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# On the sequential polynomial type of modules

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**Abstract.** Let $M$ be a finitely generated module over a Noetherian local ring $R$. The sequential polynomial type $\mathrm{sp}(M)$ of $M$ was recently introduced by Nhan, Dung and Chau, which measures how far the module $M$ is from the class of sequentially Cohen–Macaulay modules. The present paper purposes to give a parametric characterization for $M$ to have $\mathrm{sp}(M) \leq s$, where $s \geq -1$ is an integer. We also study the sequential polynomial type of certain specific rings and modules. As an application, we give an inequality between $\mathrm{sp}(S)$ and $\mathrm{sp}(S^G)$, where $S$ is a Noetherian local ring and $G$ is a finite subgroup of $\mathrm{Aut} S$ such that the order of $G$ is invertible in $S$.

## 1. Introduction.

The motivation of the present research comes from and dates back to a naive but fundamental question of what non-Cohen–Macaulay rings are. There are at least two different ways to generalize the notion of Cohen-Macaulayness; one is the generalized Cohen-Macaulayness and the other one is the sequential Cohen-Macaulayness. Our practical purpose is to find effective invariants which enable us to stratify non-Cohen–Macaulay rings and modules, describing the distance of given rings and modules from the class of Cohen–Macaulay rings and modules. For the purpose the polynomial type and the sequential polynomial type introduced by Cuong [**C1**] in 1992 and Nhan, Dung and Chau [**NDC**] in 2016 respectively are good candidates of these invariants. In the present paper we succeed to the research [**NDC**]. Our aim is to deepen the theory of the polynomial and sequential polynomial types of modules, investigating basic properties of the invariants.

Throughout this paper let $(R, \mathfrak{m})$ be a Noetherian local ring and $M$ a finitely generated $R$-module with $\dim_R M = d$. We denote by $\widehat{R}$ and $\widehat{M}$ the $\mathfrak{m}$-adic completion of $R$ and $M$ respectively. For an ideal $I$ of $R$ let $\mathrm{V}(I)$ be the set of the prime ideals of $R$ containing $I$. The polynomial type $\mathrm{p}(M)$ of $M$ is defined to be the largest value among the dimension of the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(M)$ $(i < d)$ over the ring $\widehat{R}$. In general

$$\mathrm{p}(M) \geq \max\{\dim \mathrm{nCM}(M), \dim_R D_1\}$$

and one has the equality when $R$ is a quotient of a Cohen–Macaulay local ring, where

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*Key Words and Phrases.* sequentially Cohen–Macaulay module, strict $M$-sequence in dimension $> s$, distinguished system of parameters.

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nCM($M$) is the non-Cohen–Macaulay locus of $M$ and $D_1$ is the largest $R$-submodule of $M$ with $\dim_R D_1 < d$. Therefore p($M$) naturally measures the non-Cohen-Macaulayness and the mixedness of the module $M$. The notion is originated also in the paper of Cuong [**C2**], where he developed a deep theory of $p$-standard systems of parameters. The celebrated theorem of Kawasaki [**K**] on the existence of arithmetic Macaulayfication is based on the research [**C2**]. The reader may consult [**CC**] for further developments of the theory in connection with arithmetic Macaulayfication and the annihilator theorem of Faltings.

The notion of sequential polynomial type sp($M$) was recently defined by [**NDC**] in order to understand the structure of $M$ in connection with the sequential Cohen-Macaulayness of modules. Let

$$\mathrm{H}^0_{\mathfrak{m}}(M) = D_t \subset \cdots \subset D_1 \subset D_0 = M$$

be the dimension filtration of $M$. Hence for all $1 \le i \le t$ $D_i$ is the largest $R$-submodule of $M$ such that $\dim_R D_i < \dim_R D_{i-1}$. Then $M$ is said to be a sequentially Cohen–Macaulay $R$-module, if each $D_i/D_{i+1}$ is a Cohen–Macaulay $R$-module. This notion was given in the local case by Schenzel [**Sch**] as a generalization of the concept of Cohen-Macaulayness. A sequentially Cohen–Macaulay $R$-module $M$ is necessarily Cohen–Macaulay once $\dim R/\mathfrak{p} = \dim_R M$ for all $\mathfrak{p} \in \mathrm{Ass}_R M$, which shows the concept of sequential Cohen-Macaulayness is substantially unlike from the notion of Buchsbaumness or generalized Cohen-Macaulayness of rings and modules.

In the present research we are interested to see the difference of given $R$-modules $M$ from the class of sequentially Cohen–Maculay modules. To do this let us set

$$\mathrm{sp}(M) = \max_{1 \le i \le t} \mathrm{p}(D_{i-1}/D_i)$$

and call it the sequential polynomial type of $M$. Therefore $M$ is a sequentially Cohen–Macaulay $R$-module if and only if $\mathrm{sp}(M) = -1$ and $M$ is a sequentially generalized Cohen–Macaulay $R$-module in the sense of [**CN**, Definition 4.3] if and only if $\mathrm{sp}(M) \le 0$. In general

$$\mathrm{p}(M) \ge \mathrm{sp}(M) \ge \dim \mathrm{nSCM}(M)$$

(Proposition 2.9 and [**NDC**, Proposition 3.2]), where

$$\mathrm{nSCM}(M) = \{\mathfrak{p} \in \mathrm{Supp}_R M \mid M_{\mathfrak{p}} \text{ is not a sequentially Cohen–Macaulay } R_{\mathfrak{p}}\text{-module}\}.$$

One has the equality $\mathrm{sp}(M) = \dim \mathrm{nSCM}(M)$ also, when $R$ is a quotient of a Cohen–Macaulay local ring. Thus the invariant sp($M$) measures well the non-sequential-Cohen-Macaulayness of $M$.

The reader may consult [**NDC**] for several basic properties of sp($M$). The question of whether the invariant sp($M$) is preserved under localization, completion, and being divided by a single parameter is closely studied in [**NDC**]. The authors gave among other results also the following practical method of computation.

THEOREM 1.1 ([**NDC**, Theorem 4.7]). *Suppose that $R$ is a quotient of a Gorenstein local ring and let $\mathrm{K}^j(M)$ be the $j$-deficiency module of $M$. We set $q_1 = \max_{j \notin \mathrm{D}(M)} \dim_R \mathrm{K}^j(M)$ and $q_2 = \max_{j \in \mathrm{D}(M)} \mathrm{p}(\mathrm{K}^j(M))$, where $\mathrm{D}(M) = \{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}_R M\}$. Then*

$$\mathrm{sp}(M) = \max\{q_1, q_2\}.$$

It is well-known that $M$ is a Cohen–Macaulay $R$-module (i.e., $\mathrm{p}(M) = -1$) if and only if every system of parameters of $M$ is an $M$-sequence. Our Theorem 3.7 generalizes this fact, showing for every integer $s \geq -1$ that $\mathrm{p}(M) \leq s$ if and only if every system of parameters of $M$ is an $M$-sequence in dimension $> s$, provided $R$ is a quotient of a Cohen–Macaulay local ring. Here a sequence $x_1, x_2, \ldots, x_n \in \mathfrak{m}$ is said to be an $M$-sequence in dimension $> s$, if for each $1 \leq i \leq n$ one has $x_i \notin \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Ass}_R M/(x_1, x_2, \ldots, x_{i-1})M$ such that $\dim R/\mathfrak{p} > s$ ([**BN2**, Definition 2.1]).

According to Theorem 3.7, it seems natural to ask for such a parametric characterization for modules $M$ to have $\mathrm{sp}(M) \leq s$. The first main result of this paper is an answer to the problem in terms of *distinguished systems of parameters* of $M/D_i$ and *strict $M/D_i$-sequences in dimension* $> s$. See Definitions 3.4 and 4.1 for the definition of the terminology and let us note here only the fact that for a finitely generated $R$-module $M$ every strict $M$-sequence in dimension $> s$ is always an $M$-sequence in dimension $> s$. We then have the following, which we shall prove in Section 4.

THEOREM 1.2 (Theorem 4.3). *Assume that $R$ is a quotient of a Cohen–Macaulay local ring and let $s \geq 0$ be an integer. Then $\mathrm{sp}(M) \leq s$ if and only if for all $1 \leq i \leq t$ every distinguished system of parameters of $M/D_i$ is a strict $M/D_i$-sequence in dimension $> s$.*

The second main topic of this paper is to study the sequential polynomial type of certain specific rings and modules such as the direct sum of a finite collection of modules, the formal power series extensions, and the localization of the polynomial rings, which we shall perform in Section 5. We will firstly apply the results in order to describe the relationship between $\mathrm{sp}(S)$ and $\mathrm{sp}(S^G)$, where $S$ is a Noetherian local ring and $G$ is a finite subgroup of Aut $S$ such that the order of $G$ is invertible in $S$ (Corollary 5.3). As the second application of the results we will show the following, which is the goal of the this paper.

THEOREM 1.3 (Theorem 5.7). *Let $q_1, q_2 \geq -1$ be given integers and choose an integer $d$ so that $d \geq \max\{q_1, q_2\} + 3$. Then there exists a finitely generated module $M$ over a Noetherian local ring $R$ which is a quotient of a Gorenstein local ring such that*

$$\dim_R M = d, \ \ q_1 = \max_{j \notin \mathrm{D}(M)} \dim_R \mathrm{K}^j(M), \ \ \text{and} \ \ q_2 = \max_{j \in \mathrm{D}(M)} \mathrm{p}(\mathrm{K}^j(M)).$$

*Hence* $\mathrm{sp}(M) = \max\{q_1, q_2\}$.

Let us now briefly explain how this paper is organized. Section 2 is devoted to some preliminaries, where we summarize some basic results on the invariants $\mathrm{p}(M)$ and $\mathrm{sp}(M)$. In Section 3 we study $\mathrm{p}(M)$ in connection with the behavior of systems of parameters, which plays a key role to prove Theorem 1.2. The proof of Theorem 1.2 (resp. Theorem 1.3) shall be given in Section 4 (resp. Section 5).

In what follows, unless otherwise specified, $(R, \mathfrak{m})$ denotes a Noetherian local ring and $M$ a finitely generated $R$-module. Let $d = \dim_R M$.

## 2. Preliminaries.

The purpose of this section is to summarize some preliminaries which we need throughout this paper.

