fu(x) = \int_a^b K(x, y, u(y))dy.$$

If the condition (2) holds, then F is a $M(b - a)$-Lipschitz operator.

Proof. For all $u, v \in H = L^2[a, b]$ we have

$$\|Fu - Fv\|^2 = \int_a^b \left(\left| \int_a^b K(x, y, u(y))dy - \int_a^b K(x, y, v(y))dy \right|^2 \right) dx$$

$$\leq \int_a^b \left(\int_a^b M |u(y) - v(y)| \, dy \right)^2 dx.$$

Due to the Cauchy-Schwarz integral inequality

$$\left(\int_a^b M |u(y) - v(y)| \, dy \right)^2 \leq \int_a^b M^2 dy \cdot \int_a^b |u(y) - v(y)|^2 \, dy = M^2(b - a) \|u - v\|^2 .$$

Now we obtain

$$\|Fu - Fv\|^2 \leq \int_a^b M^2(b-a)\,\|u-v\|^2\,dx = M^2(b-a)^2\,\|u-v\|^2\,,$$

and the proof is complete. ∎

To prove the Hyers-Ulam stability for the equation (1), it is sufficient to prove that the operator $\lambda I + F$ has the Hyers-Ulam stability(I is the identity of H).

Theorem 2.2. *If $\lambda > M(b-a)$, then the operator $T = \lambda I + F$ has the Hyers-Ulam stability and $(\lambda - M(b-a))^{-1}$ is a HUS constant for T.*

Proof. Using the Schwarz inequality in H, with Theorem 2.1, we obtain

$$|\langle Fx - Fy, x-y\rangle| \leq \|Fx - Fy\| \cdot ||x-y|| \leq M(b-a)\,\|x-y\|^2\,.$$

Consequently

$$\begin{aligned}\langle Tx - Ty, x-y\rangle &= \lambda\,\langle x-y, x-y\rangle + \langle Fx - Fy, x-y\rangle \\ &\geq \lambda\,\langle x-y, x-y\rangle - M(b-a)\,\|x-y\|^2 = (\lambda - M(b-a))\,\|x-y\|^2\,,\end{aligned}$$

for all $x, y \in H$.

Now let $v \in Rg(T)$, $\varepsilon > 0$ and $u \in H$ with $\|Tu - v\| < \varepsilon$. Clearly we have $u_0 \in H$ such that $Tu_0 = v$.

Since $\lambda > M(b-a)$, we obtain that

$$\langle Tu - Tu_0, u - u_0\rangle \geq (\lambda - M(b-a))\,\|u-u_0\|^2 > 0.$$

So $\langle Tu - Tu_0, u - u_0\rangle$ is a positive real number and, using again the Schwarz inequality, we have

$$\begin{aligned}(\lambda - M(b-a))\,\|u-u_0\|^2 &\leq \langle Tu - Tu_0, u-u_0\rangle = |\langle Tu - Tu_0, u-u_0\rangle| \\ &\leq ||Tu - Tu_0|| \cdot ||u-u_0||.\end{aligned}$$

It results that

$$\begin{aligned}||u-u_0|| &\leq (\lambda - M(b-a))^{-1}\,||Tu - Tu_0|| \\ &= (\lambda - M(b-a))^{-1}\,\|Tu - v\| < \frac{\varepsilon}{\lambda - M(b-a)}\end{aligned}$$

and the proof of Theorem 2.2 is complete. ∎

As a consequence of a result of G. Minty [9], every Lipschitz strongly monotone operator from a real Hilbert space into itself is a homeomorphism. Cf. the proof of Theorem 2.2, $T = \lambda I + F$ is a Lipschitz $(\lambda - M(b-a))$-strongly monotone operator and consequently, according to Theorem 2.2, we have

Theorem 2.3. *If $\lambda > M(b-a)$, then the nonlinear integral equation (1) has a unique solution in $L^2[a,b]$ for all $g \in L^2[a,b]$. Moreover the solution is stable by the variation of the free term g.*

The above results have as an obvious consequence the stability of some linear integral equations. So, it is obtained the stability of some linear integral equations without complicated constructive proofs.

Indeed, consider the linear integral equation

$$\lambda u(x) + \int_a^b k(x,y)u(y)dy = g(x) \ (g \in L^2[a,b]; \lambda > 0), \tag{3}$$

where the kernel $k : [a,b] \times [a,b] \to \mathbb{R}$ is a continuous function. Since $[a,b] \times [a,b]$ is compact, it results that k is bounded, and if $M = \sup_{(x,y)\in[a,b]\times[a,b]} |k(x,y)|$, then the condition (2) holds with $K(x,y,u) = k(x,y)u$. So, we are in a position to give the following result.

Theorem 2.4. *If $\lambda > (b-a) \sup_{(x,y)\in[a,b]\times[a,b]} |k(x,y)|$, then the linear integral equation (3) has the Hyers-Ulam stability, in the sense that the unique solution of equation (3) is stable by the variation of the free term g.*

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