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AN EXAMPLE ON DEFECT OF A COMPOSITE FUNCTION

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. Let $f(z)$ be a meromorphic function in the complex plane $|z| < +\infty$. We assume that the reader is familiar with the fundamental concepts of Nevanlinna's theory and in particular with the most usual of its symbols:

$\log^+$, $m(r, f)$, $n(r, f)$, $N(r, f)$, $T(r, f)$, $\delta(a, f)$ and etc..

Valiron [5] proved the following theorem.

THEOREM A. *If $f(z)$ is a meromorphic function of finite order μ and of lower order λ, and if $\mu - \lambda < 1$, then all deficiencies of $f(z)$ are invariant under a change of origin.*

Further, this theorem is generalized as follows. (See Mori [4].)

THEOREM B. *Let $g(z)$ be a polynomial of degree n and $f(z)$ a transcendental meromorphic function of order μ_f and of lower order λ_f. Assume that $\mu_f - \lambda_f < 1/n$. Then, for any w_0, it holds that*

$$\delta(w_0, f(g(z))) = \delta(w_0, f(z)) .$$

By a geometrical argument, Belinskii and Gol'dberg [1] gave an example of a meromorphic function $f(z)$ of order 1 and of lower order 0 having the following property: a deficiency of $f(z)$ varies under a change of origin.

This shows that in Valiron's theorem A cited above the condition $\mu - \lambda < 1$ can not be dropped.

In this note, following Edrei and Fuchs [2], [3], we give an example which shows that the condition $\mu_f - \lambda_f < 1/n$ in Theorem B can not be weakened. In the case $n = 1$, our argument seems to be more elementary than the one due to Belinskii and Gol'dberg.

2. In the construction of the example, we need following two lemmas.

LEMMA 1. *Let $g(z)$ be a meromorphic function of order $\mu_g < +\infty$ and $\tau\ (\neq \infty)$ a complex number. Then, for any fixed $t > 0$, there exists* *an auxiliary function* $\Phi_n(z^n)$ *such that*

$$\delta(0, \Phi_n(z^n)) = \delta(\tau, g(z))$$

and

$$(1) \qquad \delta(0, \Phi_n((z+t)^n)) = 1 - \limsup_{r\to\infty} \frac{\sum_{j=0}^{n-1} N(r, (1/g(\omega^j(z+t)) - \tau))}{\sum_{j=1}^{n-1} T(r, g(\omega^j(z+t)))}.$$

Here the auxiliary function $\Phi_n(z^n)$ *is written as*

$$(2) \qquad \Phi_n(z^n) = \prod_{j=0}^{n-1} \frac{g(\omega^j z) - \tau}{g(\omega^j z) - X},$$

where $\omega = e^{(2\pi/n)i}$ *and* X *is some complex number.*

PROOF. We follow Edrei's argument in [2]. Let Ω be the set as in [2]. Then, for any fixed $t > 0$, there exists a complex number X such that $X \notin \Omega$,

$$N\left(r, \frac{1}{g(z) - X}\right) \sim T(r, g(z))$$

and

$$N\left(r, \frac{1}{g(\omega^j(z+t)) - X}\right) \sim T(r, g(\omega^j(z+t))),$$

$(j = 0, 1, 2, \cdots, n-1)$ as $r \to \infty$, since a set of deficient values in the sense of Valiron is of capacity zero.

Now we note that in (2), a zero of one of the n functions

$$g(\omega^j z) - \tau \quad (j = 0, 1, 2, \cdots, n-1)$$

can not cancel a zero of the n functions $g(\omega^j z) - X$. Hence we have

$$\delta(0, \Phi_n(z^n)) = \delta(\tau, g(z))$$

(see Edrei [2]), and further we see

$$N(r, \Phi_n((z+t)^n)) = \sum_{j=0}^{n-1} N\left(r, \frac{1}{g(\omega^j(z+t)) - X}\right)$$
$$\sim \sum_{j=0}^{n-1} T(r, g(\omega^j(z+t))) \quad (r \longrightarrow +\infty)$$