For an Artinian $R$-module $A$ the set $\mathrm{Att}_R A$ of the attached primes of $A$ defined by MacDonald [**Mac**] is almost the dual concept of the set of the associated primes for finitely generated modules. Let $\dim_R A = \dim R/\mathrm{Ann}_R A$. We then have $\mathrm{Min}\,\mathrm{Att}_R A = \mathrm{Min}\,\mathrm{V}(\mathrm{Ann}_R A)$, so that

$$\dim_R A = \max\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Att}_R A\}.$$

When we naturally regard $A$ as an Artinian $\widehat{R}$-module, we have

$$\mathrm{Att}_R A = \{\mathfrak{P} \cap R \mid \mathfrak{P} \in \mathrm{Att}_{\widehat{R}} A\}$$

([**BS**, 8.2.4, 8.2.5]). Hence $\dim_R A \geq \dim_{\widehat{R}} A$ in general.

Remark 2.1. Let $i \geq 0$ be an integer and let $\mathrm{H}^i_{\mathfrak{m}}(M)$ denote the $i$-th local cohomology module of $M$ with respect to $\mathfrak{m}$. Then $\mathrm{H}^i_{\mathfrak{m}}(M)$ is an Artinian $R$-module and $\mathrm{H}^i_{\mathfrak{m}}(M) \cong \mathrm{H}^i_{\mathfrak{m}\widehat{R}}(\widehat{M})$ as $\widehat{R}$-modules. If $R$ is a quotient of a Cohen–Macaulay local ring, then $\dim_R \mathrm{H}^i_{\mathfrak{m}}(M) \leq i$ ([**NH**, Lemma 4.1]) and

$$\mathrm{Att}_{\widehat{R}} \mathrm{H}^i_{\mathfrak{m}}(M) = \bigcup_{\mathfrak{p} \in \mathrm{Att}_R \mathrm{H}^i_{\mathfrak{m}}(M)} \mathrm{Ass}_{\widehat{R}}\, \widehat{R}/\mathfrak{p}\widehat{R}$$

([**NQ**, Theorem 1.1]). Under the same assumption as above we always have

$$\dim_R \mathrm{H}^i_{\mathfrak{m}}(M) = \dim_{\widehat{R}} \mathrm{H}^i_{\mathfrak{m}}(M),$$

which is however not the case in general, although $\dim_{\widehat{R}} \mathrm{H}^i_{\mathfrak{m}}(M) \leq i$ since $\mathrm{H}^i_{\mathfrak{m}}(M) \cong \mathrm{H}^i_{\mathfrak{m}\widehat{R}}(\widehat{M})$ as $\widehat{R}$-modules.

The notion of polynomial type was introduced by Cuong [**C1**]. Let $\underline{x} = x_1, x_2, \ldots, x_d$ be a system of parameters of $M$ and $\underline{n} = n_1, n_2, \ldots, n_d$ a sequence of positive integers. Set

$$\mathrm{I}_{M,\underline{x}}(\underline{n}) = \ell_R\big(M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_d^{n_d})M\big) - n_1 n_2 \ldots n_d \cdot \mathrm{e}(\underline{x}; M),$$

where $\mathrm{e}(\underline{x}; M)$ denotes the multiplicity of $M$ with respect to the system $\underline{x}$. Then in general $\mathrm{I}_{M,\underline{x}}(\underline{n})$ which is considered as a function in $n_1, n_2, \ldots, n_d$ is not a polynomial in $n_1, n_2, \ldots, n_d \gg 0$ but it takes non-negative values and bounded above by polynomials.

Definition 2.2. The least degree of the polynomials bounding above the function

$\mathrm{I}_{M,\underline{x}}(\underline{n})$ does not depend on the choice of systems $\underline{x}$ of parameters of $M$ ([**C1**, Theorem 2.3]), which is called the polynomial type of $M$ and denoted by $\mathrm{p}(M)$.

Here we stipulate the degree of the zero polynomial to be $-1$.

It is clear that $M$ is a Cohen–Macaulay $R$-module if and only if $\mathrm{p}(M) = -1$ and $M$ is a generalized Cohen–Macaulay $R$-module if and only if $\mathrm{p}(M) \leq 0$. Let us denote by $\mathrm{nCM}(M)$ the non-Cohen–Macaulay locus

$$\mathrm{nCM}(M) = \{\mathfrak{p} \in \operatorname{Supp}_R M \mid M_{\mathfrak{p}} \text{ is not a Cohen–Macaulay } R_{\mathfrak{p}}\text{-module}\}$$

of $M$. Then $\mathrm{nCM}(M)$ is not necessarily a closed subset of $\operatorname{Spec} R$ but stable under specialization (it is actually a closed subset of $\operatorname{Spec} R$, when $R$ is a quotient of a Cohen–Macaulay local ring), so that we can define its dimension $\dim \mathrm{nCM}(M)$. With this notation we have the following, which show that $\mathrm{p}(M)$ measures the non-Cohen-Macaulayness and the mixedness of $M$.

Proposition 2.3 ([**C1**], [**NNK**]). *Let $D_1$ be the largest $R$-submodule of $M$ of dimension less than $d$. Then the following assertions hold true.*

(1) $\mathrm{p}(M) = \mathrm{p}(\widehat{M}) = \max_{i<d} \dim_{\widehat{R}} \mathrm{H}^i_{\mathfrak{m}}(M)$. *Hence* $\mathrm{p}(M) \leq d-1$.

(2) $\mathrm{p}(M) \geq \max\{\dim \mathrm{nCM}(M), \dim_R D_1\}$, *where the equality holds when $R$ is a quotient of a Cohen–Macaulay local ring. If in addition $M$ is equi-dimensional, then* $\mathrm{p}(M) = \dim \mathrm{nCM}(M)$.

Lemma 2.4. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated $R$-modules and suppose that* $\dim_R M'' \leq s$. *Then* $\mathrm{p}(M) > s$ *if and only if* $\mathrm{p}(M') > s$.

Proof. Set $\dim_R M' = d$. We may assume $d > s$. Hence $\dim_R M = d$ and $\mathrm{H}^j_{\mathfrak{m}}(M) \cong \mathrm{H}^j_{\mathfrak{m}}(M')$ for all $j \geq s+2$. Consider the exact sequence

$$\mathrm{H}^s_{\mathfrak{m}}(M'') \to \mathrm{H}^{s+1}_{\mathfrak{m}}(M') \to \mathrm{H}^{s+1}_{\mathfrak{m}}(M) \to 0$$

and remember that $\dim_{\widehat{R}} \mathrm{H}^j_{\mathfrak{m}}(M) \leq j$ for all $j$ (Remark 2.1). Consequently $\dim_{\widehat{R}} \mathrm{H}^j_{\mathfrak{m}}(M) > s$ for some $s < j < d$ if and only if $\dim_{\widehat{R}} \mathrm{H}^j_{\mathfrak{m}}(M') > s$ for some $s < j < d$. Hence by Proposition 2.3 $\mathrm{p}(M) > s$ if and only if $\mathrm{p}(M') > s$. $\square$

The notion of dimension filtration was introduced by Schenzel [**Sch**]. Removing the repeated components, Cuong and the second author [**CN**] have modified his definition. Let us maintain throughout this paper the following definition given by [**CN**].

Definition 2.5. A filtration $\mathrm{H}^0_{\mathfrak{m}}(M) = D_t \subset \cdots \subset D_1 \subset D_0 = M$ of $M$ is said to be the dimension filtration of $M$, if for each $1 \leq i \leq t$ $D_i$ is the largest $R$-submodule of $M$ such that $\dim_R D_i < \dim_R D_{i-1}$.

We then have the following.

Proposition 2.6 ([**Sch**]). *Set $d_i = \dim_R D_i$ for $0 \le i \le t-1$. Then the following assertions hold true.*

(1) $\operatorname{Ass}_R D_i = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \dim R/\mathfrak{p} \le d_i\}$ *for* $0 \le i \le t-1$.

(2) $\operatorname{Ass}_R D_{i-1}/D_i = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \dim R/\mathfrak{p} = d_{i-1}\}$ *for* $1 \le i \le t$.

(3) $D_i = \mathrm{H}^0_{\mathfrak{a}_i}(M)$*, where* $\mathfrak{a}_i$ *is the intersection of the associated primes* $\mathfrak{p}$ *of* $M$ *such that* $\dim R/\mathfrak{p} \le d_i$.

Let us recall the definition of sequential polynomial type.

Definition 2.7 ([**NDC**]). The sequential polynomial type $\mathrm{sp}(M)$ of $M$ is defined by

$$\mathrm{sp}(M) = \max_{1 \le i \le t} \mathrm{p}(D_{i-1}/D_i).$$

We have $\mathrm{sp}(M) = -1$ if and only if $M$ is a sequentially Cohen–Macaulay $R$-module, while $\mathrm{sp}(M) \le 0$ if and only if $M$ is a sequentially generalized Cohen–Macaulay $R$-module. In this sense the sequential polynomial type $\mathrm{sp}(M)$ measures how far the $R$-module $M$ is from the class of sequentially Cohen–Macaulay $R$-modules. Let

$$\mathrm{nSCM}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \text{ is not a sequentially Cohen–Macaulay } R_{\mathfrak{p}}\text{-module}\}.$$

Then $\mathrm{nSCM}(M)$ is stable under specialization and we consider its dimension. If $R$ is a catenary ring, then $\mathrm{sp}(M) \ge \dim \mathrm{nSCM}(M)$. The equality holds and $\mathrm{nSCM}(M)$ is a closed subset of $\operatorname{Spec} R$, if $R$ is a quotient of a Cohen–Macaulay local ring ([**NDC**, Proposition 3.2]).

From now on, throughout this paper let us maintain the following notation and assumptions.

Setting 2.8. Let

$$\mathrm{H}^0_{\mathfrak{m}}(M) = D_t \subset \cdots \subset D_1 \subset D_0 = M$$

denote the dimension filtration of $M$. Set $d_i = \dim_R D_i$ for $0 \le i \le t$, where $\dim_R D_t = -1$ if $D_t = 0$. Let $\mathrm{D}(M) = \{d_0, d_1, \ldots, d_t\}$. Hence

$$\mathrm{D}(M) \setminus \{-1\} = \{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_R M\}.$$

Let $s \ge -1$ be an integer. We denote by $\mathrm{D}(s)$ the largest $R$-submodule of $M$ of dimension at most $s$. If $s \ge 0$, then there exists an integer $t(s) \le t$ such that $D(s) = D_{t(s)}$. By removing the submodules $D_{t(s)+1}, \ldots, D_t$ from the dimension filtration of $M$, we obtain the subfiltration

$$D(s) = D_{t(s)} \subset \cdots \subset D_1 \subset D_0 = M,$$

which we call the dimension filtration of $M$ in dimension $> s$.

We note the following.

Proposition 2.9. $\mathrm{sp}(M) \leq \mathrm{p}(M)$.

