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ON SUMMATION PROCESSES OF FOURIER EXPANSIONS IN BANACH SPACES. I: COMPARISON THEOREMS

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. Introduction. The origin of the present investigation goes back to two lectures presented by Jean Favard [9, 10], the first of which the birthday celebrant as well as the authors were fortunate enough to be able to attend at the Oberwolfach Conference on "Approximation Theory" in 1963. A first formulation of the problem may be stated as follows:

Let X be an arbitrary (real or complex) Banach space and $[X]$ be the Banach algebra of all bounded linear operators of X into itself. Let $\{T(\rho)\}_{\rho>0}\subset[X]$ be a strong approximation process (on X for $\rho\to\infty$), i.e.,

$$\lim_{\rho\to\infty}\|T(\rho)f-f\|=0 \qquad (f\in X)\,. \tag{1.1}$$

Let $\{G(\rho)\}_{\rho>0}\subset[X]$ be a further strong approximation process. The problem is to find *direct* estimates between the quantities $\|T(\rho)f-f\|$ and $\|G(\rho)f-f\|$, thus to establish, for instance, the existence of a constant $A>0$ such that

$$\|T(\rho)f-f\|\leqq A\,\|G(\rho)f-f\| \qquad (f\in X;\ \rho>0)\,. \tag{1.2}$$

In this event, the process $\{T(\rho)\}$ is said to be *better* than $\{G(\rho)\}$. If $\{T(\rho)\}$ is better than $\{G(\rho)\}$ and the latter is in turn better than $\{T(\rho)\}$, then the processes are said to be *equivalent*, in notation

$$\|T(\rho)f-f\|\sim\|G(\rho)f-f\| \qquad (f\in X)\,.$$

First contributions of the participants of the two Favard lectures to this problem have been made by Shapiro [17], Boman-Shapiro [4], and the authors [6] (compare also the comments given in [5; p. 507]). Whereas in [4, 17] the concrete case of approximation processes representable as Fourier convolution integrals of Fejér's type is considered in Euclidean n-space (or n-dimensional torus), in [6] the problem is discussed in the setting of abstract Hilbert spaces.

In this paper the problem is studied in the setting originally envisaged

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by Favard [9, 10]. Thus the approximation processes in question will be given as summation processes of Fourier expansions corresponding to general decompositions (cf. [12; p. 86]) of Banach spaces. The proof of the general comparison theorem to be presented will depend upon a basic uniform multiplier condition (see (2.8)). Conditions of this type were studied in some basic work by G. Sunouchi [20] in connection with the related saturation problem for summation processes of (trigonometric) Fourier series, particularly employing the uniform quasi-convexity of scalar sequences.

To this end, Sec. 2 gives the formal definitions as well as the comparison theorem. To deal with condition (2.8), Sec. 3 studies sufficient conditions upon multiplier classes in connection with the uniform boundedness of the partial sums or of the Cesàro means of the expansion of f. The final section is devoted to applications.

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2. **A comparison theorem.** Denote by $f, g, \cdots$ the elements of the Banach space X with norm $\|\cdot\|$, and by X^* its dual; further, let $\boldsymbol{Z}, \boldsymbol{P}, \boldsymbol{N}$ be the sets of all, of all non-negative, of all positive integers, respectively. Let $\{P_k\}_{k=0}^{\infty}$ be a total sequence of mutually orthogonal continuous projections on X, i.e., i) $P_k \in [X]$ for each $k \in \boldsymbol{P}$, ii) $P_k f = 0$ for all $k \in \boldsymbol{P}$ implies $f = 0$ (total), iii) $P_j P_k = \delta_{jk} P_k$, δ_{jk} being Kronecker's symbol (orthogonal). Then with each $f \in X$ one may associate its (formal) Fourier series expansion

$$f \sim \sum_{k=0}^{\infty} P_k f \qquad (f \in X) . \tag{2.1}$$

With s the set of all sequences $\alpha = \{\alpha_k\}_{k=0}^{\infty}$ of scalars, $\alpha \in s$ is called a multiplier for X (corresponding to $\{P_k\}$), if for each $f \in X$ there exists an element $f^\alpha \in X$ such that $\alpha_k P_k f = P_k f^\alpha$ for all $k \in \boldsymbol{P}$, thus

$$f^\alpha \sim \sum_{k=0}^{\infty} \alpha_k P_k f . \tag{2.2}$$

Note that f^α is uniquely determined by f since $\{P_k\}$ is total. The set of all multipliers is denoted by $M = M(X; \{P_k\})$. With the natural vector operations, coordinatewise multiplication and norm

$$\|\alpha\|_M = \sup\{\|f^\alpha\|; f \in X, \|f\| \leqq 1\} , \tag{2.3}$$

M is a commutative Banach algebra containing the identity $\{1\} \in s$. An operator $T \in [X]$ is called a multiplier operator if there exists a sequence

