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Correlation functions of the shifted Schur measure

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Abstract. The shifted Schur measure introduced in [**TW2**] is a measure on the set of all strict partitions, which is defined by Schur Q-functions. The main aim of this paper is to calculate the correlation function of this measure, which is given by a pfaffian. As an application, we prove that a limit distribution of parts of partitions with respect to a shifted version of the Plancherel measure for symmetric groups is identical with the corresponding distribution of the original Plancherel measure. In particular, we obtain a limit distribution of the length of the longest ascent pair for a random permutation. Further we give expressions of the mean value and the variance of the size of partitions with respect to the measure defined by Hall-Littlewood functions.

1. Introduction.

Let π be a permutation in the symmetric group $\mathfrak{S}_N$ and $\ell(\pi)$ the length of the longest increasing subsequence in π. Concerning a limit distribution of $\ell(\pi)$ with respect to the uniform measure $\mathrm{P}_{\mathrm{uniform},N}$ on $\mathfrak{S}_N$, it is proved in [**BDJ**] that

$$\lim_{N\to\infty} \mathrm{P}_{\mathrm{uniform},N}\left(\frac{\ell(\pi)-2\sqrt{N}}{N^{1/6}}<s\right)=F_2(s), \tag{1.1}$$

where $F_2(s)$ is the Tracy-Widom distribution. The Tracy-Widom distribution is defined by the Fredholm determinant for the Airy kernel. Namely, let $\mathrm{Ai}(x)$ be the Airy function

$$\mathrm{Ai}(x)=\frac{1}{2\pi\sqrt{-1}}\int_{\infty e^{-\pi\sqrt{-1}/3}}^{\infty e^{\pi\sqrt{-1}/3}} e^{z^3/3-xz}\mathrm{d}z \tag{1.2}$$

and $K_{\mathrm{Airy}}(x,y)$ the Airy kernel

$$K_{\mathrm{Airy}}(x,y)=\int_0^\infty \mathrm{Ai}(x+z)\mathrm{Ai}(z+y)\mathrm{d}z. \tag{1.3}$$

Then the Tracy-Widom distribution $F_2(s)$ is defined by

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$$\begin{aligned} F_2(s) &= \det(I - K_{\mathrm{Airy}})|_{L^2([s,\infty))} \qquad (1.4)\\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{[s,\infty)^k} \det(K_{\mathrm{Airy}}(x_i, x_j))_{1\le i,j\le k} \mathrm{d}x_1 \dots \mathrm{d}x_k \end{aligned}$$

and gives a limit distribution of the scaled largest eigenvalue of a Hermitian matrix from the Gaussian Unitary Ensemble (GUE), see [**TW1**].

As we see below, the Plancherel measure for partitions is related to the distribution of the length $\ell(\pi)$. Let f^λ be the number of standard tableaux of shape λ. The Plancherel measure assigns to each $\lambda \vdash N$ the probability

$$\mathrm{P}_{\mathrm{Plan},N}(\{\lambda\}) = \frac{(f^\lambda)^2}{N!}. \qquad (1.5)$$

Then it follows from the Robinson-Schensted correspondence that

$$\mathrm{P}_{\mathrm{uniform},N}(\{\pi \in \mathfrak{S}_N | \ell(\pi) = h\}) = \mathrm{P}_{\mathrm{Plan},N}(\{\lambda \in \mathscr{P}_N | \lambda_1 = h\}), \qquad (1.6)$$

where $\mathscr{P}_N$ is the set of all partitions of N (see e.g. [**S**]). Hence the equation (1.1) also describes a limit distribution of λ_1 with respect to Plancherel measures. This result has been extended in [**BOO**], [**J3**], [**O1**] to the other rows λ_j's in a general position of a partition. The key of the proof in [**BOO**] is a calculation of *correlation functions* of the poissonization of the Plancherel measures. We can see the other asymptotics with respect to the Plancherel measure in e.g. [**Ho**].

On the other hand, the Schur measure introduced in [**O2**] is a measure which assigns to each partition the product of two Schur functions. Okounkov [**O2**] calculated the correlation function of the Schur measure by using the infinite wedge. The correlation function of the poissonized Plancherel measure is obtained as a specialization of the one for the Schur measure.

The main aim of this paper is to calculate the correlation function of the shifted Schur measure (see Theorem 3.1). The shifted Schur measure, introduced in [**TW2**], is a measure on the set of all strict partitions, which is defined by Schur Q-functions instead of Schur functions. The correlation function is expressed as a pfaffian and is actually calculated by operators on the exterior algebra in place of the infinite wedge in [**O2**]. Further, as an application, we obtain a shifted version of the corresponding result for a limit distribution of λ_j's in [**BOO**], [**J3**], [**O1**] (see Theorem 4.1). In particular, we find that a limit distribution of the length of the longest ascent pair for a random permutation is given by the Tracy-Widom distribution (see Corollary 4.2). Since the proof is similar to the one in [**BOO**], we only discuss its main point.

In the final section, we study about a measure defined by Hall-Littlewood functions. The measure is considered as a natural extension of the Schur measure and the shifted Schur measure. We obtain expressions of the mean value $\boldsymbol{E}(|\lambda|)$ and the variance $\mathrm{Var}(|\lambda|)$ of the size $|\lambda|$ with respect to this measure explicitly. Actually, each value is written as a sum of the product of certain power-sum functions (see Theorem 5.1). This expression of $\boldsymbol{E}(|\lambda|)$ naturally leads us a similar study of $\boldsymbol{E}(\lambda_1)$. By observing various examples, in the end of the section, we remark that there is a certain common property of expressions of $\boldsymbol{E}(\lambda_1)$ among these examples.

2. Shifted Schur measures.

We recall the Schur Q-function and the shifted Schur measure. The following facts are known in [**Mac**, III-8] and [**TW2**].

A non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers is called a partition of N if the size $|\lambda| := \sum_{j \geq 1} \lambda_j$ equals N. We denote the number of non-zero parts of λ by $\ell(\lambda)$ and we call it the length of λ. A partition λ is called *strict* if and only if all parts of λ are distinct and then we write $\lambda \vDash N$. Let $\mathscr{D}_N$ be the set of all strict partitions of N and $\mathscr{D}$ the set of all strict partitions, i.e., $\mathscr{D} = \cup_{N=0}^{\infty} \mathscr{D}_N$.