Proof. Set $\mathrm{sp}(M) = h$. If $h \leq \dim_R D_1$, then $h \leq \mathrm{p}(M)$ by Proposition 2.3 (2). Assume that $h > \dim_R D_1$. Then by Proposition 2.3 (1)

$$\mathrm{sp}(D_1) = \max_{2\leq i\leq t} \mathrm{p}(D_{i-1}/D_i) \leq \max_{2\leq i\leq t} \dim_R D_{i-1} - 1 \leq d_1 - 1.$$

Hence $h = \mathrm{p}(M/D_1)$, so that by Proposition 2.3 (1)

$$\dim_{\widehat{R}} \mathrm{H}^j_{\mathfrak{m}}(M/D_1) = h$$

for some $j < d$. Notice that $j \geq h > \dim_R D_1$ by Remark 2.1 and we have

$$\mathrm{H}^j_{\mathfrak{m}}(M/D_1) \cong \mathrm{H}^j_{\mathfrak{m}}(M).$$

Thus $\mathrm{p}(M) \geq h$ by Proposition 2.3 (1). $\square$

The following two results play an important role throughout this paper.

Proposition 2.10 ([**NDC**, Proposition 3.3]). *$\mathrm{sp}(M) \leq s$ if and only if there exists a filtration*

$$D(s) = N_k \subset \cdots \subset N_1 \subset N_0 = M$$

*of $M$ such that $\dim N_i < \dim N_{i-1}$ and $\mathrm{p}(N_{i-1}/N_i) \leq s$ for all $1 \leq i \leq k$. When this is the case, the following assertions hold true.*

(1) *$k = t(s)$ and $\dim_R D_i/N_i \leq s$ for all $i \leq t(s)$.*

(2) *$\max_{i\leq t(s)} \mathrm{p}(N_{i-1}/N_i) = s$ if and only if $\max_{i\leq t(s)}\{\mathrm{sp}(M), \dim_R D_i/N_i\} = s$.*

Theorem 2.11 ([**NDC**, Theorem 4.7, Corollary 4.8]). *Suppose that $R$ is a quotient of a Gorenstein local ring $(S, \mathfrak{n})$ with $\dim S = n$. Let $\mathrm{K}^j(M) = \mathrm{Ext}_S^{n-j}(M, S)$ ($j \in \mathbb{Z}$) be the $j$-deficiency module of $M$ and set $q_1 = \max_{j\notin \mathrm{D}(M)} \dim_R \mathrm{K}^j(M)$, $q_2 = \max_{j\in\mathrm{D}(M)} \mathrm{p}(\mathrm{K}^j(M))$. Then*

$$\mathrm{sp}(M) = \max\{q_1, q_2\}.$$

*Therefore $\mathrm{sp}(M) \leq s$ if and only if for all $j \in \mathbb{Z}$*

(1) *$\dim_R \mathrm{K}^j(M) \leq s$ if $j \notin \mathrm{D}(M)$ and*

(2) *$\dim_R \mathrm{K}^j(M) = j$ and $\mathrm{p}(\mathrm{K}^j(M)) \leq s$ if $j \in \mathrm{D}(M)$.*

## 3. Strict $M$-sequences in dimension $> s$.

The purpose of this section is to summarize some basic results on strict $M$-sequences in dimension $> s$. We firstly consider $M$-sequences in dimension $> s$. For a subset $T$ of $\operatorname{Spec} R$ and an integer $i$ we set

$$(T)_i = \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} = i\} \quad \text{and} \quad (T)_{>i} = \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} > i\}.$$

Let $s \geq -1$ be an integer.

DEFINITION 3.1 ([**BN2**]). An element $x \in \mathfrak{m}$ is said to be $M$-regular in dimension $> s$, if $x \notin \mathfrak{p}$ for any $\mathfrak{p} \in (\operatorname{Ass}_R M)_{>s}$. A sequence $x_1, x_2, \ldots, x_n \in \mathfrak{m}$ is said to be an $M$-sequence in dimension $> s$, if $x_i$ is $M/(x_1, x_2, \ldots, x_{i-1})M$-regular in dimension $> s$ for all $1 \leq i \leq n$.

We then have the following.

PROPOSITION 3.2. *Suppose that* $\mathrm{p}(M) \leq s$. *Let* $x \in \mathfrak{m}$ *be a parameter of* $M$. *Then the following assertions hold true.*

(1) *Either* $\dim R/\mathfrak{p} = d$ *or* $\dim R/\mathfrak{p} \leq s$ *for all* $\mathfrak{p} \in \operatorname{Ass}_R M$.

(2) $x$ *is* $M$*-regular in dimension* $> s$ *and* $\mathrm{p}(M/xM) \leq s$.

PROOF. (1) Let $\mathfrak{p} \in \operatorname{Ass}_R M$ and set $\dim R/\mathfrak{p} = r$. Suppose that $r < d$. We choose $\mathfrak{P} \in \operatorname{Ass}_{\widehat{R}} \widehat{R}/\mathfrak{p}\widehat{R}$ so that $\dim \widehat{R}/\mathfrak{P} = r$. Then $\mathfrak{P} \in \operatorname{Ass}_{\widehat{R}} \widehat{M}$, whence $\mathfrak{P} \in \operatorname{Att}_{\widehat{R}} \mathrm{H}^r_{\mathfrak{m}}(M)$ ([**BS**, 11.3.3]). Therefore $\dim_{\widehat{R}} \mathrm{H}^r_{\mathfrak{m}}(M) = r$ by Remark 2.1, so that $r \leq \mathrm{p}(M) \leq s$ by Proposition 2.3.

(2) By assertion (1) $x$ is $M$-regular in dimension $> s$; hence $\dim_R(0 :_M x) \leq s$. Consequently $\mathrm{H}^i_{\mathfrak{m}}(M) \cong \mathrm{H}^i_{\mathfrak{m}}(M/(0 :_M x))$ for all $i > s$, so that by the exact sequence

$$0 \to M/(0 :_M x) \xrightarrow{.x} M \to M/xM \to 0,$$

we have the exact sequence

$$0 \to \mathrm{H}^i_{\mathfrak{m}}(M)/x\mathrm{H}^i_{\mathfrak{m}}(M) \to \mathrm{H}^i_{\mathfrak{m}}(M/xM) \to (0 :_{\mathrm{H}^{i+1}_{\mathfrak{m}}(M)} x) \to 0$$

for all $i > s$. Because $\mathrm{p}(M) \leq s$, by Proposition 2.3 and the above exact sequence we get

$$\dim_{\widehat{R}} \mathrm{H}^i_{\mathfrak{m}}(M/xM) \leq s$$

for all $s < i < d-1$. Therefore by Remark 2.1

$$\dim_{\widehat{R}} \mathrm{H}^i_{\mathfrak{m}}(M/xM) \leq s$$

for all $i \leq s$, whence $\mathrm{p}(M/xM) \leq s$ by Proposition 2.3. □

We need the following fact in Section 4.

LEMMA 3.3. *With the notation of Setting* 2.8 *let* $i \in \{0, 1, \ldots, t\}$ *such that* $d_i \geq s$. *If* $x \in \mathfrak{m}$ *is* $M$*-regular in dimension* $> s$, *then* $D_i \cap xD_j = xD_i$ *for all* $j \leq i$.

PROOF. We may assume $j < i$. Let $m \in D_i \cap xD_j$ and we will show $m \in xD_i$. We write $m = xm'$ for some $m' \in D_j$ and set $\dim_R Rm' = r$. If $r = 0$, then $m' \in D_t \subseteq D_i$. Hence $m \in xD_i$. Assume that $r > 0$ and that $m' \notin D_i$; hence $d_i < r$. Then because $xm' \in D_i$, we get $\dim_R x(Rm') \leq d_i < r$, so that the exact sequence

$$0 \to x(Rm') \to Rm' \to Rm'/x(Rm') \to 0$$

shows $\dim_R Rm'/x(Rm') = r$. Since $\mathrm{Ass}_R Rm' \subseteq \mathrm{Ass}_R M$ and $x$ is $M$-regular in dimension $> s$, we have $x \notin \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Ass}_R Rm'$ with $\dim R/\mathfrak{p} > s$. Hence $\dim_R Rm'/x(Rm') = r - 1$ as $r > s$, which is a contradiction. Thus $m' \in D_i$, whence $D_i \cap xD_j = xD_i$. $\square$

Let us recall the notion of strict $M$-sequence in dimension $s > 0$.

Definition 3.4 ([**NH**]). An element $x \in \mathfrak{m}$ is called strictly $M$-regular in dimension $> s$, if $x \notin \mathfrak{p}$ for any $\mathfrak{p} \in \bigcup_{i=0}^{d} \mathrm{Att}_R \mathrm{H}^i_{\mathfrak{m}}(M)$ with $\dim R/\mathfrak{p} > s$. A sequence $x_1, x_2, \ldots, x_n$ of elements in $\mathfrak{m}$ is said to be a strict $M$-sequence in dimension $> s$, if for all $1 \le i \le n$ $x_i$ is strictly $M/(x_1, x_2, \ldots, x_{i-1})M$-regular in dimension $> s$.

A strict $M$-sequence in dimension $> s$ is naturally an $M$-sequence in dimension $> s$, because $\mathrm{Ass}_R N \subseteq \bigcup_{i=0}^{\ell} \mathrm{Att}_R \mathrm{H}^i_{\mathfrak{m}}(N)$ for every finitely generated $R$-module $N$ of dimension $\ell$ ([**BS**, 11.3.9]). Notice that the converse is not true in general. In fact, consider the formal power series ring $R = K[[x, y, z]]$ over a field $K$ and set $\mathfrak{p} = (x, y)$, $\mathfrak{m} = (x, y, z)$. We choose $M = \mathfrak{p}$. Then $\mathfrak{p} = (x, y) \in \mathrm{Att}_R H^2_{\mathfrak{m}}(M)$ and $x \in \mathfrak{p}$, so that $x$ is $M$-regular in dimension $> 0$ but not strictly $M$-regular in dimension $> 0$.

Assume that $R$ is a quotient of a Gorenstein local ring. Then the local duality theorem says that

$$\mathrm{H}^j_{\mathfrak{m}}(M) \cong \mathrm{Hom}_R(\mathrm{K}^j(M), \mathrm{E}_R(R/\mathfrak{m})),$$

where $\mathrm{E}_R(R/\mathfrak{m})$ denotes the injective hull of $R/\mathfrak{m}$. Hence

$$\mathrm{Ass}_R \mathrm{K}^j(M) = \mathrm{Att}_R \mathrm{H}^j_{\mathfrak{m}}(M)$$

for all $j \in \mathbb{Z}$ ([**BS**, 10.2.20]). We furthermore have the following.