$\tau \in s$ such that $P_k T f = \tau_k P_k f$ for all $f \in X$, $k \in \boldsymbol{P}$, i.e., one has the formal expansion

$$Tf \sim \sum_{k=0}^{\infty} \tau_k P_k f \qquad (f \in X) . \tag{2.4}$$

Thus, by definition, with each multiplier operator T there is associated a multiplier sequence $\tau \in M$ and vice versa, and since $\| T \|_{[X]} = \| \tau \|_M$ by definition (cf. (2.3)), M can be identified with the subspace of multiplier operators in $[X]$.

REMARK. The expansion (2.1) represents a slight generalization of the concept of Fourier series in a Banach space X associated with a total, biorthogonal system $\{f_k, f_k^*\}$. Here $\{f_k, f_k^*\}$ consists of two sequences $\{f_k\} \subset X$, $\{f_k^*\} \subset X^*$ such that i) $\{f_k^*\}$ is total, i.e., $f_k^*(f) = 0$ for all $k \in \boldsymbol{P}$ implies $f = 0$ and ii) $f_j^*(f_k) = \delta_{jk}$ for all $j, k \in \boldsymbol{P}$. Then (2.1) and (2.4) read

$$f \sim \sum_{k=0}^{\infty} f_k^*(f) f_k , \qquad Tf \sim \sum_{k=0}^{\infty} \tau_k f_k^*(f) f_k , \tag{2.5}$$

respectively; $P_k(X)$ is the one-dimensional linear space spanned by f_k. For these definitions and results compare Marti [12; p. 86 ff], see also Singer [18; pp.1–49], Milman [13].

Denoting the null manifold of a linear operator T by $N(T) = \{f \in X; Tf = 0\}$ and the identity mapping of X into X by I, we may formulate

THEOREM 2.1. *Let $\{T(\rho)\}$, $\{G(\rho)\} \subset [X]$ be two families of multiplier operators with associated multiplier sequences $\{\tau_k(\rho)\}$, $\{\gamma_k(\rho)\}$, respectively. Let*

$$N(G(\rho) - I) \subseteqq N(T(\rho) - I) \qquad (\rho > 0) . \tag{2.6}$$

Furthermore, if $G(\rho) = \{k \in \boldsymbol{P}; \gamma_k(\rho) = 1\}$, let $\delta(\rho) = \{\delta_k(\rho)\}_{k=0}^{\infty} \in s$, $\rho > 0$, be defined by

$$\delta_k(\rho) = \begin{cases} \dfrac{\tau_k(\rho) - 1}{\gamma_k(\rho) - 1} , & k \notin \boldsymbol{G}(\rho) \\ \quad 1 , & k \in \boldsymbol{G}(\rho) , \end{cases}$$

and assume $\delta(\rho)$ to be a multiplier for each $\rho > 0$. Then, for fixed $\rho > 0$,

$$\| T(\rho) f - f \| \leqq \| \delta(\rho) \|_M \| G(\rho) f - f \| \qquad (f \in X) . \tag{2.7}$$

If, furthermore, there exists a constant $A > 0$ such that

$$\| \delta(\rho) \|_M \leqq A \tag{2.8}$$

uniformly for all $\rho > 0$, then the process $\{T(\rho)\}$ is better than $\{G(\rho)\}$.

PROOF. Let $f \in X$ be arbitrary and $k \notin G(\rho)$; then

$$P_k(T(\rho)f - f) = \delta_k(\rho)\{\gamma_k(\rho) - 1\}P_kf = \delta_k(\rho)P_k(G(\rho)f - f) . \tag{2.9}$$

If $k \in G(\rho)$, then $P_kf \in N(G(\rho) - I) \subseteqq N(T(\rho) - I)$, and (2.9) holds trivially. Thus with multiplier operator $U^{\delta(\rho)}$ associated with $\delta(\rho) \in M$ one has

$$T(\rho)f - f = U^{\delta(\rho)}[G(\rho)f - f]$$

for each $f \in X$, $\rho > 0$ since $\{P_k\}$ is total. This completes the proof.