Let $X = (X_1, X_2, \dots)$ and $Y = (Y_1, Y_2, \dots)$ be infinite many variables. The symmetric functions $q_n(X)$ $(n \geq 0)$ are defined via the generating function

$$Q(z) = Q_X(z) = \prod_{i=1}^{\infty} \frac{1 + X_i z}{1 - X_i z} = \sum_{n=0}^{\infty} q_n(X) z^n.$$

In particular, we have $q_0 = 1$. Since

$$\log \prod_{i=1}^{\infty} \frac{1 + X_i z}{1 - X_i z} = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} X_i^n z^n = \sum_{n=1,3,5,\dots} \frac{2}{n} p_n(X) z^n,$$

where $p_n(X) = \sum_{i=1}^{\infty} X_i^n$ is the power-sum function, the function $Q(z)$ is also expressed as

$$Q(z) = \exp\left(\sum_{n=1,3,5,\dots} \frac{2}{n} p_n(X) z^n \right). \tag{2.1}$$

For $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{D}$ of length $\leq m$, the Schur Q-function $Q_\lambda(X)$ is defined as the coefficient of $z^\lambda = z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_m^{\lambda_m}$ in

$$Q(z_1, z_2, \dots, z_m) = \prod_{i=1}^{m} Q(z_i) \prod_{1 \leq i < j \leq m} \frac{z_i - z_j}{z_i + z_j}. \tag{2.2}$$

For $r > s \geq 0$, we define

$$Q_{(r,s)} = q_r q_s + 2 \sum_{i=1}^{s} (-1)^i q_{r+i} q_{s-i}$$

and $Q_{(r,s)} = -Q_{(s,r)}$ for $r \leq s$. We may write λ in the form $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2n})$ where $\lambda_1 > \lambda_2 > \cdots > \lambda_{2n} \geq 0$. Then the $2n \times 2n$ matrix

$$M_\lambda = (Q_{(\lambda_i,\lambda_j)})_{1\le i,j\le 2n}$$

is skew symmetric, and the Schur Q-function Q_λ is also given by

$$Q_\lambda = \mathrm{Pf}(M_\lambda), \tag{2.3}$$

where Pf stands for the pfaffian. The Schur P-function P_λ is defined by $P_\lambda = 2^{-\ell(\lambda)}Q_\lambda$.

The *shifted Schur measure* is a (formal) probability measure on $\mathscr{D}$ defined by

$$\mathrm{P}_{\mathrm{SS}}(\{\lambda\}) = \frac{1}{Z_{\mathrm{SS}}}Q_\lambda(X)P_\lambda(Y) \tag{2.4}$$

for each $\lambda \in \mathscr{D}$. Here the normalization constant Z_{SS} is determined by

$$Z_{\mathrm{SS}} = \sum_{\lambda\in\mathscr{D}} Q_\lambda(X)P_\lambda(Y) = \prod_{i,j=1}^{\infty} \frac{1+X_iY_j}{1-X_iY_j},$$

where the second equality is the Cauchy identity for Schur Q-functions ([**Mac**, p. 255]). Further, from (2.1), the constant Z_{SS} is also expressed as

$$Z_{\mathrm{SS}} = \exp\left(\sum_{n=1,3,5,\dots} \frac{2}{n}p_n(X)p_n(Y)\right). \tag{2.5}$$

3. Correlation functions of the shifted Schur measure.

In this section, we prove the main theorem. We identify each strict partition $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_\ell)$ $(\lambda_1>\lambda_2>\cdots>\lambda_\ell>0)$ with the finite set $\{\lambda_1,\lambda_2,\dots,\lambda_\ell\}$ of positive integers. Define the correlation function of the shifted Schur measure P_{SS} by

$$\rho_{\mathrm{SS}}(A) := \mathrm{P}_{\mathrm{SS}}(\{\lambda\in\mathscr{D} \mid \lambda\supset A\}) = \frac{1}{Z_{\mathrm{SS}}}\sum_{\lambda\supset A} Q_\lambda(X)P_\lambda(Y) \tag{3.1}$$

for a finite subset $A \subset \boldsymbol{Z}_{>0}$. The function $\rho_{\mathrm{SS}}(A)$ has a pfaffian expression.

Theorem 3.1. *For a finite subset $A = \{k_1,\dots,k_N\} \subset \boldsymbol{Z}_{>0}$, we have*

$$\rho_{\mathrm{SS}}(A) = \mathrm{Pf}(M(A)_{i,j})_{1\le i<j\le 2N}, \tag{3.2}$$

where the entry $M(A)_{i,j}$ of the skew symmetric matrix $M(A)$ is given by

$$M(A)_{i,j} = \begin{cases} \boldsymbol{K}(k_i,k_j), & \text{for} \quad 1\le i<j\le N, \\ \boldsymbol{K}(k_i,-k_{2N-j+1}), & \text{for} \quad 1\le i\le N<j\le 2N, \\ \boldsymbol{K}(-k_{2N-i+1},-k_{2N-j+1}), & \text{for} \quad N<i<j\le 2N, \end{cases}$$

and $\boldsymbol{K}(u,v)$ *is defined as* $\epsilon(u,v)$ *times the coefficient of* $z^u w^v$ *in the formal series*

$$\frac{1}{2}\boldsymbol{J}(z;X,Y)\boldsymbol{J}(w;X,Y)\frac{z-w}{z+w}.$$

Here $\boldsymbol{J}(z;X,Y)$ *is defined by*

$$\boldsymbol{J}(z;X,Y) := Q_X(z)Q_Y(-z^{-1}) = \prod_{i=1}^{\infty}\frac{1+X_i z}{1-X_i z}\frac{1-Y_i z^{-1}}{1+Y_i z^{-1}} \tag{3.3}$$

and $\epsilon(u,v)$ *is given by*

$$\epsilon(u,v) = \begin{cases} 1, & \textit{for} \quad u,v>0, \\ (-1)^v, & \textit{for} \quad u>0, v<0, \\ (-1)^{u+v}, & \textit{for} \quad u,v<0. \end{cases} \tag{3.4}$$

REMARK 3.1. The correlation function of the Schur measure is given by a determinant, see Theorem 1 in [**O2**].

We prove Theorem 3.1 by employing the exterior algebra. Let V be a module on $\boldsymbol{Z}[X_1, X_2, \dots, Y_1, Y_2, \dots]$ spanned by $\boldsymbol{e}_k$ $(k = 1, 2, \dots)$. The exterior algebra $\bigwedge V$ is spanned by vectors

$$\boldsymbol{v}_\lambda = \boldsymbol{e}_{\lambda_1} \wedge \boldsymbol{e}_{\lambda_2} \wedge \cdots \wedge \boldsymbol{e}_{\lambda_\ell},$$

where $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathscr{D}$ $(\lambda_1 > \cdots > \lambda_\ell \geq 1)$. In particular, we have $\boldsymbol{v}_\varnothing = 1$. We give $\bigwedge V$ the inner product

$$\langle \boldsymbol{v}_\lambda, \boldsymbol{v}_\mu \rangle = \delta_{\lambda,\mu} 2^{-\ell(\lambda)}.$$

Putting $\boldsymbol{e}_k^\vee = 2\boldsymbol{e}_k$ and $\boldsymbol{v}_\lambda^\vee = \boldsymbol{e}_{\lambda_1}^\vee \wedge \cdots \wedge \boldsymbol{e}_{\lambda_\ell}^\vee = 2^\ell \boldsymbol{v}_\lambda$, the bases $(\boldsymbol{v}_\lambda)_{\lambda\in\mathscr{D}}$ and $(\boldsymbol{v}_\lambda^\vee)_{\lambda\in\mathscr{D}}$ are dual to each other.