Theorem 3.5. *Let $R$ be a quotient of a Gorenstein local ring. A sequence $x_1, x_2, \ldots, x_n$ of elements in $\mathfrak{m}$ is a strict $M$-sequence in dimension $> s$ if and only if it is a $\mathrm{K}^j(M)$-sequence in dimension $> s$ for all $0 \le j \le d$.*

Proof. Suppose that $x_1, x_2, \ldots, x_n$ is a strict $M$-sequence in dimension $> s$. Then for each $0 \le j \le d$ and $1 \le k \le n$ we have by [**NH**, Theorem 1.2 (ii)] and by the local duality theorem that

$$\begin{aligned}
&\left( \bigcup_{j=0}^{d} \mathrm{Att}_R \mathrm{H}^j_{\mathfrak{m}}(M/(x_1, x_2, \ldots, x_{k-1})M) \right)_{>s} \\
&\quad = \left( \bigcup_{j=0}^{d} \mathrm{Att}_R \big(0 :_{\mathrm{H}^{j+k-1}_{\mathfrak{m}}(M)} (x_1, x_2, \ldots, x_{k-1})R\big) \right)_{>s}
\end{aligned}$$

$$= \left( \bigcup_{j=0}^{d} \mathrm{Ass}_R \, \mathrm{K}^{j+k-1}(M)/(x_1, x_2, \ldots, x_{k-1})\mathrm{K}^{j+k-1}(M) \right)_{>s}.$$

Hence $x_1, x_2, \ldots, x_n$ is a $\mathrm{K}^j(M)$-sequence in dimension $> s$ for all $0 \le j \le d$.

Conversely, suppose that $x_1, x_2, \ldots, x_n$ is a $\mathrm{K}^j(M)$-sequence in dimension $> s$ for all $0 \le j \le d$. Then since $\mathrm{Ass}_R \, \mathrm{K}^j(M) = \mathrm{Att}_R \mathrm{H}^j_{\mathfrak{m}}(M)$ for all $j$, $x_1$ is strictly $M$-regular in dimension $> s$. Therefore similarly as above, thanks to [**NH**, Theorem 1.2 (ii)] and the local duality theorem, we get

$$\begin{aligned} \left( \bigcup_{j=0}^{d} \mathrm{Att}_R \mathrm{H}^j_{\mathfrak{m}}(M/x_1 M) \right)_{>s} &= \left( \bigcup_{j=0}^{d} \mathrm{Att}_R \big(0 :_{\mathrm{H}^{j+1}_{\mathfrak{m}}(M)} x_1\big) \right)_{>s} \\ &= \left( \bigcup_{j=0}^{d} \mathrm{Ass}_R \, \mathrm{K}^{j+1}(M)/x_1 \mathrm{K}^{j+1}(M) \right)_{>s}. \end{aligned}$$

Hence $x_2 \notin \mathfrak{p}$ for any $\mathfrak{p} \in \left( \bigcup_{j=0}^{d} \mathrm{Att}_R \mathrm{H}^j_{\mathfrak{m}}(M/x_1 M) \right)_{>s}$. Therefore $x_1, x_2$ is a strict $M$-sequence in dimension $> s$. Repeating this procedure, by [**NH**, Theorem 1.2 (ii)] we readily get that $x_1, x_2, \ldots, x_n$ is a strict $M$-sequence in dimension $> s$. $\square$

We are now interested in the question of what happens when $\mathrm{sp}(M) \le s$. We need the following result later in Section 4.

Proposition 3.6. *Let $R$ be a quotient of a Gorenstein ring and suppose that $\mathrm{sp}(M) \le s$. Let $x \in \mathfrak{m}$ be $M$-regular in dimension $> s$. Then the following assertions holds true.*

(1) *Either* $\dim R/\mathfrak{p} = j$ *or* $\dim R/\mathfrak{p} \le s$ *for all* $\mathfrak{p} \in \mathrm{Ass}_R \, \mathrm{K}^j(M)$ *and* $j \ge 0$.

(2) *$x$ is strictly $M$-regular in dimension $> s$ and* $\mathrm{sp}(M/xM) \le s$.

Proof. (1) Let $j \ge 0$ be an integer and $\mathfrak{p} \in \mathrm{Ass}_R \, \mathrm{K}^j(M)$. We set $\dim R/\mathfrak{p} = r$. Then $r \le j$ by Remark 2.1. If $j \notin \mathrm{D}(M)$, then by Theorem 2.11

$$r = \dim R/\mathfrak{p} \le \dim_R \mathrm{K}^j(M) \le \mathrm{sp}(M) \le s.$$

Suppose that $j \in \mathrm{D}(M)$. Then Theorem 2.11 shows that $\dim_R \mathrm{K}^j(M) = j$ and $\mathrm{p}(\mathrm{K}^j(M)) \le \mathrm{sp}(M) \le s$. Therefore $\mathfrak{p} \in (\mathrm{Ass}_R \, \mathrm{K}^j(M))_r$, so that by [**BS**, 11.3.3] $\mathfrak{p} \in \mathrm{Att}_R \mathrm{H}^r_{\mathfrak{m}}(\mathrm{K}^j(M))$. Suppose $r < j$. Then since $\dim_R \mathrm{K}^j(M) = j$,

$$s \ge \mathrm{p}(\mathrm{K}^j(M)) \ge \dim_{\widehat{R}} \mathrm{H}^r_{\mathfrak{m}}(\mathrm{K}^j(M)) = \dim_R \mathrm{H}^r_{\mathfrak{m}}(\mathrm{K}^j(M)) \ge \dim R/\mathfrak{p} = r$$

by Remark 2.1 and Proposition 2.3.

(2) Let $j \ge 0$ be an integer and choose $\mathfrak{p} \in (\mathrm{Ass}_R \, \mathrm{K}^j(M))_{>s}$. Then $\dim R/\mathfrak{p} = j$ by assertion (1). Hence $\mathfrak{p} \in (\mathrm{Ass}_R \, \mathrm{K}^j(M))_j$, so that $\mathfrak{p} \in (\mathrm{Ass}_R \, M)_j$ by [**Sch**, Proposition 3.2]. Since $j > s$ and $x$ is $M$-regular in dimension $> s$, we have $x \notin \mathfrak{p}$. Thus by Theorem 3.5 $x$ is strictly $M$-regular in dimension $> s$. Therefore $\dim_R(0 :_M x) \le s$, since $x$ is

$M$-regular in dimension $> s$. Hence

$$\mathrm{H}^j_{\mathfrak{m}}(M) \cong \mathrm{H}^j_{\mathfrak{m}}(M/(0 :_M x))$$

for all $j > s$, so that by the local duality theorem and the exact sequence

$$0 \to M/(0 :_M x) \xrightarrow{\cdot x} M \to M/xM \to 0$$

we get the exact sequence

$$(\sharp) \quad 0 \to \mathrm{K}^{j+1}(M)/x\mathrm{K}^{j+1}(M) \to \mathrm{K}^j(M/xM) \to (0 :_{\mathrm{K}^j(M)} x) \to 0$$

for all $j > s$. Let $j \le d-1$ be an integer such that $\dim_R \mathrm{K}^j(M/xM) > s$. Then $j > s$ by Remark 2.1. Because $x$ is strictly $M$-regular in dimension $> s$, it is $\mathrm{K}^j(M)$-regular in dimension $> s$, whence $\dim_R(0 :_{\mathrm{K}^j(M)} x) \le s$. Therefore exact sequence $(\sharp)$ above shows $\dim_R \mathrm{K}^{j+1}(M) > s$. Since $\mathrm{sp}(M) \le s$, it follows by Theorem 2.11 that $j+1 \in \mathrm{D}(M)$; hence $\dim_R \mathrm{K}^{j+1}(M) = j+1$. Since $j > s$, $x$ is a parameter of $\mathrm{K}^{j+1}(M)$, whence $\dim_R \mathrm{K}^{j+1}(M)/x\mathrm{K}^{j+1}(M) = j$. Therefore exact sequence $(\sharp)$ shows $j > s$, so that $\dim_R \mathrm{K}^j(M/xM) = j$. Thus $(\mathrm{Ass}_R(M/xM))_j \ne \emptyset$ and $j \in \mathrm{D}(M/xM)$. Because $j+1 \in \mathrm{D}(M)$, we get $\mathrm{p}(\mathrm{K}^{j+1}(M)) \le s$ by Theorem 2.11, whence $\mathrm{p}(\mathrm{K}^{j+1}(M)/x\mathrm{K}^{j+1}(M)) \le s$ by Proposition 3.2. Therefore Lemma 2.4 shows $\mathrm{p}(\mathrm{K}^j(M/xM)) \le s$. Thus $\mathrm{sp}(M/xM) \le s$ by Theorem 2.11. $\square$

We close this section with the following characterization for $M$ to have $\mathrm{p}(M) \le s$.

THEOREM 3.7. *Assume that $R$ is a quotient of a Cohen–Macaulay local ring. Then the following conditions are equivalent.*

(1) $\mathrm{p}(M) \le s$.

(2) *Every system of parameters of $M$ is a strict $M$-sequence in dimension $> s$.*

(3) *Every system of parameters of $M$ is an $M$-sequence in dimension $> s$.*

PROOF. (1) $\Rightarrow$ (2) We assume that $d = \dim_R M > 0$ and that our implication holds true for $d-1$. Let $x_1, x_2, \ldots, x_d$ be a system of parameters of $M$. Then by Proposition 3.2 $x_1$ is $M$-regular in dimension $> s$ and $\mathrm{p}(M/x_1M) \le s$. Hence $x_1$ is $\widehat{M}$-regular in dimension $> s$. Because $\mathrm{sp}(\widehat{M}) = \mathrm{sp}(M)$ ([**NDC**, Theorem 3.5]), we get by hypothesis (1) and Proposition 2.9 that $\mathrm{sp}(\widehat{M}) \le s$. Therefore by Proposition 3.6 $x_1$ is strictly $\widehat{M}$-regular in dimension $> s$, so that $x_1$ is strictly $M$-regular in dimension $> s$. Because $\mathrm{p}(M/x_1M) \le s$, the hypothesis of induction shows $x_2, x_3, \ldots, x_d$ is a strict $M/x_1M$-sequence in dimension $> s$, whence $x_1, x_2, \ldots, x_d$ is a strict $M$-sequence in dimension $> s$.

(2) $\Rightarrow$ (3) This is clear; see [**BS**, 11.3.9].