Obviously, (2.6) is natural for an estimate of type (1.2) and easy to verify. On the other hand, the multiplier condition, in particular the uniform one (2.8), is strong and intricate; its verification in the applications is the actual problem. Therefore the next section is devoted to establishing convenient criteria concerning (uniformly bounded) multipliers.

3. Some multiplier classes. By the representation (2.2) it is almost obvious that a necessary condition for $\alpha \in s$ to be a multiplier is the boundedness of the coefficients α_k, i.e., that

$$M \subset l^\infty = \{\alpha \in s; \|\alpha\|_\infty = \sup_k |\alpha_k| < \infty\} .$$

In the case of a total biorthogonal system $\{f_k, f_k^*\}$ with $\{f_k\}$ being an unconditional basis for X the converse statement $l^\infty \subset M$ is also valid. In this instance, $\alpha \in l^\infty$ is a necessary and sufficient condition for $\alpha \in s$ to be a multiplier ([18; p. 484], [12; p. 110]).

But the case of unconditional bases corresponds to a very particular situation in the applications. Therefore one makes use of weaker conditions upon $\{P_k\}$ in connection with a characterization of its multiplier class. To this end, consider the nth partial sum operator S_n defined by

$$S_nf = \sum_{k=0}^{n} P_kf \qquad (f \in X) \tag{3.1}$$

and assume that S_n is uniformly bounded in n, i.e.,

$$\|S_nf\| \leqq B\|f\| \qquad (f \in X) , \tag{3.2}$$

the constant B being independent of $n \in \boldsymbol{P}$ and $f \in X$. Let us note that in this case $\{P_k(X)\}$ is called a Schauder decomposition of X if (additionally) the linear span of $\bigcup_{k=0}^{\infty} P_k(X)$ is dense in X (see [12; p. 89]). Then it is known [12; p. 109] that with $\Delta\alpha_k = \alpha_k - \alpha_{k+1}$, $k \in \boldsymbol{P}$,

$$bv = \{\alpha \in s; \|\alpha\|_{bv} = \sum_{k=0}^{\infty} |\Delta\alpha_k| + \lim_{n\to\infty} |\alpha_n| < \infty\} \tag{3.3}$$

is continuously embedded in the multiplier class corresponding to the Schauder decomposition. But the density of the linear span of $\{P_k(X)\}$

is not essential as the following theorem shows; its proof being standard is only given for the sake of completeness.

THEOREM 3.1. *Let* $\{P_k\}_{k=0}^{\infty} \subset [X]$ *be a total sequence of mutually orthogonal projections and let* $S_n = \sum_{k=0}^{n} P_k$ *satisfy* (3.2). *Then every* $\alpha \in bv$ *is a multiplier and*

$$\|\alpha\|_M \leqq B\|\alpha\|_{bv} . \tag{3.4}$$

PROOF. For each $f \in X$ set

$$f^\alpha = \sum_{k=0}^{\infty} \Delta\alpha_k^0 S_k f + \alpha_\infty f ,$$

where $\alpha_k^0 = \alpha_k - \alpha_\infty$, $k \in \boldsymbol{P}$, $\alpha_\infty = \lim_{n\to\infty} \alpha_n$. Then f^α exists in X since by (3.2)

$$\|f^\alpha\| \leqq B\|f\| \sum_{k=0}^{\infty} |\Delta\alpha_k^0| + |\alpha_\infty| \, \|f\| \leqq B\|\alpha\|_{bv}\|f\| .$$

Thus it remains to show that $f^\alpha \sim \sum \alpha_k P_k f$. But this follows since for $P_k \in [X]$ one has $P_n S_k f = P_n f$ if $k \geqq n$ and zero otherwise, and therefore

$$P_n f^\alpha = \sum_{k=n}^{\infty} \Delta\alpha_k^0 P_n f + \alpha_\infty P_n f = \alpha_n P_n f .$$

REMARK. In the case of a total biorthogonal system $\{f_k, f_k^*\}$ in X, $\{f_k\}$ being fundamental in X (i.e., the linear combinations of f_k are dense in X), it is clear by the Banach-Steinhaus theorem that (3.2) is equivalent to the assumption that $\{f_k\}$ is a Schauder basis, i.e., for every $f \in X$

$$\lim_{n\to\infty} \Big\| \sum_{k=0}^{n} f_k^*(f) f_k - f \Big\| = 0 .$$

Then Theorem 3.1 as well as its converse is contained in [18; p. 40]. Concerning this statement for Schauder decompositions see e.g. [12; p. 109].