We define the operator ψ_k $(k \geq 1)$ on $\bigwedge V$ by

$$\psi_k \boldsymbol{v}_\lambda = \boldsymbol{e}_k \wedge \boldsymbol{v}_\lambda$$

and let ψ_k^* be the adjoint operator of ψ_k with respect to the inner product defined above. The operator ψ_k^* is then explicitly given by

$$\psi_k^* \boldsymbol{v}_\lambda = \sum_{i=1}^{\ell(\lambda)} \frac{(-1)^{i-1}}{2} \delta_{k,\lambda_i} \boldsymbol{e}_{\lambda_1} \wedge \cdots \wedge \widehat{\boldsymbol{e}_{\lambda_i}} \wedge \cdots \wedge \boldsymbol{e}_{\lambda_\ell}.$$

These operators satisfy the following commutation relations

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{i,j} \tfrac{1}{2}, \quad \psi_i \psi_j = -\psi_j \psi_i, \quad \psi_i^* \psi_j^* = -\psi_j^* \psi_i^*. \tag{3.5}$$

Since

$$\psi_k \psi_k^* \boldsymbol{v}_\lambda = \begin{cases} \frac{1}{2} \boldsymbol{v}_\lambda, & \text{if } k \in \lambda, \\ 0, & \text{otherwise,} \end{cases} \tag{3.6}$$

we see that $\left(\prod_{k \in A} 2\psi_k \psi_k^*\right) \boldsymbol{v}_\lambda$ is equal to $\boldsymbol{v}_\lambda$ if $A \subset \lambda$ and to 0 otherwise.

Define the self-adjoint operator S by $S\boldsymbol{v}_\lambda = (-1)^{\ell(\lambda)} \boldsymbol{v}_\lambda$ for any $\lambda \in \mathscr{D}$. The operators satisfy the relations

$$S^2 = 1, \quad \psi_k S = -S\psi_k, \quad \psi_k^* S = -S\psi_k^*. \tag{3.7}$$

For each odd positive integer n, we define the operators α_n and α_{-n} by

$$\alpha_n := 2 \sum_{j=1}^{\infty} \psi_j \psi_{n+j}^* + S\psi_n^* + 2 \sum_{j=1}^{\frac{n-1}{2}} (-1)^j \psi_j^* \psi_{n-j}^*,$$

$$\alpha_{-n} := \alpha_n^* = 2 \sum_{j=1}^{\infty} \psi_{n+j} \psi_j^* + \psi_n S + 2 \sum_{j=1}^{\frac{n-1}{2}} (-1)^j \psi_{n-j} \psi_j.$$

It follows from (3.5) and (3.7) that

$$[\alpha_n, \alpha_m] = \frac{n}{2} \delta_{n,-m} \tag{3.8}$$

for any odd integers n and m, where $[\ ,\ ]$ is the commutator; $[a,b] = ab - ba$.

If we put

$$\widetilde{\psi}_k = \begin{cases} \psi_k, & \text{for} \quad k \geq 1, \\ \dfrac{S}{2}, & \text{for} \quad k = 0, \\ (-1)^k \psi_{-k}^*, & \text{for} \quad k \leq -1, \end{cases} \tag{3.9}$$

and $\psi(z) = \sum_{k \in \boldsymbol{Z}} z^k \widetilde{\psi}_k$, then by (3.5) and (3.7) we see that

$$[\alpha_n, \psi(z)] = z^n \psi(z) \quad \text{for any odd integer } n \tag{3.10}$$

and

$$\langle \widetilde{\psi}_k \widetilde{\psi}_l \boldsymbol{v}_\varnothing, \boldsymbol{v}_\varnothing \rangle = 0 \qquad \text{unless} \qquad l = -k \geq 0. \tag{3.11}$$

It follows from (3.11) that

$$\langle \psi(z)\psi(w)\boldsymbol{v}_\varnothing, \boldsymbol{v}_\varnothing \rangle = \left\langle \left(\frac{S^2}{4} + \sum_{k\geq 1} (-1)^k z^{-k} w^k \psi_k^* \psi_k \right) \boldsymbol{v}_\varnothing, \boldsymbol{v}_\varnothing \right\rangle \tag{3.12}$$

$$= \frac{1}{4} + \sum_{k\geq 1} \frac{1}{2} \left(-\frac{w}{z} \right)^k = \frac{z-w}{4(z+w)}.$$

Note that the operator α_n is expressed as $\alpha_n = \sum_{k \in \boldsymbol{Z}} (-1)^k \widetilde{\psi}_{k-n} \widetilde{\psi}_{-k}$.

Put

$$\Gamma_\pm(X) = \exp\left(\sum_{n=1,3,5,\ldots} \frac{2p_n(X)}{n} \alpha_{\pm n} \right).$$

Observe that

$$\Gamma_+ \boldsymbol{v}_\varnothing = \boldsymbol{v}_\varnothing, \tag{3.13}$$

$$\Gamma_\pm^* = \Gamma_\mp, \tag{3.14}$$

$$\Gamma_+(X)\Gamma_-(Y) = Z_{\mathrm{SS}} \Gamma_-(Y)\Gamma_+(X). \tag{3.15}$$

The equality (3.15) is obtained from (3.8) and (2.5). By (2.1) and (3.10), we have

$$\Gamma_\pm(X)\psi(z) = Q_X(z^{\pm 1})\psi(z)\Gamma_\pm(X). \tag{3.16}$$

The Schur Q-function is given as a matrix element of Γ_- as follows.

PROPOSITION 3.2. *For each $\lambda \in \mathscr{D}$, we have*

$$\langle \Gamma_-(X)\boldsymbol{v}_\varnothing, \boldsymbol{v}_\lambda^\vee \rangle = Q_\lambda(X). \tag{3.17}$$

More generally, for $\lambda, \mu \in \mathscr{D}$,

$$\langle \Gamma_-(X)\boldsymbol{v}_\mu, \boldsymbol{v}_\lambda^\vee \rangle = Q_{\lambda/\mu}(X), \tag{3.18}$$

where $Q_{\lambda/\mu}(X)$ is a skew Schur Q-function.

PROOF. Write λ in the form $\lambda_1 > \lambda_2 > \cdots > \lambda_{2n} \geq 0$. Since $\boldsymbol{v}_\lambda^\vee = 2^{2n} \widetilde{\psi}_{\lambda_1} \cdots \widetilde{\psi}_{\lambda_{2n}} \boldsymbol{v}_\varnothing$, the left hand side in (3.17) is equal to the coefficient of $z_1^{\lambda_1} \cdots z_{2n}^{\lambda_{2n}}$ in the expansion of

$$2^{2n} \langle \Gamma_-(X)\boldsymbol{v}_\varnothing, \psi(z_1) \cdots \psi(z_{2n})\boldsymbol{v}_\varnothing \rangle. \tag{3.19}$$

It follows from (3.13), (3.14) and (3.16) that (3.19) equals

$$2^{2n}Q(z_1)\cdots Q(z_{2n})\langle\psi(z_1)\cdots\psi(z_{2n})\boldsymbol{v}_\varnothing,\boldsymbol{v}_\varnothing\rangle.$$

By (2.2), in order to prove (3.17) it is sufficient to show

$$2^{2n}\langle\psi(z_1)\cdots\psi(z_{2n})\boldsymbol{v}_\varnothing,\boldsymbol{v}_\varnothing\rangle=\mathrm{Pf}\left(\frac{z_i-z_j}{z_i+z_j}\right)=\prod_{1\le i<j\le 2n}\frac{z_i-z_j}{z_i+z_j}.$$

Note the second equality is well-known (see e.g. [**Mac**, III-8, Ex.5]). From (3.5), (3.7) and (3.11), we see that

$$\begin{aligned}&2^{2n}\langle\psi(z_1)\cdots\psi(z_{2n})\boldsymbol{v}_\varnothing,\boldsymbol{v}_\varnothing\rangle\\&=\sum_{k=2}^{2n}(-1)^k4\langle\psi(z_1)\psi(z_k)\boldsymbol{v}_\varnothing,\boldsymbol{v}_\varnothing\rangle4^{n-1}\langle\psi(z_2)\cdots\widehat{\psi(z_k)}\cdots\psi(z_{2n})\boldsymbol{v}_\varnothing,\boldsymbol{v}_\varnothing\rangle.\end{aligned}$$