(3) $\Rightarrow$ (1) Assume that $d > 0$ and that the implication holds true for $d-1$. Therefore for every parameter $x \in \mathfrak{m}$ of $M$, every system of parameters of $M/xM$ is an $M/xM$-sequence in dimension $> s$, so that $\mathrm{p}(M/xM) \le s$. We must show $\mathrm{p}(M) \le s$. Assume the contrary. Then since $R$ is a quotient of a Cohen–Macaulay ring, by Remark 2.1 and

Proposition 2.3 we have $i \geq \dim_R \mathrm{H}^i_{\mathfrak{m}}(M) > s$ for some $i < d$. Set $\dim_R \mathrm{H}^i_{\mathfrak{m}}(M) = r$. Then since $r < d$, we can choose a parameter $x$ of $M$ so that $x \in \mathrm{Ann}_R \mathrm{H}^i_{\mathfrak{m}}(M)$. Because $x \notin \mathfrak{q}$ for all $\mathfrak{q} \in (\mathrm{Ass}_R M)_{>s}$ by the hypothesis of induction, we get $\dim_R(0 :_M x) \leq s$, whence $\mathrm{H}^j_{\mathfrak{m}}(M) \cong \mathrm{H}^j_{\mathfrak{m}}(M/(0 :_M x))$ for all $j > s$. Therefore because $i \geq r > s$, the exact sequence

$$0 \to M/(0 :_M x) \xrightarrow{\cdot x} M \to M/xM \to 0$$

gives rise to the exact sequence

$$\mathrm{H}^{i-1}_{\mathfrak{m}}(M) \to \mathrm{H}^{i-1}_{\mathfrak{m}}(M/xM) \to (0 :_{\mathrm{H}^i_{\mathfrak{m}}(M)} x) \to 0.$$

We now notice that $(0 :_{\mathrm{H}^i_{\mathfrak{m}}(M)} x) = \mathrm{H}^i_{\mathfrak{m}}(M)$ by our choice of $x$ and get

$$\dim_R \mathrm{H}^{i-1}_{\mathfrak{m}}(M/xM) \geq \dim_R \mathrm{H}^i_{\mathfrak{m}}(M) = r > s.$$

Hence $\mathrm{p}(M/xM) \geq r > s$ by Proposition 2.3 because $i - 1 < d - 1$, which is a required contradiction. $\square$

## 4. A parametric characterization.

The purpose of this section is to prove Theorem 1.2. To begin with let us recall the following.

Definition 4.1 ([**Sch**]). A system of parameters $x_1, x_2, \ldots, x_d$ of $M$ is called a distinguished system of parameters of $M$, if $(x_{d_i+1}, x_{d_i+2} \ldots, x_d)D_i = 0$ for all $1 \leq i \leq t$.

Thanks to the prime avoidance theorem, distinguished systems of parameters of $M$ exist. Notice that if $x_1, x_2, \ldots, x_d$ is a distinguished system of parameters of $M$, then $x_1, x_2, \ldots, x_{d_i}$ is a distinguished system of parameters of $D_i$ for all $0 \leq i \leq t-1$.

Let us start from the following.

Theorem 4.2. *Let $R$ be a quotient of a Gorenstein local ring and let $s \geq 0$ be an integer. If $\mathrm{sp}(M) \leq s$, then every distinguished system of parameters of $M$ is a strict $M$-sequence in dimension $> s$.*

Proof. Suppose that $\mathrm{sp}(M) \leq s$. Let $x_1, x_2, \ldots, x_d$ be a distinguished system of parameters of $M$. For each $1 \leq n \leq d$ we set $M^{(n)} = M/(x_1, x_2, \ldots, x_n)M$. Let $H^{(n)}$ be the largest $R$-submodule of $M^{(n)}$ of dimension at most $s$. Let

$$H^{(n)} = H^{(n)}_{r_n} \subset \cdots \subset H^{(n)}_1 \subset H^{(n)}_0 = M^{(n)}$$

be the dimension filtration of $M^{(n)}$ in dimension $> s$ and let

$$D(s) = D_{t(s)} \subset \cdots \subset D_1 \subset D_0 = M$$

be the dimension filtration of $M$ in dimension $> s$. For each integer $i \leq t(s)$ we set

$$D_i^{(n)} = (D_i + (x_1, x_2, \dots, x_n)M)/(x_1, x_2, \dots, x_n)M$$

and $r_0 = t(s)$. We will prove by induction on $n$ ($1 \le n \le d$) that the following assertions hold true.

(a) $x_1, x_2, \dots, x_n$ is a strict $M$-sequence in dimension $> s$.

(b) $\mathrm{sp}(M/(x_1, x_2, \dots, x_n)M) \le s$.

(c) $D_i^{(n)} \subseteq H_i^{(n)}$, $\dim_R H_i^{(n)}/D_i^{(n)} \le s$, and $\dim_R H_{i-1}^{(n)} = \dim_R D_{i-1}^{(n)} = d_{i-1} - n$ for all $i \le r_n$.

(d) $r_n = r_{n-1}$ if $\dim_R D_{r_{n-1}-1}^{(n-1)} > s+1$ and $r_n = r_{n-1} - 1$ if $\dim_R D_{r_{n-1}-1}^{(n-1)} = s+1$.

Step 1: Let $n = 1$. Notice that $(\mathrm{Ass}_R M)_{>s} = \bigcup_{i=0}^{r_0-1} (\mathrm{Ass}_R D_i)_{d_i}$. Since $x_1, x_2, \dots, x_d$ is a distinguished system of parameters of $M$, $x_1$ is a parameter of $D_i$ for all $i$ such that $d_i > 0$. As $s \ge 0$, $x_1$ is $M$-regular in dimension $> s$. By Proposition 3.6 (2), $x_1$ is strictly $M$-regular in dimension $> s$ and $\mathrm{sp}(M/x_1M) \le s$. Let $i \le r_0 - 1$. Then $d_i > s$. Since $s \ge 0$ and $D_{i+1} \subseteq D_i$, we get by Lemma 3.3

$$\dim_R D_i^{(1)} = \dim_R D_i/(D_i \cap x_1M) = \dim_R D_i/x_1D_i = d_i - 1$$

and

$$\begin{aligned} D_i^{(1)}/D_{i+1}^{(1)} &\cong (D_i + x_1M)/(D_{i+1} + x_1M) \\ &\cong D_i/\left(D_i \cap (D_{i+1} + x_1M)\right) \\ &\cong D_i/(D_{i+1} + x_1D_i) \\ &\cong (D_i/D_{i+1})/x_1(D_i/D_{i+1}). \end{aligned}$$

On the other hand, because $\mathrm{sp}(M) \le s$, we have $\mathrm{p}(D_i/D_{i+1}) \le s$. As $d_i > s$ and $s \ge 0$, $x_1$ is a parameter of $D_i/D_{i+1}$. Hence $\mathrm{p}(D_i^{(1)}/D_{i+1}^{(1)}) \le s$, thanks to the above isomorphisms and Proposition 3.2. We now consider the filtration

$$H^{(1)} = H^{(1)} + D_p^{(1)} \subset \cdots \subset H^{(1)} + D_1^{(1)} \subset D_0^{(1)} = M/x_1M,$$

where $p = r_0 - 1$ if $d_{r_0-1} = s+1$ and $p = r_0$ if $d_{r_0-1} > s+1$. Then since $\dim_R H^{(1)} \le s$ and $\mathrm{p}(D_i^{(1)}/D_{i+1}^{(1)}) \le s$ for each $i \le p-1$, it follows by Lemma 2.4 that

$$\mathrm{p}\big((D_i^{(1)} + H^{(1)})/(D_{i+1}^{(1)} + H^{(1)})\big) \le s.$$

Consequently Proposition 2.10 shows that $r_1 = p$ and that $D_i^{(1)} \subseteq H_i^{(1)}$ possessing

$$\dim_R H_i^{(1)}/(D_i^{(1)} + H^{(1)}) \le s$$

for all $i \le r_1$. Therefore since $\dim_R H^{(1)} \le s$, we have

$$\dim_R H_i^{(1)}/D_i^{(1)} \le s$$

for all $i \le r_1$. If $i \le r_1 - 1$, then $\dim_R H_i^{(1)} > s$ and hence

$$\dim_R H_i^{(1)} = \dim_R D_i^{(1)} = d_i - 1.$$

Thus the assertion follows.

Step 2: Let $n \ge 2$ and assume that our assertion holds true for $n-1$. Hence

(a′) $x_1, x_2, \ldots, x_{n-1}$ is a strict $M$-sequence in dimension $> s$.

(b′) $\operatorname{sp}(M/(x_1, x_2, \ldots, x_{n-1})M) \le s$.

(c′) $D_i^{(n-1)} \subseteq H_i^{(n-1)}$, $\dim_R H_i^{(n-1)}/D_i^{(n-1)} \le s$, and $\dim_R H_{i-1}^{(n-1)} = \dim_R D_{i-1}^{(n-1)} = d_{i-1} - n + 1$ for all $i \le r_{n-1}$.

(d′) $r_{n-1} = r_{n-2}$ if $\dim_R D_{r_{n-2}-1}^{(n-2)} > s+1$ and $r_{n-1} = r_{n-2} - 1$ if $\dim_R D_{r_{n-2}-1}^{(n-2)} = s+1$.

Let $i \le r_{n-1} - 1$. Then $\dim_R H_i^{(n-1)} = \dim_R D_i^{(n-1)} = d_i - n + 1$. Since $\dim_R H_i^{(n-1)} > s$ and $\dim_R H_i^{(n-1)}/D_i^{(n-1)} \le s$, we have $d_i - n + 1 > s$ and

$$\bigl(\operatorname{Ass}_R H_i^{(n-1)}\bigr)_{d_i-n+1} = \bigl(\operatorname{Ass}_R D_i^{(n-1)}\bigr)_{d_i-n+1}.$$

Since $s \ge 0$, we have $d_i > n-1$. Notice that $x_1, x_2, \ldots, x_{d_i}$ is a system of parameters of $D_i$; hence $x_n$ is a parameter of $D_i/(x_1, x_2, \ldots, x_{n-1})D_i$. Therefore because

$$\dim_R D_i^{(n-1)} = d_i - n + 1 = \dim_R D_i/(x_1, x_2, \ldots, x_{n-1})D_i$$

and $D_i^{(n-1)}$ is a quotient module of $D_i/(x_1, \ldots, x_{n-1})D_i$, the element $x_n$ is a parameter of $D_i^{(n-1)}$. Hence $x_n \notin \mathfrak{p}$ for any $\mathfrak{p} \in (\operatorname{Ass}_R D_i^{(n-1)})_{d_i-n+1}$, so that $x_n \notin \mathfrak{p}$ for any $\mathfrak{p} \in (\operatorname{Ass}_R H_i^{(n-1)})_{d_i-n+1}$. On the other hand, since $\dim_R H_i^{(n-1)} = d_i - n + 1 > s$ for all $i \le r_{n-1} - 1$ and $\dim_R H_{r_{n-1}}^{(n-1)} \le s$, we have

$$\Bigl(\operatorname{Ass}_R M^{(n-1)}\Bigr)_{>s} = \bigcup_{i=0}^{r_{n-1}-1} \Bigl(\operatorname{Ass}_R H_i^{(n-1)}\Bigr)_{d_i-n+1},$$

whence $x_n$ is $M^{(n-1)}$-regular in dimension $> s$. Therefore since $\operatorname{sp}(M^{(n-1)}) \le s$ by assertion (b′), Proposition 3.6 shows that $x_n$ is strictly $M^{(n-1)}$-regular in dimension $> s$. Consequently by assertion (a′) $x_1, x_2, \ldots, x_n$ is a strict $M$-sequence in dimension $> s$, whence assertion (a) follows. Thanks to Step 1 and assertions (b′), (c′), and (d′) for the module $M^{(n-1)}$, we readily get assertions (b), (c), and (d). Thus $x_1, x_2, \ldots, x_d$ is a strict $M$-sequence in dimension $> s$. □

The following is the main result of this paper, which gives a parametric characterization for $M$ to have $\operatorname{sp}(M) \le s$.