However, also the uniform boundedness (3.2) of the partial sums is quite restrictive for the applications. In order to replace this assumption by a weaker one, let us introduce the nth Cesàro mean operator (of order 1)

$$\sigma_n f = \sum_{k=0}^{n} \Big(1 - \frac{k}{n+1}\Big) P_k f \qquad (f \in X) \tag{3.5}$$

and assume that σ_n is uniformly bounded in $n \in \boldsymbol{P}$, i.e.,

$$\|\sigma_n f\| \leqq C\|f\| \qquad (f \in X) , \tag{3.6}$$

the constant C being independent of $n \in \boldsymbol{P}$ and $f \in X$. Now results of the theory of trigonometric series induce one to examine the set of bounded, quasi-convex sequences

$$(3.7)\qquad bqc = \{\alpha \in l^\infty;\ \|\alpha\|_{bqc} = \sum_{k=0}^{\infty} (k+1)\,|\Delta^2\alpha_k| + \lim_{n\to\infty} |\alpha_n| < \infty\}\,,$$

where $\Delta^2\alpha_k = \alpha_k - 2\alpha_{k+1} + \alpha_{k+2}$, $k \in \boldsymbol{P}$.

THEOREM 3.2. *Let $\{P_k\} \subset [X]$ be a total sequence of mutually orthogonal projections and let the Cesàro means* (3.5) *satisfy* (3.6). *Then every $\alpha \in bqc$ is a multiplier and*

$$(3.8)\qquad \|\alpha\|_M \leqq C\,\|\alpha\|_{bqc}\,.$$

PROOF. For each $f \in X$ set

$$f^\alpha = \sum_{k=0}^{\infty} (k+1)(\Delta^2\alpha_k^0)\sigma_k f + \alpha_\infty f\,,$$

where $\alpha_k^0 = \alpha_k - \alpha_\infty$, $k \in \boldsymbol{P}$. Then f^α exists in X since by (3.6)

$$\|f^\alpha\| \leqq C\,\|f\| \sum_{k=0}^{\infty} (k+1)\,|\Delta^2\alpha_k^0| + |\alpha_\infty|\,\|f\| \leqq C\,\|\alpha\|_{bqc}\,\|f\|\,.$$

Thus it remains to show that $f^\alpha \sim \sum \alpha_k P_k f$. But this follows since for $P_n \in [X]$ one has $P_n\sigma_k f = [1 - (n/(k+1))]P_n f$ if $k \geqq n$ and zero otherwise, and hence

$$P_n f^\alpha = \sum_{k=n}^{\infty} (k+1)(\Delta^2\alpha_k^0)\Big(1 - \frac{n}{k+1}\Big)P_n f + \alpha_\infty P_n f = \alpha_n P_n f\,.$$

REMARK. In case of a total biorthogonal system $\{f_k, f_k^*\}$ in X, $\{f_k\}$ being fundamental in X, (3.6) is equivalent to the statement that $\{f_k\}$ is a Cesàro basis, i.e., for every $f \in X$

$$\lim_{n\to\infty} \Big\| \sum_{k=0}^{n} \Big(1 - \frac{k}{n+1}\Big) f_k^*(f) f_k - f \Big\| = 0\,.$$

In this case Theorem 3.2 states that bqc is contained in the multiplier class associated with $\{f_k, f_k^*\}$; the converse direction, namely bqc being contained in the latter multiplier class implies that $\{f_k\}$ is a Cesàro basis, is shown by Kadeč [11].

Concerning connections between the various multiplier classes one has $bqc \subset bv \subset l^\infty$ in the sense of continuous embedding. For, if $\alpha \in bv$, then $\alpha_k = \sum_{m=k}^{\infty} \Delta\alpha_m + \alpha_\infty$, and thus $\|\alpha\|_\infty \leqq \|\alpha\|_{bv}$. If $\alpha \in bqc$, then $\lim_{n\to\infty} \Delta\alpha_n = 0$, and hence $\Delta\alpha_n = \sum_{k=n}^{\infty} \Delta^2\alpha_k$. This implies

$$\sum_{n=0}^{\infty} |\Delta\alpha_n| \leqq \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} |\Delta^2\alpha_k| = \sum_{k=0}^{\infty} (k+1)\,|\Delta^2\alpha_k|\,,$$

thus $\|\alpha\|_{bv} \leqq \|\alpha\|_{bqc}$. For more general results in this direction see [7].