and

$$N\left(r, \frac{1}{\Phi_n((z+t)^n)}\right) = \sum_{j=0}^{n-1} N\left(r, \frac{1}{g(\omega^j(z+t)) - \tau}\right),$$

so we have

$$\delta(0, \Phi_n((z+t)^n)) = 1 - \limsup_{r\to\infty} \frac{N(r, 1/(\Phi_n((z+t)^n)))}{T(r, \Phi_n((z+t)^n))}$$
$$= 1 - \limsup_{r\to\infty} \frac{\sum_{j=0}^{n-1} N(r, 1/(g(\omega^j(z+t)) - \tau))}{\sum_{j=0}^{n-1} T(r, g(\omega^j(z+t)))} .$$

Thus we have our Lemma.

We note that $\Phi_n(z)$ is a meromorphic function of order μ_g/n and that

$$\delta(0, \Phi_n(z^n)) = \delta(0, \Phi_n(z))$$

holds.

LEMMA 2 (Edrei and Fuchs [3]). *Let* $z_1, z_2, z_3, \cdots$ $(|z_1| \leqq |z_2| \leqq |z_3| \leqq \cdots)$ *be a given sequence of distinct complex numbers having no finite point of accumulation and let* $\nu_1, \nu_2, \nu_3, \cdots$ *be a given sequence of positive integers. Finally, let* $\zeta(r)$ *be any given function of* $r(>0)$, *decreasing and strictly positive. Then, it is possible to find a meromorphic function* $F(z)$ *of the form*

$$F(z) = \sum_{k=1}^{\infty} \frac{\alpha_k}{(z - z_k)^{\nu_k}} , \quad (\alpha_k > 0, \ \sum \alpha_k < +\infty)$$

and a set E *of finite measure such that*

$$T(r, F) = N(r, F) \quad (r \notin E)$$

and

$$0 \leqq T(r, F) - N(r, F) < \zeta(r) \quad (r > r_0, r \in E) .$$

3. For our purpose it suffices to construct a meromorphic function $\Phi(z)$ in the complex plane $|z| < +\infty$ such that $\mu_\Phi = 1/n$, $\lambda_\Phi = 0$, $\delta(0, \Phi(z)) = 0$ and such that $\delta(0, \Phi((z+t)^n)) = 1$ for any fixed $t > 0$.

First we consider an integral function

$$f_1(z) = \prod_{k=1}^{\infty} \left(1 - \left(\frac{z}{r_k}\right)^{\eta_k}\right), \qquad (0 < 2r_r \leqq r_{k+1}, \ \eta_k(\geqq 1)\text{: integer}) .$$

We can see

$$N\left(r, \frac{1}{f_1}\right) \leqq T(r, f_1) \leqq N\left(r, \frac{1}{f_1}\right) + 4$$

and can take two sequences $\{r_k\}_{k=1}^{\infty}$ and $\{\eta_k\}_{k=1}^{\infty}$ such that

$$(3) \qquad T(r_k, f_1) < (\log r_k)^2$$

and

$$\eta_k = [r_k (\log r_k)^4] ,$$

where $[a]$ denotes the integral part of a. (See [1], [3]). For the sequences $\{r_k\}_{k=1}^{\infty}$, $\{\eta_k\}_{k=1}^{\infty}$ and a fixed $t > 0$, we take z_k such that

$$z_k = \alpha(r_k - s) ,$$

where, if $n = 2m + 1$, then

$$\alpha = -1 , \qquad s = t \sin^2\left(\frac{m}{2m+1}\pi\right) ,$$

or if $n = 2m$, then

$$\alpha = e^{i(1-1/(2m))\pi} , \qquad s = t \sin^2\left(\frac{2m-1}{4m}\pi\right) .$$