THEOREM 4.3. *Let $R$ be a quotient of a Cohen–Macaulay local ring and let $s \ge 0$ be an integer. Then $\operatorname{sp}(M) \le s$ if and only if for all $1 \le i \le t$ every distinguished system of parameters of $M/D_i$ is a strict $M/D_i$-sequence in dimension $> s$.*

PROOF. Assume that $\mathrm{sp}(M) \leq s$. Let $x_1, x_2, \ldots, x_d$ be a distinguished system of parameters of $M$. Then since $R$ is a quotient of a Cohen–Macaulay local ring, $\mathrm{sp}(\widehat{M}) = \mathrm{sp}(M) \leq s$ and

$$\mathrm{H}^0_{\mathfrak{m}}(M) = \widehat{D_t} \subset \cdots \subset \widehat{D_1} \subset \widehat{D_0} = \widehat{M}$$

is the dimension filtration of $\widehat{M}$ ([**NDC**, Theorem 3.5]). Therefore $x_1, x_2, \ldots, x_d$ is a distinguished system of parameters of $\widehat{M}$. Hence by Theorem 4.2 $x_1, x_2, \ldots, x_d$ is a strict $\widehat{M}$-sequence in dimension $> s$. Let $i \in \{1, 2, \ldots, d\}$ and $\mathfrak{p} \in \big(\mathrm{Att}_R \mathrm{H}^j_{\mathfrak{m}}(M/(x_1, x_2, \ldots, x_{i-1})M)\big)_{>s}$. Choose $\mathfrak{P} \in \mathrm{Ass}_{\widehat{R}} \widehat{R}/\mathfrak{p}\widehat{R}$ so that $\dim \widehat{R}/\mathfrak{P} = \dim R/\mathfrak{p}$. Then by [**NQ**, Theorem 1.1] (see Remark 2.1 also)

$$\mathfrak{P} \in \big(\mathrm{Att}_{\widehat{R}} \mathrm{H}^j_{\mathfrak{m}}(M/(x_1, x_2, \ldots, x_{i-1})M)\big)_{>s} = \big(\mathrm{Att}_{\widehat{R}} \mathrm{H}^j_{\mathfrak{m}\widehat{R}}(\widehat{M}/(x_1, x_2, \ldots, x_{i-1})\widehat{M})\big)_{>s},$$

whence $x_i \notin \mathfrak{P}$. Thus $x_i \notin \mathfrak{p}$, as $\mathfrak{p} = \mathfrak{P} \cap R$. Hence $x_1, x_2, \ldots, x_d$ is a strict $M$-sequence in dimension $> s$.

Let $i \in \{1, 2, \ldots, t\}$. Then since $\mathrm{sp}(M) \leq s$ and

$$0 \subset D_{i-1}/D_i \subset \cdots \subset D_1/D_i \subset M/D_i$$

is the dimension filtration of $M/D_i$, we get $\mathrm{sp}(M/D_i) \leq s$. Hence every distinguished system of parameters of $M/D_i$ is a strict $M/D_i$-sequence in dimension $> s$.

Conversely, assume that for all $1 \leq i \leq t$ every distinguished system of parameters of $M/D_i$ is a strict $M/D_i$-sequence in dimension $> s$. Suppose that $\mathrm{sp}(M) > s$ and we will produce a contradiction. Since $\mathrm{sp}(M) > s \geq 0$ and $\dim_R D_t \leq 0$, there exists $k \in \{1, 2, \ldots, t\}$ such that $\mathrm{p}(D_{k-1}/D_k) > s$. Remember that $R$ is a quotient of a Cohen–Macaulay local ring and we get by Proposition 2.3 and Remark 2.1 that

$$j \geq \dim_R \mathrm{H}^j_{\mathfrak{m}}(D_{k-1}/D_k) > s$$

for some integer $j < d_{k-1}$. We consider the exact sequence

$$\mathrm{H}^{j-1}_{\mathfrak{m}}(M/D_{k-1}) \to \mathrm{H}^j_{\mathfrak{m}}(D_{k-1}/D_k) \to \mathrm{H}^j_{\mathfrak{m}}(M/D_k)$$

induced from the exact sequence

$$0 \to D_{k-1}/D_k \to M/D_k \to M/D_{k-1} \to 0.$$

Then since $\dim_R \mathrm{H}^j_{\mathfrak{m}}(M/D_k) > s$ or $\dim_R \mathrm{H}^{j-1}_{\mathfrak{m}}(M/D_{k-1}) > s$, there exists $\mathfrak{p} \in \mathrm{Att}_R \mathrm{H}^j_{\mathfrak{m}}(M/D_k)$ or $\mathfrak{p} \in \mathrm{Att}_R \mathrm{H}^{j-1}_{\mathfrak{m}}(M/D_{k-1})$ such that $\dim R/\mathfrak{p} > s$. If $\mathfrak{p} \in \mathrm{Att}_R \mathrm{H}^j_{\mathfrak{m}}(M/D_k)$, then $d_{k-1} > j \geq \dim R/\mathfrak{p}$ by Remark 2.1. If $\mathfrak{p} \in \mathrm{Att}_R \mathrm{H}^{j-1}_{\mathfrak{m}}(M/D_{k-1})$, then $d_{k-1} > j > j-1 \geq \dim R/\mathfrak{p}$. So, in any case we have $\dim R/\mathfrak{p} < d_{k-1}$. Therefore, thanks to the prime avoidance theorem, there exists a part $x_{d_{k-1}+1}, x_{d_{k-1}+2}, \ldots, x_d$ of a system of parameters of $M$ such that $(x_{d_i+1}, x_{d_i+2}, \ldots, x_d)D_i = 0$ for all $1 \leq i \leq k-1$, while there exists a system $x_1, x_2, \ldots, x_{d_{k-1}}$ of parameters of $M/(x_{d_{k-1}+1}, x_{d_{k-1}+2}, \ldots, x_d)M$ such that $x_1 \in \mathfrak{p}$. Notice that since

$$0 \subset D_{k-1}/D_k \subset \cdots \subset D_1/D_k \subset M/D_k$$

is the dimension filtration of $M/D_k$, $x_1, x_2, \ldots, x_d$ is a distinguished system of parameters of $M/D_k$. We similarly have that $x_1, x_2, \ldots, x_d$ is a distinguished system of parameters of $M/D_{k-1}$. Therefore by our assumption $x_1, x_2, \ldots, x_d$ is a strict $M/D_k$-sequence and $M/D_{k-1}$-sequence in dimension $> s$, which is impossible since $x_1 \in \mathfrak{p}$. $\square$

## 5. The sequential polynomial type of some specific rings and modules.

In this section we study the sequential polynomial type of certain specific rings and modules. Our goal is Theorem 5.7 below.

Firstly we study the sequential polynomial type of finite direct sums. Let $\{M_i\}_{1\le i\le n}$ be a family of finitely generated $R$-modules. If each $M_i$ is sequentially Cohen–Macaulay (resp. sequentially generalized Cohen–Macaulay), then so is $\bigoplus_{i=1}^n M_i$ ([**CN**, Proposition 4.5]). We generalize this result in the following form.

Lemma 5.1. *Let $\{M_i\}_{1\le i\le n}$ be a family of finitely generated $R$-modules.*

(1) $\mathrm{p}(\bigoplus_{i=1}^n M_i) = \max_{1\le i\le n} \mathrm{p}(M_i)$, *if* $\dim_R M_1 = \cdots = \dim_R M_n$.

(2) $\mathrm{sp}(\bigoplus_{i=1}^n M_i) = \max_{1\le i\le n} \mathrm{sp}(M_i)$.

Proof. We may assume $n = 2$. Set $M = M_1$ and $N = M_2$. Assertion (1) is a direct consequence of Proposition 2.3, since $\mathrm{H}^i_{\mathfrak{m}}(M\oplus N) \cong \mathrm{H}^i_{\mathfrak{m}}(M)\oplus \mathrm{H}^i_{\mathfrak{m}}(N)$ for all $i \ge 0$. To prove assertion (2), we may assume that $d = \dim M \ge \dim N$. Let

$$\mathrm{H}^0_{\mathfrak{m}}(M)\oplus \mathrm{H}^0_{\mathfrak{m}}(N) = D_t \subset \cdots \subset D_1 \subset D_0 = M\oplus N$$

be the dimension filtration of $M\oplus N$. Set $\dim_R D_i = d_i$ for each $i$. Then

$$\mathrm{D}(M\oplus N) = \mathrm{D}(M)\cup \mathrm{D}(N) = \{d_0, d_1, \ldots, d_t\},$$

where $d_t \le 0$ and $d_0 = d$. If $t = 0$, then $d \le 0$, whence $\mathrm{sp}(M\oplus N) = \max\{\mathrm{sp}(M), \mathrm{sp}(N)\} = -1$. Suppose $t = 1$. Then $d > 0$ and $\dim_R D_1 \le 0$. Therefore by Proposition 2.3 we have

$$\begin{aligned}\mathrm{sp}(M\oplus N) = \mathrm{p}((M\oplus N)/D_1) &= \max_{i>0} \dim_{\widehat{R}} \mathrm{H}^i_{\mathfrak{m}}(M\oplus N)\\ &= \max_{i>0}\{\dim_{\widehat{R}} \mathrm{H}^i_{\mathfrak{m}}(M), \dim_{\widehat{R}} \mathrm{H}^i_{\mathfrak{m}}(N)\}\\ &= \max\{\mathrm{p}(M/\mathrm{H}^0_{\mathfrak{m}}(M)), \mathrm{p}(N/\mathrm{H}^0_{\mathfrak{m}}(N))\}.\end{aligned}$$

Notice that $\mathrm{D}(M)\cup\mathrm{D}(N) \subseteq \{d, 0, -1\}$. As $t = 1$ and $d > 0$, we readily get $\mathrm{sp}(M) = \mathrm{p}(M/\mathrm{H}^0_{\mathfrak{m}}(M))$. Moreover, if $\dim N > 0$, then $\mathrm{sp}(N) = \mathrm{p}(N/\mathrm{H}^0_{\mathfrak{m}}(N))$. If $\dim N \le 0$, then $N/\mathrm{H}^0_{\mathfrak{m}}(N) = 0$, so that $\mathrm{sp}(N) = -1 = \mathrm{p}(N/\mathrm{H}^0_{\mathfrak{m}}(N))$.