Let us recall that our main interest in this section lies in furnishing us with sufficient criteria for a uniform bound (in $\rho > 0$) of the multipliers involved in the comparison Theorem 2.1. In general, the problem is very difficult on account of the complex structure of the multipliers. Therefore we shall restrict ourselves to the particular, but nevertheless widely applicable case that the family $\{\tau(\rho)\}_{\rho>0}$ is of Fejer's type, i.e., $\tau_k(\rho) = t(k/\rho)$ for some function $t(x)$ defined on $[0, \infty)$. Introducing $BV[0, \infty)$ as the set of functions of bounded variation on $[0, \infty)$ one obtains

LEMMA 3.3. *Let $\{\tau(\rho)\}_{\rho>0} \subset s$ be a family of sequences for which there exists a function $t(x) \in BV[0, \infty)$ such that $\tau_k(\rho) = t(k/\rho)$ for all $k \in \boldsymbol{P}$, $\rho > 0$. Then $\tau(\rho) \in bv$ for each $\rho > 0$ and*

$$\sum_{k=0}^{\infty} |\Delta\tau_k(\rho)| \leqq \int_0^{\infty} |dt(x)| \qquad (\rho > 0) . \tag{3.9}$$

Indeed, for any $n \in \boldsymbol{P}$ and $\rho > 0$

$$\sum_{k=0}^{n} |\Delta\tau_k(\rho)| \leqq \sum_{k=0}^{n} \int_k^{k+1} |dt(x/\rho)| \leqq \int_0^{\infty} |dt(x)| .$$

Since obviously $|\tau_k(\rho)| \leqq \sup_x |t(x)|$ uniformly for $\rho > 0$, one has by Theorem 3.1

COROLLARY 3.4. *Let $\{\tau(\rho)\}_{\rho>0} \subset s$ be as in Lemma 3.3 and $\{P_k\}$ as in Theorem 3.1 satisfying (3.2). Then $\{\tau(\rho)\}_{\rho>0}$ is a family of uniformly bounded multipliers.*

For the analogous result in case of bounded, quasi-convex sequences consider the space BQC of bounded, quasi-convex functions $t(x)$ defined on $[0, \infty)$. $BQC[0, \infty)$ consists of bounded continuous functions t which are locally (i.e. on every compact subinterval) absolutely continuous on $(0, \infty)$ and whose derivatives t' are locally of bounded variation* on $(0, \infty)$ such that $\int_0^{\infty} x\,|dt'(x)| < \infty$.

LEMMA 3.5. *Let $\{\tau(\rho)\}_{\rho>0} \subset s$ be a family of sequences for which there exists a function $t(x) \in BQC[0, \infty)$ such that $\tau_k(\rho) = t(k/\rho)$ for all $k \in \boldsymbol{P}$, $\rho > 0$. Then $\tau(\rho) \in bqc$ for each $\rho > 0$ and*

$$\sum_{k=0}^{\infty} (k+1)\,|\Delta^2\tau_k(\rho)| \leqq \int_0^{\infty} x\,|dt'(x)| \qquad (\rho > 0) . \tag{3.10}$$

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* In many cases of interest t' is furthermore continuous on $(0, \infty)$, except perhaps for a finite set of discontinuities of the first kind, and absolutely continuous in every bounded subinterval of $(0, \infty)$ which does not contain any of these points. Then $\int_0^{\infty} x\,|dt'(x)| < \infty$ is satisfied if $\int_0^{\infty} x\,|t''(x)|\,dx < \infty$.

PROOF. In view of the hypothesis and the definition of BQC one has for any $k \in \boldsymbol{P}$

$$\begin{aligned}\Delta^2\tau_k(\rho) &= \int_0^{1/\rho}\left[\int_{u+(k/\rho)}^{u+((k+1)/\rho)} dt'(x)\right]du = \int_{k/\rho}^{(k+2)/\rho} dt'(x)\int_{\max\{0,x-((k+1)/\rho)\}}^{\min\{1/\rho,x-(k/\rho)\}} du \\ &= \int_{k/\rho}^{(k+1)/\rho}[x - (k/\rho)]dt'(x) + \int_{(k+1)/\rho}^{(k+2)/\rho}[((k+2)/\rho) - x]dt'(x) .\end{aligned}$$

Hence, for arbitrary $n \in \boldsymbol{P}$ and $\rho > 0$

$$\begin{aligned}\sum_{k=0}^{n}(k+1)\,|\,\Delta^2\tau_k(\rho)\,| &\leqq \sum_{k=0}^{n}\int_{k/\rho}^{(k+1)/\rho} x\,|\,dt'(x)\,| \\ &\quad + \int_{(n+1)/\rho}^{(n+2)/\rho}(n+1)[((n+2)/\rho) - x]\,|\,dt'(x)\,| \leqq \int_0^\infty x\,|\,dt'(x)\,| .\end{aligned}$$

