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HORIZONTALLY CONFORMAL SUBMERSIONS OF CR-SUBMANIFOLDS

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Abstract

It is shown that any horizontally conformal submersion of a CR-submanifold M of a Kaehler manifold $\bar{M}$ onto a Kaehler manifold N is a Riemannian submersion. Moreover, if M is mixed geodesic, then it is proved that such submersion is a harmonic map.

1. Introduction

Let $\bar{M}$ be an almost Hermitian manifold with complex structure J and M be a Riemannian manifold isometrically immersed in $\bar{M}$. Then M is called holomorphic (complex) if $J(T_pM) \subset T_pM$, for every $p \in M$, where T_pM denotes the tangent space to M at the point p. M is called totally real if $J(T_pM) \subset T_pM^{\perp}$ for every $p \in M$, where $T_pM^{\perp}$ denotes the normal space to M at the point p. As a generalization of holomorphic and totally real submanifolds, CR-submanifolds of Hermitian manifolds were introduced by A. Bejancu in [2] and [3] as follows. A Riemannian manifold M isometrically immersed in an almost Hermitian manifold $\bar{M}$ is called a CR-submanifold if there exists a differentiable distribution $\mathscr{D}$ on M satisfying the following conditions: (i) $\mathscr{D}$ is holomorphic, i.e., $J(\mathscr{D}_p) = \mathscr{D}_p$ for each $p \in M$, and (ii) the complementary orthogonal distribution $\mathscr{D}^{\perp}$ is anti-invariant, i.e., $J(\mathscr{D}_p^{\perp}) \subset T_pM^{\perp}$ for each $p \in M$. A CR-submanifold is called proper if $\mathscr{D} \neq 0$ and $\mathscr{D}^{\perp} \neq 0$. It is known that every real hypersurface of an almost Hermitian manifold is an example of a proper CR-submanifold [1]. It is also known that the distribution $\mathscr{D}^{\perp}$ is always integrable if $\bar{M}$ is a Kaehler manifold [5].

On the other hand, Riemannian submersions between Riemannian manifolds were initiated by B. O'Neill [10]. The simplest example of a Riemannian submersion is the projection of a Riemannian product manifold on one of its factors. We note that a submersion gives two distributions on total manifold called horizontal and vertical distributions. It is important to mention that the vertical distribution of a Riemannian submersion is always integrable.

Keywords. CR-submanifold, Almost Hermitian manifold, Kaehler manifold, Distribution, Horizontally conformal submersion, Harmonic map.

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S. Kobayashi observed the above similarity between the Riemannian submersion and CR-submanifold, then he introduced the submersion of a CR-submanifold as follows: Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$ and N be an almost Hermitian manifold. Kobayashi considered the submersion $\pi : M \to N$ such that

(i) $\mathscr{D}^{\perp}$ is a kernel of $d\pi$, where $d\pi$ is the derivative map.

(ii) $d\pi : \mathscr{D}_p \to T_{\pi(p)}N$ is complex isometry.

(iii) J interchanges $\mathscr{D}^{\perp}$ and $TM^{\perp}$,

where $TM^{\perp}$ is the normal bundle of M. Then he showed that if $\overline{M}$ is a Kaehler manifold under these conditions, N is also a Kaehler manifold.

Let (M^m, g_M) and (N^n, g_N) be Riemannian manifolds. Suppose that $\varphi : (M^m, g_M) \to (N^n, g_N)$ is a smooth map between Riemannian manifolds and $p \in M$. Then, φ is called horizontally weakly conformal map at p [4] if either

(a) $d\varphi_p = 0$ or

(b) $d\varphi_p$ maps the horizontal space $\mathscr{H}_p = \{ker(d\varphi_p)\}^{\perp}$ conformally onto $T_{\varphi(p)}N$, i.e., $d\varphi_p$ is surjective and there exists a number $\Lambda(p) \neq 0$ such that

$$g_N(d\varphi_p(X), d\varphi_p(Y)) = \Lambda(p) g_M(X, Y), \quad X, Y \in \mathscr{H}_p. \tag{1.1}$$

If a point p is of type (a), then it is called critical point of φ. A point p of type (b) is called regular. The number $\Lambda(p)$ is called the square dilation, it is necessarily non-negative. Its square root $\lambda(p) = \sqrt{\Lambda(p)}$ is called the dilation. The map φ is called horizontally conformal submersion if φ has no critical point. Thus, it follows that a Riemannian submersion is a horizontally conformal submersion with dilation identically one. We note that a horizontally conformal submersion $\varphi : M \to N$ is said to be horizontally homothetic if the gradient of its dilation λ is vertical, i.e.,

$$\mathscr{H}(grad\ \lambda) = 0 \tag{1.2}$$

at $p \in M$, where $\mathscr{H}$ is the projection on the horizontal space $\mathscr{H} = \{ker(d\varphi_p)\}^{\perp}$.