Therefore, by the expansion formula of a pfaffian, we obtain

$$2^{2n}\langle\psi(z_1)\cdots\psi(z_{2n})\boldsymbol{v}_\varnothing,\boldsymbol{v}_\varnothing\rangle=\mathrm{Pf}(4\langle\psi(z_i)\psi(z_j)\boldsymbol{v}_\varnothing,\boldsymbol{v}_\varnothing\rangle).$$

Hence the claim follows from (3.12). The generating function of $Q_{\lambda/\mu}$ in [**Mac**, III-8, Ex.9] yields the second formula (3.18) by a discussion similar to the above. □

From (3.6), (3.14) and (3.17), the correlation function is expressed as

$$\rho_{\mathrm{SS}}(A)=\frac{1}{Z_{\mathrm{SS}}}\sum_{\lambda\supset A}Q_\lambda(X)P_\lambda(Y)=\frac{1}{Z_{\mathrm{SS}}}\left\langle\varGamma_+(X)\left(\prod_{k\in A}2\psi_k\psi_k^*\right)\varGamma_-(Y)\boldsymbol{v}_\varnothing,\boldsymbol{v}_\varnothing\right\rangle.$$

It follows from (3.13), (3.14) and (3.15) that

$$\rho_{\mathrm{SS}}(A)=\left\langle\left(\prod_{k\in A}2\varPsi_k\varPsi_k^*\right)\boldsymbol{v}_\varnothing,\boldsymbol{v}_\varnothing\right\rangle,\tag{3.20}$$

where we put

$$\varPsi_k=\mathrm{Ad}(G)\psi_k,\quad \varPsi_k^*=\mathrm{Ad}(G)\psi_k^*,\quad G=\varGamma_+(X)\varGamma_-(Y)^{-1}.\tag{3.21}$$

Using (3.16), we have

$$\mathrm{Ad}(G)\psi(z)=\boldsymbol{J}(z;X,Y)\psi(z),\tag{3.22}$$

where $\boldsymbol{J}(z;X,Y)$ is defined in (3.3).

LEMMA 3.3. *We have*

$$\rho_{\mathrm{SS}}(A) = \mathrm{Pf}(\widetilde{M}(A)_{i,j})_{1 \le i < j \le 2N}.$$

Here the entry of the skew symmetric matrix $\widetilde{M}(A)$ *is given by*

$$\widetilde{M}(A)_{i,j} = \begin{cases} 2\langle \Psi_{k_i}\Psi_{k_j}\boldsymbol{v}_\varnothing, \boldsymbol{v}_\varnothing\rangle, & \text{for} \quad 1 \le i < j \le N, \\ 2\langle \Psi_{k_i}\Psi^*_{k_{2N-j+1}}\boldsymbol{v}_\varnothing, \boldsymbol{v}_\varnothing\rangle, & \text{for} \quad 1 \le i \le N < j \le 2N, \\ 2\langle \Psi^*_{k_{2N-i+1}}\Psi^*_{k_{2N-j+1}}\boldsymbol{v}_\varnothing, \boldsymbol{v}_\varnothing\rangle, & \text{for} \quad N < i < j \le 2N. \end{cases} \tag{3.23}$$

Proof. From (3.5) and (3.21), we have $\Psi_k\Psi_l^* = -\Psi_l^*\Psi_k$ $(k \ne l)$. Therefore we obtain

$$\left\langle \left(\prod_{k \in A} \Psi_k\Psi_k^* \right) \boldsymbol{v}_\varnothing, \boldsymbol{v}_\varnothing \right\rangle = \langle \Psi_{k_1}\Psi_{k_2}\cdots\Psi_{k_N}\Psi^*_{k_N}\cdots\Psi^*_{k_1}\boldsymbol{v}_\varnothing, \boldsymbol{v}_\varnothing\rangle.$$

By (3.22), the operator Ψ_j and Ψ_j^*, respectively, is expressed as a linear combination of $\widetilde{\psi}_n$'s over $\boldsymbol{Z}[X_1, X_2, \ldots, Y_1, Y_2, \ldots]$. Hence if we abbreviate $\widetilde{\Psi}_j = \Psi_{k_j}$ for $1 \le j \le N$ and $\widetilde{\Psi}_j = \Psi^*_{k_{2N-j+1}}$ for $N+1 \le j \le 2N$, we have

$$\langle \Psi_1\Psi_2\cdots\Psi_N\Psi_N^*\cdots\Psi_1^*\boldsymbol{v}_\varnothing, \boldsymbol{v}_\varnothing\rangle = \mathrm{Pf}(\langle \widetilde{\Psi}_i\widetilde{\Psi}_j\boldsymbol{v}_\varnothing, \boldsymbol{v}_\varnothing\rangle)_{1 \le i < j \le 2N}$$

by a discussion similar to the proof of Proposition 3.2. Thus, by (3.20), we obtain the lemma. □

Proof of Theorem 3.1. We compute entries in the right hand side of (3.23). It follows from (3.12) and (3.22) that

$$\langle 2\Psi(z)\Psi(w)\boldsymbol{v}_\varnothing, \boldsymbol{v}_\varnothing\rangle = \frac{1}{2}\boldsymbol{J}(z; X, Y)\boldsymbol{J}(w; X, Y)\frac{z-w}{z+w},$$

where $\Psi(z) = \mathrm{Ad}(G)\psi(z)$. Since the coefficient of z^k $(k \in \boldsymbol{Z} \setminus \{0\})$ in $\Psi(z)$ is equal to Ψ_k if $k > 0$ and to $(-1)^k\Psi^*_{-k}$ if $k < 0$, we can easily see the theorem from Lemma 3.3. □

Remark 3.2. Though Jing [**Ji**] obtains the expression of Schur Q-functions by vertex operators with the commutator relation (3.8) it seems very hard to obtain the result in Theorem 3.1 using these vertex operators.

4. Applications.

As an application of Theorem 3.1, we give a limit distribution of λ_j's with respect to a specialization of the shifted Schur measure.

4.1. A shifted version of the Plancherel measure.

We define a measure similar to the Plancherel measure on $\mathscr{P}_N$ by means of the shifted Robinson-Schensted-Knuth (RSK) correspondence (see e.g. [**HH**]).

A *shifted shape* $\mathrm{Sh}(\lambda)$ associated with a strict partition λ is obtained by replacing the i-th row to the right by $i-1$ boxes for $i \geq 1$ from the Young diagram λ. A *standard shifted tableau* T of the shifted shape $\lambda \vDash N$ is an assignment of $1, 2, \ldots, N$ to each box in the shifted shape $\mathrm{Sh}(\lambda)$ such that entries in T are increasing across rows and down columns. For example,

$$\begin{array}{cccc} 1 & 2 & 4 & 6 \\ & 3 & 5 & 8 \\ & & 7 & \end{array}$$

is a standard shifted tableau of shape $\lambda = (4, 3, 1)$.