COROLLARY 3.6. *Let* $\{\tau(\rho)\}_{\rho>0}\subset s$ *be as in Lemma* 3.5 *and* $\{P_k\}$ *as in Theorem* 3.2 *satisfying* (3.6). *Then* $\{\tau(\rho)\}_{\rho>0}\subset s$ *is a family of uniformly bounded multipliers.*

4. Applications.

4.1 Typical and Abel-Cartwright means. Let X be a Banach space and $\{P_k\}$ be a sequence of projections as specified in Sec. 2. We would like to compare the following means of the series (2.1): The typical means of order $\kappa > 0$

$$R_\kappa(n)f \sim \sum_{k=0}^{n} r_\kappa(k/(n+1))P_kf\,, \quad r_\kappa(x) = \begin{cases}1 - x^\kappa, & 0 \leqq x \leqq 1 \\ 0 & , x \geqq 1\end{cases} \tag{4.1}$$

with the Abel-Cartwright means of order $\kappa > 0$

$$W_\kappa(n)f \sim \sum_{k=0}^{\infty} w_\kappa(k/(n+1))P_kf\,, \quad w_\kappa(x) = \exp\{-x^\kappa)\,, \qquad x \geqq 0 . \tag{4.2}$$

Obviously, there holds equality in (4.1) since the sum is finite. In order to show that $R_\kappa(n)$, $W_\kappa(n)$ are multiplier operators of type (2.4) with discrete parameter $\rho = n + 1$, $n \to \infty$, assume that the Cesàro means σ_n are uniformly bounded (see (3.6)). Then, since $r_\kappa, w_\kappa \in BQC$ (cf. [5; Sec. 6.4]), an application of Corollary 3.6 in particular gives that $R_\kappa(n)$, $W_\kappa(n) \in [X]$ are multiplier operators of type (2.4). To obtain an estimate of type (1.2) one may apply Theorem 2.1. Concerning condition (2.6), if $f \in N(W_\kappa(n) - I)$, then

$$0 = P_k(W_\kappa(n)f - f) = (\exp\{-(k/(n+1))^\kappa\} - 1)P_kf \qquad (k \in \boldsymbol{P})\,,$$

and hence $P_kf = 0$ for every $k \in \boldsymbol{N}$. Since $\{P_k\}$ is total this implies $f = P_0f$, and since the same reasoning applies to $R_\kappa(n)$, it follows that

$$N(W_\kappa(n) - I) = N(R_\kappa(n) - I) = P_0(X) \qquad (\kappa > 0;\ n \in \boldsymbol{P}) .$$

In order to verify the uniform multiplier condition (2.8) observe that in case of the typical and Abel-Cartwright means the corresponding sequences $\{\delta(\rho)\}$ are of Fejér's type so that one has to examine $d_\kappa(x)$, $[d_\kappa(x)]^{-1}$ where

$$d_\kappa(x) = \frac{w_\kappa(x) - 1}{r_\kappa(x) - 1} = \begin{cases} \dfrac{e^{-x^\kappa} - 1}{x^\kappa}, & 0 \leqq x \leqq 1 \\ -e^{-x^\kappa}, & x > 1 . \end{cases}$$

By an elementary calculation one has $d_\kappa(x)$, $[d_\kappa(x)]^{-1} \in BQC$ for each $\kappa > 0$ so that by Corollary 3.6 the uniform multiplier condition (2.8) is verified. Analogously one has $(r_{\kappa_2}(x) - 1)(r_{\kappa_1}(x) - 1)^{-1} \in BQC$ if $\kappa_2 > \kappa_1 > 0$. Thus

THEOREM 4.1. *Let X be a Banach space, $\{P_k\}_{k=0}^\infty \subset [X]$ be a total sequence of mutually orthogonal projections and let the Cesàro means σ_n of* (3.5) *satisfy* (3.6). *Then, for each $\kappa > 0$, the typical and the Abel-Cartwright means are equivalent,* i.e.,

$$\| R_\kappa(n)f - f \| \sim \| W_\kappa(n)f - f \| \qquad (f \in X) .$$

If $\kappa_2 > \kappa_1 > 0$ then $R_{\kappa_2}(n)$ is better than $R_{\kappa_1}(n)$, i.e., *there exists a constant D such that*

$$\| R_{\kappa_2} f - f \| \leqq D \| R_{\kappa_1} f - f \|$$

for all $f \in X$, $n \in \boldsymbol{P}$.