We note that horizontally conformal maps were introduced independently by B. Fuglede [7] and T. Ishihara [8]. From the above discussion, one can conclude that the notion of horizontally conformal maps is a generalization of the concept of Riemannian submersions.

Comparing the definition of a Riemannian submersion and horizontally conformal submersion, it is natural to think that submersions of CR-submanifolds (in the sense of Kobayashi) may be generalized, using the concept of horizontally conformal submersion. Let M be a CR-submanifold of a Kaehler manifold $\overline{M}$ and N be an almost Hermitian manifold. In this paper, we consider horizontally conformal submersions of CR-submanifolds and prove that every horizontally homothetic submersion $\varphi : M \to N$ is a Riemannian submersion. Moreover we prove that if N is a Kaehler manifold, then every horizontally conformal submersion $\varphi : M \to N$ is also a Riemannian submersion. Furthermore, if M is mixed geodesic, then φ is a harmonic map.

2. Preliminaries

In this section, we give short information for some fundamental tensors associated to a distribution, harmonic maps and Gauss-Weingarten formulas for Riemannian submanifolds of Riemannian manifolds, for details, see: [4] and [11].

2.1. Fundamental tensors associated to a distribution and harmonic maps

We recall that a foliation F on a manifold is called the foliation associated to a submersion φ if, for the smooth submersion, the connected components of its fibres are the leaves of a smooth foliation. Let (M, g_M) be a Riemannian manifold and $\mathscr{V}$ be a q-dimensional distribution on M. Denote its orthogonal distribution $\mathscr{V}^{\perp}$ by $\mathscr{H}$. Then, we have

$$TM = \mathscr{V} \oplus \mathscr{H}. \tag{2.1}$$

$\mathscr{V}$ is called the vertical distribution and $\mathscr{H}$ is called the horizontal distribution. We use the same letters to denote the orthogonal projections onto these distributions.

By the unsymmetrized second fundamental form of $\mathscr{V}$, we mean the tensor field $A^{\mathscr{V}}$ defined by

$$A^{\mathscr{V}}_E F = \mathscr{H}(\nabla_{\mathscr{V}E}\mathscr{V}F), \quad E, F \in \Gamma(TM), \tag{2.2}$$

where ∇ is the Levi-Civita connection on M. The symmetrized second fundamental form $B^{\mathscr{V}}$ of $\mathscr{V}$ is given by

$$B^{\mathscr{V}}(E, F) = \frac{1}{2}\{A^{\mathscr{V}}_E F + A^{\mathscr{V}}_F E\} = \frac{1}{2}\{\mathscr{H}(\nabla_{\mathscr{V}E}\mathscr{V}F) + \mathscr{H}(\nabla_{\mathscr{V}F}\mathscr{V}E)\} \tag{2.3}$$

for any $E, F \in \Gamma(TM)$. The integrability tensor of $\mathscr{V}$ is the tensor field $I^{\mathscr{V}}$ given by

$$I^{\mathscr{V}}(E, F) = A^{\mathscr{V}}_E F - A^{\mathscr{V}}_F E - \mathscr{H}([\mathscr{V}E, \mathscr{V}F]). \tag{2.4}$$

Moreover, the mean curvature of $\mathscr{V}$ is defined by

$$\mu^{\mathscr{V}} = \frac{1}{q}\ Trace\ B^{\mathscr{V}} = \frac{1}{q}\sum_{i=1}^{q} \mathscr{H}(\nabla_{e_r} e_r), \tag{2.5}$$

where $\{e_1, \dots, e_q\}$ is a local frame of $\mathscr{V}$. By reversing the roles of $\mathscr{V}$, $\mathscr{H}$, $B^{\mathscr{H}}$, $A^{\mathscr{H}}$ and $I^{\mathscr{H}}$ can be defined similarly. For instance, $B^{\mathscr{H}}$ is defined by

