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A THEOREM ABOUT INTERPOLATION OF OPERATIONS ON INTERMEDIATE SPACES

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. Introduction. Let us consider the Hilbert transform $\tilde{f}$ of function $f \in L^1(-\infty, \infty)$, it is well known that the $\tilde{f}$ exist a.e. but the only local integrability of $|\tilde{f}|^{1-\varepsilon}(0 < \varepsilon < 1)$ holds in general. Then we shall discuss its integral estimation on the whole space from a stand point of views of interpolation of operations on intermediate spaces.

The intermediate space between two Banach spaces, was introduced by A. J. Luxemburg [6, 7]. Let us consider totally σ-finite measure space (R, μ) and the space V of equivalent classes of real valued measurable functions on R. The equivalent relation here is that of coincidence of almost everywhere. If in V we introduce a topology of convergence in measure on sets of finite measure, V becomes a topological vector space. If we take as the interpolation pair L^1_μ and $L^p_\mu(1 < p < \infty)$, then these are continuously embedded in V. We shall consider the space $L^1_\mu + L^p_\mu$ and introduce in it the norm

$$||f||[L^1_\mu + L^p_\mu] = \inf(||g||[L^p_\mu] + ||h||[L^1_\mu])$$

where the infimum is taken over all pairs $g \in L^p_\mu$ and $h \in L^1_\mu$ such that $f = g + h$, then $L^1_\mu + L^p_\mu$ also becomes a Banach space and also continuously embedded in V. We shall also consider another totally σ-finite measure space (S, ν) and the intermediate space $L^{1-\varepsilon}_\nu + L^p_\nu$ on S. Although $L^{1-\varepsilon}_\nu$ is not a Banach space, for the sake of convenience we shall use the same notations as before.

Let us consider the operation T which transforms measurable functions on R to those on S. The operation T is called quasi-linear if

(i) $T(f_1 + f_2)$ is uniquely defined whenever Tf_1 and Tf_2 are defined and

$$|T(f_1 + f_2)| \leqq \kappa(|Tf_1| + |Tf_2|)$$

where κ is a constant independent of f_1 and f_2,

(ii) $T(cf)$ is uniquely defined whenever Tf is defined and

$$|T(cf)| = |c|\ |Tf|$$

for all scalars c.

We say that the operation T is of type (p, p) $(1 < p < \infty)$ if Tf is defined for each $f \in L^p_\mu(R)$ and belongs to $L^p_\nu(S)$ such that

$$\| Tf \| [L^p_\nu] \leqq M_p \| f \| [L^p_\mu]$$

where M_p is a constant independent of f. The least admissible value of M_p is called the (p, p)-norm of operation T. Next we shall define the weak type $(1, 1)$ of operations. Given any $y > 0$, denote by $E_y = E_y[Tf]$ the set of points of the space S where $| Tf | > y$, and write $\nu(E_y)$ for the ν-measure of the set E_y. We say that the operation T is of weak type $(1, 1)$ if

$$\nu(E_y[Tf]) \leqq \frac{M_1}{y} \| f \| [L^1_\mu]$$

where M_1 is a constant independent of f. The least admissible value of M_1 is called the weak $(1, 1)$-norm of operation T.

Now we shall prove

THEOREM. *Suppose that a quasi-linear operation T is of weak type $(1, 1)$ and of type (p, p) $(1 < p < \infty)$ respectively. Then we have $Tf \in L^{1-\varepsilon}_\nu + L^p_\nu$ $(0 < \varepsilon < 1)$ for any $f \in L^1_\mu + L^p_\mu$ and*

$$\| Tf \| [L^{1-\varepsilon}_\nu + L^p_\nu] \leqq \kappa M^{p/(1-\varepsilon)} \| f \| [L^1_\mu + L^p_\mu]$$

where κ is a constant independent of f, M means the maximum value of M_1, M_p and 1.

2. **Proof of Theorem.** Let us begin to prove the following supplementary results.

PROPOSITION. *Let us suppose that a quasi-linear operation T is of weak type $(1, 1)$. Then we have if $h \in L^1_\mu$,*

$$\left(\int_{|Th| \leqq 1} | Th |^p d\nu \right)^{1/p} \leqq KM_1 \| h \| [L^1_\mu] \tag{1}$$

and

$$\left(\int_{|Th| > 1} | Th |^{1-\varepsilon} dv \right)^{1/(1-\varepsilon)} \leqq KM_1 \| h \| [L^1_\mu] \tag{2}$$

respectively.

PROOF. These are immediate consequence of results of our preceding paper, but we prefer to give a simple and direct proof. Let us denote by $n(y)$ the distribution function of $| Th |$, then

$$\int_{|Th|\leqq 1} |Th|^p d\nu = -\int_0^1 y^p dn(y) = -n(1) + p\int_0^1 y^{p-1} n(y) dy$$
$$\leqq p\int_0^1 y^{p-1}\Big(\frac{M_1}{y}||h||\Big) dy = \frac{p}{p-1} M_1 ||h|| \,.$$

The passage is justified easily (see A. Zygmund [9, vol. II, p. 112 (4.8)]). Similary we have

$$\int_{|Th|>1} |Th|^{1-\varepsilon} d\nu = -\int_1^\infty y^{1-\varepsilon} dn(y) = n(1) + (1-\varepsilon)\int_1^\infty y^{-\varepsilon} n(y) dy$$
$$\leqq M_1||h|| + (1-\varepsilon)\int_1^\infty y^{-\varepsilon}\Big(\frac{M_1}{y}||h||\Big) dy = \frac{M_1}{\varepsilon}||h|| \,.$$

Now we shall need the following lemma.