Let g^λ be the number of standard shifted tableaux of shape λ. It is known that g^λ is explicitly given by

$$g^\lambda = \frac{|\lambda|!}{\lambda_1!\lambda_2!\cdots\lambda_\ell!} \prod_{1 \leq i < j \leq \ell} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}$$

(see e.g. [**Mac**, III-8, Ex.12]). By means of the shifted RSK we can see that

$$\sum_{\lambda \vDash N} 2^{N-\ell(\lambda)} (g^\lambda)^2 = N! \tag{4.1}$$

(see [**HH**]).

In view of the equality (4.1), we define a probability measure on $\mathscr{D}_N$, that is, we assign to each $\lambda \in \mathscr{D}_N$ the probability

$$\mathrm{P}_{\mathrm{SPl},N}(\{\lambda\}) = \frac{2^{N-\ell(\lambda)}}{N!}(g^\lambda)^2. \tag{4.2}$$

This measure, which is noted in [**TW2**], can be regarded as a shifted version of the Plancherel measure defined in (1.5) in a combinatorial sense.

4.2. Ascent pairs for a permutation.

The measure defined in (4.2) is related to the so-called ascent pair for a permutation. For $\pi = (\pi(1), \pi(2), \ldots, \pi(N)) \in \mathfrak{S}_N$, an *ascent pair* for π is a pair $(\phi^{\mathrm{de}}, \phi^{\mathrm{in}})$ of a decreasing subsequence $\phi^{\mathrm{de}} = (\pi(i_1) > \cdots > \pi(i_k))$, $i_1 < \cdots < i_k$ and an increasing subsequence $\phi^{\mathrm{in}} = (\pi(j_1) < \cdots < \pi(j_l))$, $j_1 < \cdots < j_l$ of π such that the sequence

$$(\pi(i_k), \ldots, \pi(i_1), \pi(j_1), \ldots, \pi(j_l))$$

is weakly increasing (i.e. the inequality $\pi(i_1) \leq \pi(j_1)$ is satisfied). We define the length of the ascent pair $(\phi^{\mathrm{de}}, \phi^{\mathrm{in}})$ by $k + l - 1$. Denote the length of the longest ascent pair for π by $L(\pi)$.

Example 4.1. For a permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 7 & 1 & 9 & 6 & 3 & 5 & 8 & 2 \end{pmatrix}$$

the pair $(\phi^{\mathrm{de}}, \phi^{\mathrm{in}})$, where $\phi^{\mathrm{de}} = (4,3,2)$ and $\phi^{\mathrm{in}} = (4,7,9)$, is the ascent pair with length 5. Since this is the longest ascent pair for π, we have $L(\pi) = 5$.

By the shifted RSK, the distribution of $L(\pi)$ with respect to the uniform measure on $\mathfrak{S}_N$ equals the distribution of λ_1 with respect to the measure $\mathrm{P}_{\mathrm{SPl},N}$ on $\mathscr{D}_N$, i.e.,

$$\mathrm{P}_{\mathrm{uniform},N}(\{\pi \in \mathfrak{S}_N | L(\pi) = h\}) = \mathrm{P}_{\mathrm{SPl},N}(\{\lambda \in \mathscr{D}_N | \lambda_1 = h\}). \tag{4.3}$$

4.3. Limit distributions.

We consider the random point process on $\boldsymbol{R}$ (see the Appendix in [**BOO**]) whose correlation functions $\rho_{\mathrm{Airy}}(X) = \mathrm{P}_{\mathrm{Airy}}(\{Y \subset \boldsymbol{R} \mid \#Y < \infty,\ X \subset Y\})$ for any finite subset $X = \{x_1, \dots, x_k\} \subset \boldsymbol{R}$ are given by $\rho_{\mathrm{Airy}}(X) = \det(K_{\mathrm{Airy}}(x_i, x_j))_{1 \le i,j \le k}$. Here K_{Airy} is the Airy kernel defined in (1.3). Let $\zeta = (\zeta_1 > \zeta_2 > \cdots) \in \boldsymbol{R}^\infty$ be its random configuration. The random variables ζ_i's are called the *Airy ensemble*. It is known that the Airy ensemble describes the behavior of the largest eigenvalue of a GUE matrix, the 2nd largest one, and so on, see [**TW1**].

Theorem 4 in [**BOO**] (see also [**J3**], [**O1**]) asserts that the random variables

$$\frac{\lambda_i - 2\sqrt{N}}{N^{1/6}}, \quad i = 1, 2, \dots, \quad \lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}_N \tag{4.4}$$

with respect to the Plancherel measure defined by (1.5) converge, in the joint distribution, to the Airy ensemble as $N \to \infty$. The following theorem is a shifted version of this result.

THEOREM 4.1. *As $N \to \infty$, the random variables*

$$\frac{\lambda_i - 2\sqrt{2N}}{(2N)^{1/6}}, \quad i = 1, 2, \dots \tag{4.5}$$

with respect to the measure $\mathrm{P}_{\mathrm{SPl},N}$ on $\mathscr{D}_N$ converge to the Airy ensemble, in joint distributions.

Compare (4.5) with (4.4). Especially, since the distribution of ζ_1 in the Airy ensemble is given by the Tracy-Widom distribution, we immediately see the following result from (4.3).

COROLLARY 4.2. *We have*

$$\lim_{N \to \infty} \mathrm{P}_{\mathrm{uniform},N}\left(\frac{L - 2\sqrt{2N}}{(2N)^{1/6}} < s \right) = F_2(s).$$

Compare with (1.1). Theorem 4.1 is proved by computing the correlation function of the so-called *poissonization* of the measure $\mathrm{P}_{\mathrm{SPl},N}$. For $\xi > 0$, we define the poissonization

$\mathrm{P}^{\xi}_{\mathrm{PSP}}$ of the measure $\mathrm{P}_{\mathrm{SPl},N}$ by

$$\mathrm{P}^{\xi}_{\mathrm{PSP}}(\{\lambda\}) = e^{-\xi}\sum_{N=0}^{\infty}\frac{\xi^N}{N!}\mathrm{P}_{\mathrm{SPl},N}(\{\lambda\}) = e^{-\xi}\xi^{|\lambda|}2^{|\lambda|-\ell(\lambda)}\left(\frac{g^{\lambda}}{|\lambda|!}\right)^2 \tag{4.6}$$

for $\lambda \in \mathscr{D}$. Here $\mathrm{P}_{\mathrm{SPl},N}(\{\lambda\}) = 0$ unless $\lambda \vDash N$. Then we have the

Theorem 4.3. *For any fixed $M \geq 1$ and any $a_1, \dots, a_M \in \boldsymbol{R}$ we have*

$$\begin{aligned}\lim_{\xi\to\infty}\mathrm{P}^{\xi}_{\mathrm{PSP}}&\left(\left\{\lambda \in \mathscr{D} \;\middle|\; \frac{\lambda_i - 2\sqrt{2\xi}}{(2\xi)^{\frac{1}{6}}} < a_i,\ 1 \leq i \leq M\right\}\right) \\ &= \mathrm{P}_{\mathrm{Airy}}(\zeta_i < a_i,\ 1 \leq i \leq M),\end{aligned} \tag{4.7}$$

where $\zeta_1 > \zeta_2 > \cdots$ is the Airy ensemble.

Since Theorem 4.1 can be proved from Theorem 4.3 by using the *depoissonization technique* developed in [**J1**], we omit the proof, see [**BOO**].