$$B^{\mathscr{H}}(E, F) = \frac{1}{2}\{\mathscr{V}(\nabla_{\mathscr{H}E}\mathscr{H}F) + \mathscr{V}(\nabla_{\mathscr{H}F}\mathscr{H}E)\} \tag{2.6}$$

and, hence we have

$$\mu^{\mathscr{H}} = \frac{1}{m-q}\ Trace\ B^{\mathscr{H}} = \frac{1}{m-q}\sum_{s=1}^{m-q} \mathscr{V}(\nabla_{E_s} E_s), \tag{2.7}$$

where $E_1, \dots, E_{m-q}$ is a local frame of $\mathscr{H}$.

Notice that if $\varphi : M \to N$ is a horizontally conformal submersion, the $\mathscr{V}$ is the foliation associated to φ and it is integrable. So $I^{\mathscr{V}} = 0$ and $A^{\mathscr{V}}$ is symmetric. Moreover, we have the following:

LEMMA 2.1 [4]. *Let $\varphi : M \to N$ be a horizontally conformal submersion between Riemannian manifolds. Denote its dilation by $\lambda : M \to (0, \infty)$. Then for the associated foliation F:*

(i) *The horizontal distribution has mean curvature*

$$\mu^{\mathscr{H}} = \mathscr{V}(grad \ln \lambda) = \frac{1}{2}\mathscr{V}(grad \ln|d\varphi|^2). \tag{2.8}$$

(ii) *If φ is horizontally homothetic, i.e. $\mathscr{H}(grad \ln \lambda) = 0$, then*

$$\mu^{\mathscr{H}} = grad \ln \lambda = \frac{1}{2}(grad \ln|d\varphi|^2). \tag{2.9}$$

Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \to N$ is a smooth mapping between them. Then the differential $d\varphi$ of φ can be viewed a section of the bundle $Hom(TM, \varphi^{-1}TN) \to M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $(\varphi^{-1}TN)_p = T_{\varphi(p)}N$, $p \in M$. $Hom(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$\nabla d\varphi(X, Y) = \nabla^{\varphi}_X d\varphi(Y) - d\varphi(\nabla^M_X Y) \tag{2.10}$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M, g_M) \to (N, g_N)$ is said to be harmonic if *trace* $\nabla\, d\varphi = 0$. On the other hand, the tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = div\ d\varphi = \sum_{i=1}^{m} \nabla\, d\varphi(e_i, e_i), \tag{2.11}$$

where $\{e_1, \dots, e_m\}$ is the orthonormal frame on M. Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$. For a horizontally conformal submersion, we have the following:

LEMMA 2.2 [4]. *Let $\varphi : (M^m, g_M) \to (N^n, g_N)$ be a smooth horizontally conformal submersion between Riemannian manifolds. Let $\lambda : M \to (0, \infty)$ denote the dilation of φ and $\mu^{\mathscr{V}}$ the mean curvature vector field of its fibres. Then the tension field of φ is given by*

$$\tau(\varphi) = -(n-2)\, d\varphi(grad \ln \lambda) - (m-n)\, d\varphi(\mu^{\mathscr{V}}). \tag{2.12}$$

2.2. Riemannian submanifolds

Let $(\overline{M}, g)$ be an almost Hermitian manifold [11]. This means that $\overline{M}$ admits a tensor field J of type $(1,1)$ on $\overline{M}$ such that, $\forall X, Y \in \Gamma(T\overline{M})$, we have

$$J^2 = -I, \quad g(X, Y) = g(JX, JY), \tag{2.13}$$

where g is the Riemannian metric. If J is parallel with respect the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$, i.e.,

$$(\overline{\nabla}_X J)Y = 0 \tag{2.14}$$

then $\overline{M}$ is called a Kaehler manifold.

Let M be a Riemannian manifold isometrically immersed in $\overline{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(TM^{\perp})$ the set of all vector fields normal to M. Denote by ∇ the Levi-Civita connection of M. Then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.15}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.16}$$

for any $X, Y \in \Gamma(TM)$ and any $N \in \Gamma(TM^{\perp})$, where $\nabla^{\perp}$ is the connection in the normal bundle $TM^{\perp}$, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form and the shape operator A are related by

$$g(A_N X, Y) = g(h(X, Y), N). \tag{2.17}$$

We recall from [1], a CR-submanifold M is called mixed totally geodesic if h satisfies $h(X, Z) = 0$, for $X \in \Gamma(\mathscr{D})$ and $Z \in \Gamma(\mathscr{D}^{\perp})$.