4.2 Trigonometric system. Let $X_{2\pi} = L_{2\pi}^p$, $1 \leqq p \leqq \infty$, or $C_{2\pi}$ be the Banach space of 2π-periodic functions with standard norms $\| \cdot \|_{X_{2\pi}}$

$$\left\{\int_{-\pi}^{\pi} |f(x)|^p dx\right\}^{1/p} (1 \leqq p < \infty),\ \text{ess. sup}\,|f(x)|,\ \max |f(x)| ,$$

respectively. Defining $\{P_k\}$ by

$$P_0 f(x) = f^\wedge(0) , \quad P_k f(x) = f^\wedge(k)e^{ikx} + f^\wedge(-k)e^{-ikx} \qquad (k \in \boldsymbol{N}) , \tag{4.3}$$

$f^\wedge(k)$ being the usual Fourier coefficients

$$f^\wedge(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx \qquad (k \in \boldsymbol{Z}) ,$$

it is obvious that $\{P_k\}$ is a sequence of orthogonal projections which are total on account of the uniqueness property for Fourier coefficients. The famous theorem of M. Riesz states that $S_n = \sum_{k=0}^n P_k$ is uniformly bounded in n provided $1 < p < \infty$, but not for $p = 1$ and $p = \infty$. Furthermore, the theorem of Fejér implies the uniform boundedness of the Cesàro means σ_n in every $X_{2\pi}$ which in particular shows that (3.6) does not imply (3.2).

Rewriting $R_\kappa(n)$ and $W_\kappa(n)$ in the usual fashion

$$R_\kappa(n)f(x) = \sum_{k=-n}^{n}\left(1 - \left(\frac{|k|}{n+1}\right)^\kappa\right)f^\wedge(k)e^{ikx} \tag{4.4}$$

$$W_\kappa(n)f(x) = \sum_{k=-\infty}^{\infty} \exp\{-(|k|/(n+1))^\kappa\}f^\wedge(k)e^{ikx} \tag{4.5}$$

one obtains from Theorem 4.1

COROLLARY 4.2. *Let $X_{2\pi}$ and $\{P_k\}$ be given as above. Then*

i) $\| R_\kappa(n)f - f\|_{X_{2\pi}} \sim \| W_\kappa(n)f - f\|_{X_{2\pi}}$ $\qquad (f \in X_{2\pi};\ \kappa > 0)$,

ii) $\| R_{\kappa_2}(n)f - f\|_{X_{2\pi}} \leqq D\| R_{\kappa_1}(n)f - f\|_{X_{2\pi}}$ $\qquad (f \in X_{2\pi};\ \kappa_2 > \kappa_1 > 0)$.

Let us note that Corollary 4.2 does not assert the convergence of $R_\kappa(n)f$ (or of $W_\kappa(n)f$) towards f as $n \to \infty$. This convergence is only guaranteed if $\bigcup_{k=0}^\infty P_k(X_{2\pi})$ is dense in $X_{2\pi}$, i.e. for $L^p_{2\pi}$, $1 \leqq p < \infty$, and $C_{2\pi}$. Hence $\{P_k(L^p_{2\pi})\}$ is in particular a Schauder decomposition of $L^p_{2\pi}$, $1 < p < \infty$. Let us also mention that the above equivalence relations imply some particular results of Žuk [21] who obtained these with the aid of estimates in terms of moduli of continuity.

REMARK. Formulae (4.3) and (2.4) indicate that our approach only admits symmetric operators in $L^p_{2\pi}$, $1 < p < \infty$. But it is immediately clear that this is not necessary. Indeed, a sufficient multiplier condition corresponding to Theorem 3.1 for a two-a-way sequence $\{P_k\}_{k=-\infty}^\infty$ of projections reads for $\{\alpha_k\}_{k=-\infty}^\infty$

$$|\alpha_k| \leqq M \quad (k \in \boldsymbol{Z}), \sum_{k=-\infty}^{\infty} |\Delta\alpha_k| \leqq C .$$

This condition was weakened by Marcinkiewicz (cf. [1]) to

$$|\alpha_k| \leqq M \quad (k \in \boldsymbol{Z}), \sum_{|k|=2^N}^{2^{N+1}} |\Delta\alpha_k| \leqq C^* \qquad (N \in \boldsymbol{P}) ;$$

for a discrete analog see Sunouchi [19].

Now let us briefly indicate the connection on $X_{2\pi}$ between multiplier operators and operators of Fourier convolution type.