4.4. The proof of Theorem 4.3.

The measure $\mathrm{P}^{\xi}_{\mathrm{PSP}}$ can be obtained by a specialization of the shifted Schur measure. Actually, since the Schur Q-function can be expanded as (see [**Mac**])

$$Q_{\lambda}(X) = \sum_{\rho=1^{m_1}3^{m_3}\cdots} 2^{\ell(\rho)}X^{\lambda}_{\rho}(-1)\prod_{i:\mathrm{odd}}\frac{p_k(X)^{m_i}}{m_i!i^{m_i}},$$

where $X^{\lambda}_{(1^{|\lambda|})}(-1) = g^{\lambda}$ (see [**Mac**, III-8, Ex.12]), if we make a specialization such as $p_k(X) = p_k(Y) = \sqrt{\frac{\xi}{2}}\delta_{k1}$ $(k \geq 1)$, then we have

$$Q_{\lambda} = (2\xi)^{\frac{|\lambda|}{2}}\frac{g^{\lambda}}{|\lambda|!}.$$

Hence the shifted Schur measure in (2.4) becomes the measure $\mathrm{P}^{\xi}_{\mathrm{PSP}}$ in (4.6).

Let $\rho^{\xi}_{\mathrm{PSP}}$ be the correlation function of the measure $\mathrm{P}^{\xi}_{\mathrm{PSP}}$.

Proposition 4.4. *We have*

$$\begin{aligned}\lim_{\xi\to+\infty}&(2\xi)^{N/6}\rho^{\xi}_{\mathrm{PSP}}(\{[2\sqrt{2\xi}+(2\xi)^{1/6}x_1], \dots, [2\sqrt{2\xi}+(2\xi)^{1/6}x_N]\}) \\ &= \det(K_{\mathrm{Airy}}(x_i, x_j))_{1\leq i,j\leq N}.\end{aligned}$$

The limit is uniform for $(x_1, \dots, x_N)$ on a compact set of $\boldsymbol{R}^N$.

This proposition follows immediately from Theorem 3.1 and the following lemma.

Lemma 4.5. *We have*

$$(2\xi)^{\frac{1}{6}}\boldsymbol{K}_{\mathrm{B}}(2\sqrt{2\xi}+x(2\xi)^{\frac{1}{6}},2\sqrt{2\xi}+y(2\xi)^{\frac{1}{6}})\to 0, \tag{4.8}$$

$$(2\xi)^{\frac{1}{6}}\boldsymbol{K}_{\mathrm{B}}(2\sqrt{2\xi}+x(2\xi)^{\frac{1}{6}},-(2\sqrt{2\xi}+y(2\xi)^{\frac{1}{6}}))\to K_{\mathrm{Airy}}(x,y), \tag{4.9}$$

$$(2\xi)^{\frac{1}{6}}\boldsymbol{K}_{\mathrm{B}}(-(2\sqrt{2\xi}+x(2\xi)^{\frac{1}{6}}),-(2\sqrt{2\xi}+y(2\xi)^{\frac{1}{6}}))\to 0, \tag{4.10}$$

as $\xi\to\infty$, uniformly in x and y on compact sets in $\boldsymbol{R}$.

Proof. By the specialization $p_k(X)=p_k(Y)=\sqrt{\xi/2}\delta_{k1}$, the function $\boldsymbol{J}(z;X,Y)$ in (3.3) becomes $e^{\sqrt{2\xi}(z-z^{-1})}$, which is the generating function of Bessel functions. Therefore, in order to prove Lemma 4.5, we evaluate integrals of the form

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)\iint e^{2\xi(z-z^{-1}+w-w^{-1})}\frac{z-w}{z+w}\frac{\mathrm{d}z\mathrm{d}w}{z^{u+1}w^{v+1}},$$

where the contours are two unit circles and $u=\pm(2\sqrt{2\xi}+x(2\xi)^{\frac{1}{6}})$ and $v=\pm(2\sqrt{2\xi}+y(2\xi)^{\frac{1}{6}})$. Then Lemma 4.5 is obtained by a similar discussion in [**TW2**]. We leave the detail for readers. □

Proof of Theorem 4.3. The proof follows from Proposition 4.4 and the discussion in [**BOO**]. □

4.5. The α-specialized shifted Schur measure.

Let α be a real number such that $0<\alpha<1$ and let m and n be positive integers. We put $X_i=Y_j=\alpha$ for $1\le i\le m$ and $1\le j\le n$, and let the rest be zero in the definition of the shifted Schur measure. This is called the *α-specialization*, see [**TW2**] and [**M**]. Using Theorem 3.1, we also give a limit distribution of λ_i's with respect to the α-specialized shifted Schur measure. Denote by $\mathrm{P}_{\mathrm{SS},\sigma}$ the α-specialized shifted Schur measure, where $\sigma=(m,n,\alpha)$ denotes the set of parameters above, and put $\tau=m/n$.

Theorem 4.6. *There exist positive constants $c_1=c_1(\alpha,\tau)$ and $c_2=c_2(\alpha,\tau)$ such that*

$$\lim_{n\to\infty}\mathrm{P}_{\mathrm{SS},\sigma}\left(\left\{\lambda\in\mathscr{D}\mid\frac{\lambda_i-c_1n}{c_2n^{1/3}}<a_i,\ 1\le i\le M\right\}\right)=\mathrm{P}_{\mathrm{Airy}}(\zeta_i<a_i,\ 1\le i\le M)$$

holds for any $M\ge1$ and any $a_1,\dots,a_M\in\boldsymbol{R}$.

When $M=1$, this theorem gives the result in [**TW2**]. Although they assume that α and τ satisfy the relation $\alpha^2<\tau<\alpha^{-2}$, we can remove this assumption as they expect in the footnote of that paper.

We write ρ_{SS}, M, $\boldsymbol{J}(z;X,Y)$ and $\boldsymbol{K}$ in Theorem 3.1 after making the α-specialization by ρ_σ, M_σ, $\boldsymbol{J}_\sigma(z)$ and $\boldsymbol{K}_\sigma$, respectively. Let c_1, c_2 and z_0 be positive constants depending on α and τ given in [**TW2**]. These constants are not explicitly given for $\tau\ne1$, see Section 1 in [**TW2**]. Employing the following proposition, we can prove Theorem 4.6 as Theorem 4.1 and so we omit the proof.

Proposition 4.7. *We have*

$$\lim_{n\to\infty}(c_2 n^{\frac{1}{3}})^N \rho_\sigma(\{[c_1 n + c_2 n^{\frac{1}{3}} x_1], \dots, [c_1 n + c_2 n^{\frac{1}{3}} x_N]\}) = \det(K_{\mathrm{Airy}}(x_i, x_j))_{1 \le i,j \le N}.$$

The limit is uniform for $(x_1, \dots, x_N)$ *in a compact set of* $\boldsymbol{R}^N$.