LEMMA. *Let us suppose that the inequality between three non-negative numbers A, B and C such that $A \leqq \kappa(B + C)$, $\kappa \geqq 1$. Then we have the following inequalities*

(i) *if* $0 \leqq A \leqq 1$,

$$A \leqq \begin{cases} \kappa(B + C), \text{ if } 0 \leqq C \leqq 1 \\ \kappa(B + C^{(1-\varepsilon)/p}), \text{ if } C > 1 \end{cases}$$

(ii) *if* $A > 1$,

$$A \leqq \begin{cases} (2\kappa)^{p/(1-\varepsilon)}(B^{p/(1-\varepsilon)} + C^{p/(1-\varepsilon)}), \text{ if } 0 \leqq C \leqq 1 \\ (2\kappa)^{p/(1-\varepsilon)}(B^{p/(1-\varepsilon)} + C), \text{ if } C > 1 \,. \end{cases}$$

End of proof of Theorem. Without loss of generality we can assume that $||f|| = 1$. For any given positive number η there exists a decomposition of f such that $f = g + h$ and $||g|| + ||h|| \leqq ||f|| + \eta = 1 + \eta$. Let us denote by S_1 and S_2 the set of points $|Tf| > 1$ and $|Tf| \leqq 1$ respectively. Let us also denote by S_{11} and S_{12} the set of points $|Th| > 1$ and $|Th| \leqq 1$ respectively. Then if we apply the first part of Lemma to $|Tf| = A$, $|Tg| = B$ and $|Th| = C$ and integrating over S_2

$$\Big(\int_{|Tf|\leqq 1} |Tf|^p d\nu\Big)^{1/p}$$
$$\leqq \kappa\Big(\int_{S_2\cap S_{12}} (|Tg| + |Th|)^p d\nu\Big)^{1/p} + \kappa\Big(\int_{S_2\cap S_{11}} (|Tg| + |Th|^{(1-\varepsilon)/p})^p d\nu\Big)^{1/p} \,.$$

If we apply the Minkowsky inequality and substitute estimations (1), (2) of Proposition and $||Tg|| \leqq M_p||g||$ then we shall obtain

$$\Big(\int_{|Tf|\leqq 1} |Tf|^p d\nu\Big)^{1/p}$$
$$\leqq 2\kappa\Big(\int_S |Tg|^p d\nu\Big)^{1/p} + \kappa\Big(\int_{S_{12}} |Th|^p d\nu\Big)^{1/p} + \kappa\Big(\int_{S_{11}} |Th|^{1-\varepsilon} d\nu\Big)^{1/p}$$

$$\leqq KM_2\|g\| + KM_1\|h\| + KM_1^{(1-\varepsilon)/p}\|h\|^{(1-\varepsilon)/p} \leqq KM(1+\eta)\,.$$

By the same way we get

$$\begin{aligned}&\left(\int_{|Tf|>1}|Tf|^{1-\varepsilon}d\nu\right)^{1/(1-\varepsilon)}\\&\quad\leqq KM_2^{p/(1-\varepsilon)}\|g\|^{p/(1-\varepsilon)} + KM_1^{p(1-\varepsilon)}\|h\|^{p/(1-\varepsilon)} + KM_1\|h\|\\&\quad\leqq KM^{p/(1-\varepsilon)}(1+\eta)^{p/(1-\varepsilon)}\,.\end{aligned}$$

If we let the η tend to zero we have

$$\begin{aligned}\|Tf\|[L_\nu^{1-\varepsilon} + L_\nu^p] &\leqq \left(\int_{|Tf|\leqq 1}|Tf|^p d\nu\right)^{1/p} + \left(\int_{|Tf|>1}|Tf|^{1-\varepsilon}d\nu\right)^{1/(1-\varepsilon)}\\&\leqq KM^{p/(1-\varepsilon)} = KM^{p/(1-\varepsilon)}\|f\|[L_\mu^1 + L_\mu^p]\,.\end{aligned}$$

From the Marcinkiewicz-Zygmund theorem and this one we can derive:

COROLLARY 1. *Let us suppose the same hypothesis those of theorem and let us write*

$$r = (1-\theta) + \theta p \qquad 0 < \theta \leqq 1\,.$$

Then we have $Tf \in L_\nu^{1-\varepsilon} + L_\nu^r (0 < \varepsilon < 1)$ *for any* $f \in L_\mu^1 + L_\mu^r$ *and*

$$\|Tf\|[L_\nu^{1-\varepsilon} + L_\nu^r] \leqq KM^{r/(1-\varepsilon)}\|f\|[L_\mu^1 + L_\mu^r]\,,$$

where K *is a constant independent of* f *and* M *means the maximum value of* M_1, M_p *and* 1.

Moreover we can prove the following proposition which corresponds to the case $\theta = 0$.

COROLLARY 2. *Let us suppose that a quasi-linear operation* T *is of weak type* (1, 1). *Then we have* $Tf \in L_\nu^{1-\varepsilon} + L_\nu^{1+\varepsilon} (0 < \varepsilon < 1)$ *for any* $f \in L_\mu^1$ *and*

$$\|Tf\|[L_\nu^{1-\varepsilon} + L_\nu^{1+\varepsilon}] \leqq KM_1\|f\|[L_\mu^1]\,.$$

The above discussions suggest us that results are naturally generalized by the same method developed here.

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