We next take a sequence $\{\nu_k\}_{k=1}^{\infty}$ such that $\nu_k = [r_k(\log r_k)^3]$. Then by Lemma 2 we can find a meromorphic function

$$f_2(z) = \sum_{k=1}^{\infty} \frac{\alpha_k}{(z - z_k)^{\nu_k}} , \qquad (\alpha_k > 0, \sum \alpha_k < +\infty)$$

such that $T(r, f_2) = N(r, f_2) + O(1)$. We now put

$$f(z) = f_1(z) \cdot f_2(z) .$$

Then we have

$$T(r, f(z)) = T(r, f_1(z) \cdot f_2(z)) \leqq T(r, f_1) + T(r, f_2)$$
$$\leqq N\left(r, \frac{1}{f_1}\right) + N(r, f_2) + O(1)$$

and

$$T(r, f(z)) \geqq \max\left(N\left(r, \frac{1}{f_1}\right), N(r, f_2)\right) .$$

By (3) and by construction of $f(z)$, we obtain

$$N(r_k, f) = N(r_k, f_2) \sim \nu_k \log \frac{r_k}{r_k - s} + O((\log r_k)^2)$$
$$\sim s(\log r_k)^3 .$$

Thus we see

$$T(r_k, f) \sim s(\log r_k)^3$$

as $r_k \to +\infty$ and $s > 0$. Therefore

$$1 \geqq \limsup_{r\to\infty} \frac{N(r, f(z))}{T(r, f(z))} \geqq \limsup_{r_k\to\infty} \frac{N(r_k, f(z))}{T(r_k, f(z))} = 1 .$$

Hence we have

$$\delta(\infty, f(z)) = 0 .$$

On the other hand, for any sufficiently large r such that $r_k \leqq r < r_{k+1}$, we have

$$T(r, f(z+t)) \geqq N\left(r, \frac{1}{f_1(z+t)}\right)$$
$$> K \cdot \eta_k \log \frac{r}{r_k - t} + O((\log r_k)^2)\,, \quad K > \frac{1}{3}$$

and

$$N(r, f(\omega^{[n/2]}(z+t))) = N(r, f_2(\omega^{[n/2]}(z+t)))$$
$$\leqq \nu_k \log^+ \frac{r}{x_k} + O((\log r_k)^2)\,,$$

where $x_k = |\,\omega^{-[n/2]}\alpha(r_k - s) - t\,| > r_k - t$. These two estimates are obtained by the quite same argument as in [1]. Hence

$$\limsup_{r\to\infty} \frac{N(r, f(\omega^{[n/2]}(z+t)))}{T(r, f(z+t))} = 0\,.$$

Further we see, from the above construction,

$$N(r, f(\omega^j(z+t))) \leqq N(r, f(\omega^{[n/2]}(z+t)))\,, \quad (j = 0, 1, 2, \cdots, n-1)\,.$$

Thus we have

$$\limsup_{r\to\infty} \frac{\sum_{j=0}^{n-1} N(r, f(\omega^j(z+t)))}{\sum_{j=0}^{n-1} T(r, f(\omega^j(z+t)))} = 0\,.$$

In (2), we put $\tau = 0$ and $g(z) = 1/f(z)$. Then we have, by Lemma 1,

$$\delta(0, \Phi_n(z)) = \delta(0, g(z)) = \delta(\infty, f(z)) = 0$$

and

$$1 \geqq \delta(0, \Phi_n((z+t)^n)) \geqq 1 - \limsup_{r\to\infty} \frac{\sum_{j=0}^{n-1} N(r, 1/(g(\omega^j(z+t))))}{\sum_{j=0}^{n-1} T(r, g(w^j(z+t)))} = 1\,.$$

Therefore, for any fixed $t > 0$, there exists a meromorphic function $\Phi_n(z)$ of order $1/n$ and of lower order 0 satisfying

$$\delta(0, \Phi_n(z)) = 0 \quad \text{and} \quad \delta(0, \Phi_n((z+t)^n)) = 1\,.$$

REMARK. In the case $1 < \mu_f < +\infty$, we can also construct a similar example by an analogous argument.

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