3. Horizontally conformal submersions of CR-submanifolds

Let M be a CR-submanifold of a Kaehler manifold $(\overline{M}, \bar{g}_{\overline{M}}, J)$ and (N, g_N, J^*) be an almost Hermitian manifold. Let $\varphi : M \to N$ be a horizontally conformal submersion such that

$$\mathscr{D}^{\perp} = Ker\, d\varphi, \quad \mathscr{D} = \mathscr{H}, \quad J(\mathscr{D}^{\perp}) = TM^{\perp} \quad \text{and} \quad J^* o\, d\varphi = d\varphi o J. \tag{3.1}$$

It is obvious that φ is a generalization of submersions of CR-submanifolds (in the sense of Kobayashi). In this section, we show that there are some restrictions for φ.

Theorem 3.1. *Let M be a CR-submanifold of a Kaehler manifold $\overline{M}$ and N be an almost Hermitian manifold. Suppose that $\varphi : M \to N$ is a horizontally homothetic submersion under the assumptions in* (3.1). *Then φ is a Riemannian submersion up to a scale. Moreover, φ is a harmonic map if M is mixed geodesic.*

Proof. From (1.2) we have $\mathscr{H}(grad \ln \lambda) = 0$. On the other hand, from (2.6) we get

$$g(B^{\mathscr{H}}(X, X), V) = g(\nabla_X X, V)$$

for any $X \in \Gamma(\mathscr{D})$ and $V \in \Gamma(\mathscr{D}^{\perp})$, where g is the Riemannian metric on M induced from $\bar{g}_{\overline{M}}$ and ∇ is the Levi-Civita connection on M induced from the Levi-Civita connection $\overline{\nabla}$ of $\overline{M}$. Using (2.15), we obtain $g(B^{\mathscr{H}}(X,X),V) = g(\overline{\nabla}_X X, V)$. Thus, from (2.13) and (2.14), we have

$$g(B^{\mathscr{H}}(X,X),V) = g(\overline{\nabla}_X JX, JV).$$

Hence, we arrive at

$$g(B^{\mathscr{H}}(X,X),V) = g([X,JX] + \overline{\nabla}_{JX} X, JV).$$

Since $[X,JX] \in \Gamma(TM)$ and $JV \in \Gamma(TM^{\perp})$, we derive

$$g(B^{\mathscr{H}}(X,X),V) = g(\overline{\nabla}_{JX} X, JV).$$

Using again (2.13) and (2.14), we get

$$g(B^{\mathscr{H}}(X,X),V) = -g(\overline{\nabla}_{JX} JX, V).$$

Then, (2.15) implies that

$$g(B^{\mathscr{H}}(X,X),V) = -g(\nabla_{JX} JX, V).$$

for $X \in \Gamma(\mathscr{D})$ and $V \in \Gamma(\mathscr{D}^{\perp})$. Hence, we have

$$g(B^{\mathscr{H}}(X,X),V) = -g(B^{\mathscr{H}}(JX,JX),V). \tag{3.2}$$

Since $\mathscr{D}$ is an almost complex distribution, we choose a local orthonormal $\{E_1,\ldots,E_s,F_1,\ldots,F_s\}$ for $\mathscr{D}$ with $JE_i = F_i$. Hence, using (2.7), we can write

$$g(\mu^{\mathscr{H}},V) = \frac{1}{2s}\sum_{i=1}^{s} g(B^{\mathscr{H}}(E_i,E_i),V) + g(B^{\mathscr{H}}(F_i,F_i),V).$$

Then, from (3.2) we obtain

$$g(\mu^{\mathscr{H}},V) = \frac{1}{2s}\sum_{i=1}^{s} g(B^{\mathscr{H}}(E_i,E_i),V) - g(B^{\mathscr{H}}(E_i,E_i),V) = 0$$

which implies that

$$\mu^{\mathscr{H}} = 0. \tag{3.3}$$

Then, considering (2.9) and (3.3) we conclude that $grad \ln \lambda = 0$. Hence, it follows that λ is a constant on M. Thus, φ is a Riemannian submersion up to scale. On the other hand, from (2.2) and (2.3), we have