Proof. From Theorem 3.1, we have

$$\rho_\sigma(\{k_1, \dots, k_N\}) = \sqrt{\det(M_\sigma(\{k_1, \dots, k_N\}))}.$$

Then we may write the skew matrix M_σ in the form

$$M_\sigma = \begin{pmatrix} M_1 & M_2 \\ -{}^t M_2 & M_3 \end{pmatrix},$$

where we put $N \times N$ matrices $M_1 = (\boldsymbol{K}_\sigma(k_i, k_j))_{1 \le i,j \le N}$, $M_2 = (\boldsymbol{K}_\sigma(k_i, -k_{N-j+1}))_{1 \le i,j \le N}$ and $M_3 = (\boldsymbol{K}_\sigma(-k_{N-i+1}, -k_{N-j+1}))_{1 \le i,j \le N}$. Let D be an $N \times N$ diagonal matrix whose i-th entry is given by $\boldsymbol{J}_\sigma(z_0)^{-1} z_0^i$. Then ρ_σ is expressed as

$$\begin{aligned}
\rho_\sigma(\{k_1, \dots, k_N\}) &= \sqrt{(-1)^N \det\begin{pmatrix} M_2 & M_1 \\ M_3 & -{}^t M_2 \end{pmatrix}} \\
&= \sqrt{(-1)^N \det\left(\begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}\begin{pmatrix} M_2 & M_1 \\ M_3 & -{}^t M_2 \end{pmatrix}\begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix}\right)} \\
&= \sqrt{(-1)^N \det\begin{pmatrix} DM_2D^{-1} & DM_1D \\ D^{-1}M_3D^{-1} & -D^{-1}\,{}^t M_2 D \end{pmatrix}} \\
&= \mathrm{Pf}\begin{pmatrix} DM_1D & DM_2D^{-1} \\ -D^{-1}\,{}^t M_2 D & D^{-1}M_3D^{-1} \end{pmatrix}.
\end{aligned}$$

Thus we immediately obtain the proposition from the following lemma. □

Lemma 4.8. *We have*

$$\boldsymbol{J}_\sigma(z_0)^{-2} z_0^{2c_1 n + c_2 n^{\frac{1}{3}}(x+y)} n^{\frac{1}{3}} \boldsymbol{K}_\sigma(c_1 n + c_2 n^{\frac{1}{3}} x, c_1 n + c_2 n^{\frac{1}{3}} y) \to 0,$$

$$z_0^{c_2 n^{\frac{1}{3}}(x-y)} n^{\frac{1}{3}} \boldsymbol{K}_\sigma(c_1 n + c_2 n^{\frac{1}{3}} x, -(c_1 n + c_2 n^{\frac{1}{3}} y)) \to c_2^{-1} K_{\mathrm{Airy}}(x, y),$$

$$\boldsymbol{J}_\sigma(z_0)^2 z_0^{-(2c_1 n + c_2 n^{\frac{1}{3}}(x+y))} n^{\frac{1}{3}} \boldsymbol{K}_\sigma(-(c_1 n + c_2 n^{\frac{1}{3}} x), -(c_1 n + c_2 n^{\frac{1}{3}} y)) \to 0,$$

as $n \to \infty$*, uniformly in* x *and* y *on compact sets in* $\boldsymbol{R}$.

The proof of this lemma is obtained by the discussion in Section 6.4 of [**TW2**]. Since the assumption $\alpha^2 < \tau < \alpha^{-2}$ is not used in that section, we do not need this assumption in Theorem 4.6.

5. Hall-Littlewood measures.

Let $\mathscr{P}$ be the set of all partitions. In this section, we consider the so-called Hall-Littlewood measure on $\mathscr{P}$, defined by Hall-Littlewood functions. It is considered as a natural extension of the Schur measure and the shifted Schur measure. Let $Q_\lambda(X;t)$ (respectively $P_\lambda(X;t)$) be the Hall-Littlewood Q-(respectively P-)function for a partition λ (see [**Mac**, III]). We define the *Hall-Littlewood measure* by

$$\mathrm{P}_{\mathrm{HL},X,Y,t}(\{\lambda\}) = \frac{1}{Z} Q_\lambda(X;t) P_\lambda(Y;t).$$

Here the constant $Z = Z(X,Y;t)$ is determined by

$$Z := \sum_{\lambda \in \mathscr{P}} Q_\lambda(X;t) P_\lambda(Y;t) = \prod_{i,j=1}^{\infty} \frac{1 - tX_iY_j}{1 - X_iY_j},$$

where the second equality is the Cauchy identity for Hall-Littlewood functions. Since $Q_\lambda(X;t)$ is the Schur function $s_\lambda(X)$ at $t=0$ and the Schur Q-function $Q_\lambda(X)$ at $t=-1$, the Hall-Littlewood measure gives the Schur measure at $t=0$ and the shifted Schur measure at $t=-1$.

The mean value and the variance of the size $|\lambda|$ of a partition λ are given explicitly as follows.

Theorem 5.1. *The mean value* $\boldsymbol{E}(|\lambda|)$ *and the variance* $\mathrm{Var}(|\lambda|)$ *of the size* $|\lambda|$ *of a partition with respect to the Hall-Littlewood measure are given by*

$$\boldsymbol{E}(|\lambda|) = \sum_{k=1}^{\infty} (1-t^k) p_k(X) p_k(Y), \tag{5.1}$$

$$\mathrm{Var}(|\lambda|) = \sum_{k=1}^{\infty} k(1-t^k) p_k(X) p_k(Y). \tag{5.2}$$

Here $p_k(X)$ *is the* k*-th power sum function.*

Proof. Define a differential operator Δ_X by

$$\Delta_X = \sum_{k=1}^{\infty} k p_k(X) \frac{\partial}{\partial p_k(X)}.$$

Since $Q_\lambda(X;t) = \sum_{\rho:|\rho|=|\lambda|} z_\rho(t)^{-1} X_\rho^\lambda(t) \prod_{k=1}^{\infty} p_k(X)^{m_k(\rho)}$ (see [**Mac**, III-(7.51)]) we have

$$\Delta_X Q_\lambda(X;t) = |\lambda| Q_\lambda(X;t).$$

Therefore we obtain $\boldsymbol{E}(|\lambda|) = \frac{1}{Z}\sum_\lambda |\lambda| Q_\lambda(X;t)P_\lambda(Y;t) = \Delta_X(\log Z)$. On the other hand, since $Z = \exp\left(\sum_{k=1}^\infty \frac{1-t^k}{k} p_k(X)p_k(Y)\right)$ (see [**Mac**, p. 223]) we have $\Delta_X(\log Z) = \sum_{k=1}^\infty (1-t^k)p_k(X)p_k(Y)$ so that we get (5.1).

In general, we see that $\boldsymbol{E}(|\lambda|^n) = \frac{1}{Z}\Delta_X^n Z$. In particular, it follows that

$$\begin{aligned}
\boldsymbol{E}(|\lambda|^2) &= \frac{1}{Z}\Delta_X^2(Z) = \frac{1}{Z}\Delta_X\left(Z\frac{1}{Z}\Delta_X(Z)\right) = \frac{1}{Z}\Delta_X(Z\Delta_X(\log Z)) \\
&= \frac{1}{Z}\left\{\Delta_X(Z)\cdot\Delta_X(\log Z) + Z\Delta_X^2(\log Z)\right\} \\
&= (\Delta_X(\log Z))^2 + \Delta_X^2(\log Z).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\mathrm{Var}(|\lambda|) &= \boldsymbol{E}(|\lambda|^2) - \boldsymbol{E}(|\lambda|)^2 = \Delta_X^2(\log Z) \\
&= \Delta_X\left(\sum_{k=1}^\infty (1-t^k)p_k(X)p_k(Y)\right) \\
&= \sum_{k=1}^\infty k(1-t^k)p_k(X)p_k(Y).
\end{aligned}$$

This completes the proof of the theorem. □

Remark 5.1. The mean value $\boldsymbol{E}(|\lambda|)$ with respect to the Schur measure is given in [**O3**].