Let $t > 1$ and assume that our assertion holds true for $t-1$. Notice that $D_i = E_i \oplus F_i$ for all $i \in \{0, 1, \ldots, t\}$, where $E_i$ (resp. $F_i$) is the largest submodule of $M$ (resp. $N$) of dimension at most $d_i$. We set $M' = M/E_{t-1}$ and $N' = N/F_{t-1}$. Then

$$\mathrm{D}(M' \oplus N') = \{-1, d_{t-2}, \ldots, d_1, d_0 = d\},$$

so that

$$\mathrm{sp}((M \oplus N)/D_{t-1}) = \mathrm{sp}(M' \oplus N') = \max\{\mathrm{sp}(M'), \mathrm{sp}(N')\}$$

by the hypothesis of induction, while because $D_{t-1} = E_{t-1} \oplus F_{t-1}$ and $D_t \subset D_{t-1}$ is the dimension filtration of $D_{t-1}$, we have

$$\mathrm{sp}(D_{t-1}) = \max\{\mathrm{sp}(E_{t-1}), \mathrm{sp}(F_{t-1})\}$$

by the case where $t = 1$. Thus

$$\mathrm{sp}(M \oplus N) = \max\{\mathrm{sp}(M), \mathrm{sp}(N)\}$$

as wanted. $\square$

Assume that $R$ is a subring of a Noetherian local ring $S$ such that $S$ is a finitely generated $R$-module. Let $M$ be a finitely generated $S$-module. Then $M$ is a finitely generated $R$-module with $\dim_S M = \dim_R M$ and the dimension filtration of the $S$-module $M$ is also the dimension filtration of the $R$-module $M$ (cf. e.g., [**TPDA**, Section 3]). Hence

$$\mathrm{sp}_R(M) = \mathrm{sp}_S(M),$$

where $\mathrm{sp}_R(M)$ (resp. $\mathrm{sp}_S(M)$) denotes the sequential polynomial type of $M$ as an $R$-module (resp. as an $S$-module). Therefore by Lemma 5.1 we readily get the following.

Proposition 5.2. *Let $S$ be a Noetherian local ring and let $R$ be a subring of $S$ such that $S$ is a finitely generated $R$-module. If $R$ is a direct summand of $S$ as an $R$-module, then* $\mathrm{sp}(S) \geq \mathrm{sp}(R)$.

Let $S$ be a Noetherian local ring and let $G$ be a finite subgroup of $\mathrm{Aut}\, S$. Assume that the order $g = |G|$ of $G$ is invertible in $S$ and let $R = S^G$ denote the ring of invariants. Then $R$ is a Noetherian local ring and $S$ is a module-finite extension of $R$ (see [**Br**] and reduce the general case to the case where $S$ is a reduced ring) such that $R$ is a direct summand of $S$. Hence we have the following.

Corollary 5.3. *Suppose that $S$ is a Noetherian local ring and $G$ is a finite subgroup of* $\mathrm{Aut} S$ *such that the order of $G$ is invertible in $S$. Then* $\mathrm{sp}(S) \geq \mathrm{sp}(S^G)$.

Secondly, we study the sequential polynomial type of canonical modules.

Proposition 5.4. *Suppose that $R$ is a quotient of a Gorenstein local ring. Denote by* $\mathrm{K}(M)$ *the canonical module of $M$. Then* $\mathrm{p}(\mathrm{K}(M)) \leq \mathrm{p}(M)$ *and* $\mathrm{sp}(\mathrm{K}(M)) \leq \mathrm{sp}(M)$.

Proof. Set $d = \dim_R M$ and $N = M/D_1$, where $D_1$ is the largest $R$-submodule of $M$ with $\dim_R D_1 < d$. Then $\mathrm{K}(M) \cong \mathrm{K}(N)$ and $\mathrm{Ass}_R \mathrm{K}(N) = (\mathrm{Ass}_R M)_d$. Let $\mathfrak{p} \in \mathrm{nCM}(\mathrm{K}(N))$; hence $(\mathrm{K}(N))_{\mathfrak{p}}$ is not a Cohen–Macaulay $R_{\mathfrak{p}}$-module. Choose $\mathfrak{q} \in$

$\operatorname{Ass}_R \mathrm{K}(N)$ so that $\mathfrak{q} \subseteq \mathfrak{p}$. Then since $\dim R/\mathfrak{q} = d$ and the ring $R$ is catenary, we have

$$d = \dim R/\mathfrak{p} + \dim_{R_\mathfrak{p}} N_\mathfrak{p}.$$

Hence $(\mathrm{K}(N))_\mathfrak{p} \cong \mathrm{K}(N_\mathfrak{p})$, so that $\mathrm{K}(N_\mathfrak{p})$ is not a Cohen–Macaulay $R_\mathfrak{p}$-module. Consequently $N_\mathfrak{p}$ cannot be Cohen–Macaulay, whence $\mathfrak{p} \in \mathrm{nCM}(N)$, which shows

$$\mathrm{nCM}(\mathrm{K}(N)) \subseteq \mathrm{nCM}(N).$$

Therefore because the $R$-module $\mathrm{K}(N)$ is equi-dimensional, by Proposition 2.3 we get

$$\begin{aligned} \mathrm{sp}(\mathrm{K}(M)) = \mathrm{sp}(\mathrm{K}(N)) = \mathrm{p}(\mathrm{K}(N)) &= \dim_R \mathrm{nCM}(\mathrm{K}(N)) \\ &\leq \dim_R \mathrm{nCM}(N) \leq \mathrm{p}(N) \leq \mathrm{sp}(M). \end{aligned}$$

Thus $\mathrm{p}(\mathrm{K}(M)) = \mathrm{sp}(\mathrm{K}(M)) \leq \mathrm{sp}(M) \leq \mathrm{p}(M)$, which proves the assertion. $\square$

We now study the sequential polynomial type of polynomial extensions and formal power series extensions. We begin with the following.

Lemma 5.5. *Let $T = R[x_1, x_2, \ldots, x_r]$ be the polynomial ring. Set $\mathfrak{n} = (\mathfrak{m}, x_1, x_2, \ldots, x_r)$ in $T$. If $\mathrm{sp}(R) \geq 0$, then $\mathrm{sp}(T_\mathfrak{n}) = \mathrm{sp}(R) + r$.*

Proof. We may assume $r = 1$. Let $x_1 = x$ and $S = (R[x])_\mathfrak{n}$, where $\mathfrak{n} = (\mathfrak{m}, x)$. Then the canonical map $R \to S$ is a flat local homomorphism and every fiber ring $S/\mathfrak{p}S$ with $\mathfrak{p} \in \operatorname{Spec} R$ is an integral domain. Hence the dimension filtration $\mathrm{H}^0_\mathfrak{m}(M) = D_t \subset \cdots \subset D_1 \subset D_0 = R$ of $R$ induces the dimension filtration

$$\mathrm{H}^0_\mathfrak{m}(R) \otimes_R S = D_t \otimes_R S \subset \cdots \subset D_1 \otimes_R S \subset D_0 \otimes_R S = S$$

of $S$ (cf. e.g., [**GHS**, Proposition 2.6]). We need the following.

Claim 1. *Let $k \in \{0, 1, \ldots, t-1\}$. If $\mathrm{p}(D_k/D_{k+1}) \geq 0$, then $\mathrm{p}((D_k/D_{k+1}) \otimes_R S) = \mathrm{p}(D_k/D_{k+1}) + 1$.*

Proof of Claim. Let $L = D_k/D_{k-1}$. Then $\dim_R L = d_k$. Since $S/\mathfrak{m}S$ is a Cohen–Macaulay ring, there exists a system $a_1, a_2, \ldots, a_{d_k}$ of parameters of $L$ such that $x, a_1, a_2, \ldots, a_{d_k}$ is a system of parameters of the $S$-module $L \otimes_R S$ and

$$\begin{gathered} \mathrm{e}(x^n, a_1^{n_1}, a_2^{n_2}, \ldots, a_{d_k}^{n_{d_k}}; L \otimes_R S) = n \cdot \mathrm{e}(x; S/\mathfrak{m}S) \cdot \mathrm{e}(a_1^{n_1}, a_2^{n_2}, \ldots, a_{d_k}^{n_{d_k}}; L); \\ \ell_S\bigl(L \otimes_R S/(x^n, a_1^{n_1}, a_2^{n_2}, \ldots, a_{d_k}^{n_{d_k}}) L \otimes_R S\bigr) = n \cdot \mathrm{e}(x; S/\mathfrak{m}S) \cdot \ell_R(L/a_1^{n_1}, a_2^{n_2}, \ldots, a_{d_k}^{n_{d_k}} L) \end{gathered}$$

for all positive integers $n, n_1, \ldots, n_{d_k}$ ([**CCT**, Lemmas 6.2, 6.3]). Then the difference

$$\ell_R(L/(a_1^{n_1}, a_2^{n_2}, \ldots, a_{d_k}^{n_{d_k}})L) - \mathrm{e}(a_1^{n_1}, a_2^{n_2}, \ldots, a_{d_k}^{n_{d_k}}; L)$$

considering as a function in $n_1, n_2, \ldots, n_{d_k}$ is bounded above by polynomials and the least degree of those polynomials is $\mathrm{p}(L)$ ([**C1**, Theorem 2.3]). Since $\mathrm{p}(L) \geq 0$, the difference

$$\ell_R(L \otimes_R S/(x^n, a_1^{n_1}, a_2^{n_2}, \dots, a_{d_k}^{n_{d_k}})(L \otimes_R S)) - \mathrm{e}(x^n, a_1^{n_1}, a_2^{n_2}, \dots, a_{d_k}^{n_{d_k}}; L \otimes_R S)$$

considering as a function in $n, n_1, n_2, \dots, n_{d_k}$ is bounded above by polynomials and the least degree of such polynomials is equal to $\mathrm{p}(L) + 1$. Thus

$$\mathrm{p}(L \otimes_R S) = \mathrm{p}(L) + 1$$

as claimed. $\square$

Let $X = \{k \le t \mid \mathrm{p}(D_{k-1}/D_k) \ge 0\}$. Then since $\mathrm{sp}(R) \ge 0$, $X \neq \emptyset$ and therefore

$$\begin{aligned} \mathrm{sp}(S) = \max_{1 \le k \le t} \mathrm{p}\big((D_{k-1} \otimes_R S)/(D_k \otimes_R S)\big) &= \max_{k \in X} \mathrm{p}\big((D_{k-1}/D_k) \otimes_R S\big) \\ &= \max_{k \in X} \mathrm{p}(D_{k-1}/D_k) + 1 = \mathrm{sp}(R) + 1 \end{aligned}$$

by Claim 1. $\square$

Let us use Proposition 5.5 to study the sequential polynomial type of the formal power series ring $R[[x_1, x_2, \dots, x_r]]$.