$$g(B^{\mathscr{V}}(Z,Z),X) = g(\nabla_Z Z, X).$$

for $Z \in \Gamma(\mathscr{D}^{\perp})$ and $X \in \Gamma(\mathscr{D})$. Then, using (2.13) and (2.14) we obtain

$$g(B^{\mathscr{V}}(Z,Z),X) = g(\overline{\nabla}_Z JZ, JX).$$

Thus, from (2.16), we get

$$g(B^{\mathscr{V}}(Z,Z),X) = -g(A_{JZ}Z, JX).$$

Hence, using (2.17), we arrive at

$$g(B^{\mathscr{V}}(Z,Z),X) = -g(h(Z,JX),JZ). \tag{3.4}$$

Then, since M is mixed geodesic, from (2.5) and (3.4), we have

$$g(\mu^{\mathscr{V}},X) = -\frac{1}{q}\sum_{j=1}^{q} g(h(e_j,JX),Je_j) = 0. \tag{3.5}$$

Then, the harmonicity of φ follows from Lemma 2.2, (1.2) and (3.5). Hence, proof is complete.

If N is a Kaehler manifold, we have the following result.

Theorem 3.2. *Let M be a CR-submanifold of a Kaehler manifold $\overline{M}$ and N be a Kaehler manifold. Suppose that $\varphi : M \to N$ is a horizontally conformal submersion under the assumptions in* (3.1). *Then φ is a Riemannian submersion up to a scale. Moreover, φ is a harmonic map if M is mixed geodesic.*

Proof. From ([4], Lemma 4.5.1, page: 119), we have

$$\nabla\, d\varphi(X,Y) = X(\ln\lambda)\, d\varphi(Y) + Y(\ln\lambda)\, d\varphi(X) - g(X,Y)\, d\varphi(grad \ln\lambda) \tag{3.6}$$

for $X, Y \in \Gamma(\mathscr{D})$. On the other hand, from (2.10), we have

$$\nabla\, d\varphi(X,JY) = \nabla^{\varphi}_X\, d\varphi(JY) - d\varphi(\nabla_X JY) = \nabla^{\varphi}_X\, d\varphi(JY) - d\varphi(\overline{\nabla}_X JY - h(X,JY))$$

for $X, Y \in \Gamma(\mathscr{D})$. Since $\overline{M}$ is a Kaehler manifold, we get

$$\begin{aligned}\nabla\, d\varphi(X,JY) &= \nabla^{\varphi}_X\, d\varphi(JY) - d\varphi(J\overline{\nabla}_X Y)\\ &= \nabla^{N}_{d\varphi(X)}\, d\varphi(JY) - d\varphi(J\mathscr{H}\nabla_X Y).\end{aligned}$$

Since φ is holomorphic, i.e., $J^* o\, d\varphi = d\varphi o J$, we have

$$\nabla\, d\varphi(X,JY) = \nabla^{N}_{d\varphi(X)} J^*\, d\varphi(Y) - J^*\, d\varphi(\nabla_X Y).$$

Then, Kaehler N implies that

$$\nabla\, d\varphi(X,JY) = J^* \nabla^{N}_{d\varphi(X)}\, d\varphi(Y) - J^*\, d\varphi(\nabla_X Y).$$

Hence

$$\nabla\, d\varphi(X,JY) = J^* \nabla\, d\varphi(X,Y).$$

Using symmetry of $\nabla\, d\varphi$, we derive

$$\nabla\, d\varphi(JX,JX) = -\nabla\, d\varphi(X,X). \tag{3.7}$$

Then, from (3.7), we obtain

$$\begin{aligned} Trace^{H}\, \nabla\, d\varphi &= \sum_{j=1}^{2s} \nabla\, d\varphi(E_j,E_j)\\ &= \sum_{j=1}^{s} \nabla\, d\varphi(E_j,E_j) + \nabla\, d\varphi(JE_j,JE_j) = 0.\end{aligned}$$

Hence

$$Trace^H \nabla d\varphi = 0. \tag{3.8}$$

On the other hand, from (3.6), we obtain

$$Trace^H \nabla d\varphi = -2(p-1)\, d\varphi(grad \ln \lambda). \tag{3.9}$$

Then, (3.8) and (3.9) imply that

$$d\varphi(grad \ln \lambda) = 0$$

which shows that $grad \ln \lambda$ is vertical, i.e., φ is horizontally homothetic. Then, proof follows from Theorem 3.1.

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