Next we consider the mean value $\boldsymbol{E}(\lambda_1)$ of λ_1, the first row of a partition λ. Based on the fact in Theorem 5.1, we now examine whether $\boldsymbol{E}(\lambda_1)$ has an expression similar to $\boldsymbol{E}(|\lambda|)$. Assume $X = Y$ in the definition of the Hall-Littlewood measure and let $M(t,X) := 2\sum_{k=1}^\infty (1-t^k)p_k(X)$.

Example 5.1. The poissonized Plancherel measure for symmetric groups is obtained from the Hall-Littlewood measure by putting $t=0$ and the exponential specialization $p_k(X) = p_k(Y) = \sqrt{\xi}\delta_{1k}$. Hence it follows that $M(t,X) = 2\sqrt{\xi}$. Since it is known that $\boldsymbol{E}(\lambda_1) \sim 2\sqrt{\xi}$ as $\xi \to +\infty$ (see e.g. [**BOO**]), we have $\lim \frac{\boldsymbol{E}(\lambda_1)}{M(t,X)} = 1$.

Example 5.2. The α-specialized Schur measure is obtained by putting $t=0$ and the α-specialization $X = Y = (\overbrace{\alpha,\dots,\alpha}^{n}, 0, 0, \dots)$. We have hence $M(t,X) = 2\sum_{k=1}^\infty n\alpha^k = \frac{2\alpha}{1-\alpha}n$. Since it is proved in [**J2**] that $\boldsymbol{E}(\lambda_1) \sim \frac{2\alpha n}{1-\alpha}$ as $n \to +\infty$, we have $\lim \frac{\boldsymbol{E}(\lambda_1)}{M(t,X)} = 1$.

Example 5.3. The poissonization of the shifted version of the Plancherel measure for symmetric groups is obtained by putting $t=-1$ and $p_k(X) = p_k(Y) = \sqrt{\frac{\xi}{2}}\delta_{1k}$. We have hence $M(t,X) = 4\sqrt{\frac{\xi}{2}} = 2\sqrt{2\xi}$. Since we have proved that $\boldsymbol{E}(\lambda_1) \sim 2\sqrt{2\xi}$ as $\xi \to +\infty$ in Section 4, we have $\lim \frac{\boldsymbol{E}(\lambda_1)}{M(t,X)} = 1$.

EXAMPLE 5.4. The α-specialized shifted Schur measure is obtained by putting $t=-1$ and the α-specialization $X=Y=(\overbrace{\alpha,\dots,\alpha}^{n},0,0,\dots)$. Hence we have $M(t,X)=2\sum_{k\ge 1:\mathrm{odd}} 2n\alpha^k=\frac{4\alpha n}{1-\alpha^2}$. Since it is proved in [**TW2**] that $\boldsymbol{E}(\lambda_1)\sim\frac{4\alpha n}{1-\alpha^2}$ as $n\to+\infty$, we have $\lim\frac{\boldsymbol{E}(\lambda_1)}{M(t,X)}=1$.

In view of the examples above, we might expect that $\frac{\boldsymbol{E}(\lambda_1)}{M(t,X)}$ always converges to 1. However, we encounter an example that $\frac{\boldsymbol{E}(\lambda_1)}{M(t,X)}$ does not converge to 1 as follows.

EXAMPLE 5.5. Suppose $0<t<1$. We make the principal specialization $X=Y=(t,t^2,\dots,t^n,0,0,\dots)$ and $n\to+\infty$. Then we have

$$Q_\lambda=t^{n(\lambda)+|\lambda|},\qquad P_\lambda=\frac{t^{n(\lambda)+|\lambda|}}{\prod_{j\ge1}(t;t)_{m_j(\lambda)}},$$

where we put $n(\lambda)=\sum_{j\ge1}(j-1)\lambda_j=\sum_{j\ge1}\binom{\lambda'_j}{2}$, $(a;q)_m=\prod_{j=0}^{m-1}(1-aq^j)$ and denote the multiplicity of j in λ by $m_j(\lambda)$ (see [**Mac**, III-2 Ex.1]). Further we obtain

$$Z=\prod_{i,j=1}^{\infty}\frac{1-t^{i+j+1}}{1-t^{i+j}}=\prod_{r=2}^{\infty}\frac{1}{1-t^r}.$$

Since $2n(\lambda)+|\lambda|=\sum_{j\ge1}\lambda'_j(\lambda'_j-1)+\sum_{j\ge1}\lambda'_j=\sum_{j\ge1}(\lambda'_j)^2$, the Hall-Littlewood measure becomes

$$\mathrm{P}_{t,\mathrm{Prin}}(\lambda):=\prod_{r=2}^{\infty}(1-t^r)\frac{t^{\sum_{j\ge1}(\lambda'_j)^2+|\lambda|}}{\prod_{j\ge1}(t;t)_{m_j(\lambda)}}.$$

This measure is studied by Fulman [**F1**].

We calculate the distribution function of λ_1. Since for a positive integer h

$$\sum_{\lambda:\lambda_1<h}\frac{t^{\sum_{j\ge1}(\lambda'_j)^2+|\lambda|}}{\prod_{j\ge1}(t;t)_{m_j(\lambda)}}=\sum_{\mu:\ell(\mu)<h}\frac{t^{\sum_{j=1}^{h-1}(\mu_j)^2+|\mu|}}{\prod_{j=1}^{h-1}(t;t)_{\mu_j-\mu_{j+1}}}=\prod_{\substack{r=1\\ r\not\equiv 0,\pm1\ (\mathrm{mod}\ 2h+1)}}^{\infty}\frac{1}{1-t^r},$$

(the second equality is proved by Andrews [**A**], see also [**F2**]) we have

$$\begin{aligned}\mathrm{P}_{t,\mathrm{Prin}}(\lambda_1<h)&=\prod_{\substack{r\ge2\\ r\equiv0,\pm1\ (\mathrm{mod}\ 2h+1)}}(1-t^r)\\ &=\prod_{k=1}^{\infty}(1-t^{(2h+1)k})(1-t^{(2h+1)k+1})(1-t^{(2h+1)k-1}).\end{aligned}\tag{5.3}$$

The mean value $\boldsymbol{E}(\lambda_1)$ is given by $\boldsymbol{E}(\lambda_1)=\sum_{h=1}^{\infty}h(\mathrm{P}_{t,\mathrm{Prin}}(\lambda_1<h+1)-\mathrm{P}_{t,\mathrm{Prin}}(\lambda_1<h))$.

It follows from (5.3) that $\boldsymbol{E}(\lambda_1) = t^2 + O(t^3)$ as $t \to +0$. On the other hand, it is easy to see that

$$M(t, X) = 2\sum_{k=1}^{\infty}(1-t^k)\sum_{j=1}^{\infty} t^{jk} = 2\sum_{k=1}^{\infty}(1-t^k)\frac{t^k}{1-t^k} = \frac{2t}{1-t}$$

and therefore $M(t, X) = 2t + 2t^2 + O(t^3)$ as $t \to +0$. Therefore $\boldsymbol{E}(\lambda_1)$ is not equal to $M(t, X)$.

Thus it is interesting to determine when the ratio $\frac{\boldsymbol{E}(\lambda_1)}{M(t,X)}$ converges to 1. We will study this problem in future.

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