PROPOSITION 5.6. *Suppose that $R$ is a quotient of a Cohen–Macaulay local ring. If $\mathrm{sp}(R) \ge 0$, then $\mathrm{sp}(R[[x_1, x_2, \dots, x_r]]) = \mathrm{sp}(R) + r$.*

PROOF. We may assume $r = 1$. Since $R$ is a quotient of a Cohen–Macaulay local ring, we get $\mathrm{sp}(R) = \mathrm{sp}(\widehat{R})$ and $\mathrm{sp}(R[[x]]) = \mathrm{sp}(\widehat{R}[[x]])$ ([**NDC**, Theorem 3.5]). Let $\mathfrak{n} = (\mathfrak{m}, x)$ be the maximal ideal of $R[x]$. We then have extensions

$$R[x] \subset (R[x])_{\mathfrak{n}} \subset R[[x]]$$

of rings and when $R$ is a complete local ring, $R[[x]]$ is exactly the completion of the local ring $(R[x])_{\mathfrak{n}}$. Hence $\mathrm{sp}(R[[x]]) = \mathrm{sp}(\widehat{R}) + r = \mathrm{sp}(R) + r$ by Lemma 5.5. $\square$

Suppose that $R$ is a quotient of a Gorenstein local ring and let $M$ be a finitely generated $R$-module. We write $\mathrm{D}(M) = \{d_0, d_1, \dots, d_t\}$ and set

$$q_1 = \max_{j \notin \mathrm{D}(M)} \dim \mathrm{K}^j(M) \quad \text{and} \quad q_2 = \max_{j \in \mathrm{D}(M)} \mathrm{p}(\mathrm{K}^j(M)).$$

We then have by [**NDC**, Theorem 4.7] that

$$\mathrm{sp}(M) = \max\{q_1, q_2\}.$$

It seems natural to ask whether for given integers $q_1, q_2 \ge -1$, there always exists a finitely generated module $M$ over a Noetherian local ring $R$ which is a quotient of a Gorenstein local ring, possessing $q_1 = \max_{j \notin \mathrm{D}(M)} \dim_R \mathrm{K}^j(M)$ and $q_2 = \max_{j \in \mathrm{D}(M)} \mathrm{p}(\mathrm{K}^j(M))$. We close this paper with an affirmative answer to this question.

THEOREM 5.7. *Let $q_1, q_2 \ge -1$ be given integers and choose an integer $d$ so that $d \ge \max\{q_1, q_2\} + 3$. Then there exists a finitely generated module $M$ over a Noetherian local ring $R$ which is a quotient of a Gorenstein local ring such that*

$$\dim M = d,\ \ q_1 = \max_{j \notin \mathrm{D}(M)} \dim_R \mathrm{K}^j(M),\ \ \textit{and}\ \ q_2 = \max_{j \in \mathrm{D}(M)} \mathrm{p}(\mathrm{K}^j(M)).$$

*Hence* $\mathrm{sp}(M) = \max\{q_1, q_2\}$.

Proof. We consider two cases.

(1) *The case where* $q_1 = q_2$. If $q_1 = q_2 = -1$, we have nothing to prove, because every sequentially Cohen–Macaulay module of dimension $d$ satisfies our requirements. Assume that $q_1 = q_2 \geq 0$ and set $k = d - q_1$. Let us choose a Buchsbaum complete local ring $(R, \mathfrak{m})$ such that $\dim R = k$, $\mathrm{H}^0_{\mathfrak{m}}(R) = 0$, and $\mathrm{H}^j_{\mathfrak{m}}(R) \neq 0$ for some $j \in \{2, \ldots, k-1\}$ (this choice is possible; see [**G**, Theorem 1.1]), whence $\mathrm{sp}(R) = \mathrm{p}(R) = 0$ and $\dim_R \mathrm{K}^j(R) = 0$ for some $j \in \{2, \ldots, k-1\}$. Therefore the canonical module $\mathrm{K}(R)$ of $R$ is not a Cohen–Macaulay $R$-module ([**BN1**, Corollary 2.7]); hence $\mathrm{p}(\mathrm{K}(R)) = 0$. If $q_1 = 0$, the ring $R$ satisfies our requirements. Assume that $q_1 \geq 1$. Let $S = (R[x_1, x_2, \ldots, x_{q_1}])_{\mathfrak{n}}$, where $R[x_1, x_2, \ldots, x_{q_1}]$ is the polynomial ring and $\mathfrak{n} = (\mathfrak{m}, x_1, x_2, \ldots, x_{q_1})$. Then $\dim S = d$ and $\mathrm{sp}(S) = q_1$ by Lemma 5.5. Since $0 \subset R$ is the dimension filtration of $R$, $0 \subset S$ is the dimension filtration of $S$; hence $\mathrm{D}(S) = \{-1, d\}$. As $\mathrm{p}(R) = 0$, we have $\mathrm{p}(S) = q_1$ (see Claim 1 in the proof of Lemma 5.5), whence by Proposition 2.3 and Remark 2.1 we get $\dim_R \mathrm{K}^t(S) = q_1$ for some $t < d$ such that $t \notin \mathrm{D}(S)$. Thus the ring $S$ satisfies all our requirements.

(2) *The case where* $q_1 \neq q_2$. Let $R = K[[x_1, x_2, \ldots, x_d]]$ be the formal power series ring over a field $K$ and $\mathfrak{m} = (x_1, x_2, \ldots, x_d)$. For $i \in \{1, 2\}$ satisfying $q_i \geq 0$, we set $M_i = (x_1, x_2, \ldots, x_{d-q_i})R$, $N_i = R/(x_1, x_2, \ldots, x_{d-q_i-1})R$, and $\mathfrak{p}_i = (x_1, x_2, \ldots, x_{d-q_i-1})R$. Then $N_i$ is a Cohen–Macaulay $R$-module of dimension $q_i + 1$ with $\mathrm{Ass}_R N_i = \{\mathfrak{p}_i\}$, while $R/M_i$ is a Cohen–Macaulay local ring of dimension $q_i$ with $\mathrm{Ass}_R M_i = \{0\}$. Therefore $\mathrm{K}^j(N_i) = 0$ for all $j \neq q_i + 1$, while $\mathrm{K}^{q_i+1}(N_i)$ is a Cohen–Macaulay $R$-module of dimension $q_i + 1$. Besides, thanks to the exact sequence

$$0 \to M_i \to R \to R/M_i \to 0,$$

we get $\mathrm{K^j(M_i)} = 0$ for all $j \neq q_i + 1, d$, $\mathrm{K}^{q_i+1}(M_i) \cong \mathrm{K}^{q_i}(R/M_i)$, and $\mathrm{K}^d(M_i) \cong \mathrm{K}^d(R)$. Hence $\mathrm{K}^{q_i+1}(M_i)$ is a Cohen–Macaulay $R$-module of dimension $q_i$ and $\mathrm{K}^d(M_i)$ is a Cohen–Macaulay $R$-module of dimension $d$.

Let us now consider the following three cases separately. Firstly suppose that $q_2 = -1$. Then $q_1 \geq 0$. Let $M = M_1$. Then $\mathrm{Ass}_R M = \{0\}$. We have $\mathrm{K}^j(M) = 0$ for all $j \neq q_1 + 1, d$ and $\dim_R \mathrm{K}^{q_1+1}(M) = q_1$, $\dim_R \mathrm{K}^d(M) = d$, and $\mathrm{p}(\mathrm{K}^d(M)) = -1$. Hence $M$ satisfies our requirements.

Suppose that $q_1 = -1$. Then $q_2 \geq 0$. Let $M = M_2 \oplus N_2$. Then $\dim M = d$ and $\mathrm{Ass}_R M = \{0, \mathfrak{p}_2\}$. We have $\mathrm{K}^j(M) = 0$ for all $j \neq q_2 + 1, d$, $\mathrm{K}^d(M) \cong \mathrm{K}(R)$, and

$$\mathrm{K}^{q_2+1}(M) \cong \mathrm{K}^{q_2}(R/M_2) \oplus \mathrm{K}^{q_2+1}(N_2).$$

Hence $\dim_R \mathrm{K}^{q_2+1}(M) = q_2 + 1$ and $\mathrm{p}(\mathrm{K}^{q_2+1}(M)) = q_2$, while $\dim_R \mathrm{K}^d(M) = d$ and $\mathrm{p}(\mathrm{K}^d(M)) = -1$. Thus $M$ satisfies our requirements.

Let us consider the last case where $q_1 \geq 0$ and $q_2 \geq 0$. Let $M = M_1 \oplus M_2 \oplus N_2$. Then $\mathrm{Ass}_R M = \{0, \mathfrak{p}_2\}$. As $q_1 \neq q_2$, we have $\mathrm{K}^j(M) = 0$ for $j \neq q_1 + 1, q_2 + 1, d$. Notice that $\dim_R \mathrm{K}^{q_1+1}(M) = q_1$, $\dim_R \mathrm{K}^{q_2+1}(M) = q_2 + 1$, and $\mathrm{p}(\mathrm{K}^{q_2+1}(M)) = q_2$, while $\dim_R \mathrm{K}^d(M) = d$ and $\mathrm{p}(\mathrm{K}^d(M)) = -1$. Thus $M$ satisfies our requirements, which completes the proof of Theorem 5.7. □

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