For $\kappa = 1$ the operator $R_\kappa(n)$ coincides with the Cesàro mean operator σ_n and admits the closed representation

$$\sigma_n f(x) = \frac{1}{2\pi(n+1)}\int_{-\pi}^{\pi} f(x-u)\left[\frac{\sin((n+1)u/2)}{\sin(u/2)}\right]^2 du$$

which is Fejér's singular integral. $W_\kappa(n)$ reduces for $\kappa = 1$ to the classical Abel means and also admits the closed representation

$$P_r f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\frac{1-r^2}{1-2r\cos u + r^2}du \qquad (r = e^{-1/n})$$

which is the singular integral of Abel-Poisson. Thus Corollary 4.2 states the equivalence of the (approximation) processes σ_n and P_r on $X_{2\pi}$. Generally, operators of type (2.4) in $X_{2\pi}$ may be reformulated as Fourier convolution type integrals,

$$Tf(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)d\mu(u) \qquad (f \in X_{2\pi}),$$

with appropriate "2π-periodic" kernel μ.

4.3 Legendre polynomials. Let $X = L^p(-1, 1)$, $1 \leqq p < \infty$, or $C[-1, 1]$ with norm $\|\cdot\|$

$$\left\{\int_{-1}^{1} |f(x)|^p dx\right\}^{1/p},\ 1 \leqq p < \infty, \qquad \max |f(x)|,$$

respectively. Consider the Legendre polynomials defined by

$$C_k(x) = (-1)^k[2^k k!]^{-1}(d/dx)^k[(1-x^2)^k] \qquad (k \in \boldsymbol{P}).$$

Since

$$\int_{-1}^{1} C_k(x)C_m(x)dx = \left(k + \frac{1}{2}\right)^{-1}\delta_{km},$$

the projections $\{P_k\}$,

$$P_k f(x) = \left[\left(k + \frac{1}{2}\right)\int_{-1}^{1} f(u)C_k(u)du\right]C_k(x) \qquad (k \in \boldsymbol{P})$$

are mutually orthogonal.

Pollard [16] has shown that the corresponding partial sum operators $\{S_n\}$ are uniformly bounded and approximate $f \in L^p(-1, 1)$ provided $4/3 < p < 4$. On the other hand, Askey-Hirschman [2] have proved that the Cesàro mean operators $\{\sigma_n\}$ are uniformly bounded and approximate $f \in X$ for every X. Thus, $\{P_k\}$ is total and $\bigcup_{k=0}^{\infty} P_k(X)$ is dense in X; in particular, $\{C_k\}$ is a Schauder basis in $L^p(-1, 1)$, $4/3 < p < 4$, and a Cesàro basis in every X. Hence, on account of Theorem 4.1

COROLLARY 4.3. *Let X, $\{P_k\}$ be as above and $R_\kappa(n)$, $W_\kappa(n)$ be given by* (4.1), (4.2), *respectively. Then, for each $f \in X$,*

i) $\|R_\kappa(n)f - f\| \sim \|W_\kappa(n)f - f\| \qquad (\kappa > 0)$,

ii) $\|R_{\kappa_2}(n)f - f\| \leqq D\|R_{\kappa_1}(n)f - f\| \qquad (\kappa_2 > \kappa_1 > 0)$.

Statements analogous to this Corollary may be derived for ultraspherical polynomials of order $\lambda \geqq 0$

$$C_k^\lambda(x) = M_{k,\lambda}(1 - x^2)^{-\lambda+1/2}(d/dx)^k[(1 - x^2)^{k+\lambda-1/2}] , \tag{4.6}$$

$M_{k,\lambda}$ being a suitable constant. They coincide for the particular instance $\lambda = 1/2$ with the Legendre polynomials, for $\lambda = 0$ with the Tchebychef polynomials of the first kind, and for $\lambda = 1$ with the Tchebychef polynomials of the second kind. Furthermore, one has with respect to the weight function $(1 - x^2)^{\lambda-1/2}$

$$\int_{-1}^{1} C_k^\lambda(x)C_m^\lambda(x)(1 - x^2)^{\lambda-1/2}dx = M_{k,\lambda}^*\delta_{km} .$$

Thus, the projections $\{P_k\}$,

$$P_k f(x) = \left[(M_{k,\lambda}^*)^{-1}\int_{-1}^{1} f(u)C_k^\lambda(u)(1 - u^2)^{\lambda-1/2}du\right]C_k^\lambda(x) , \tag{4.7}$$

are mutually orthogonal in

$$X^{\lambda,p} = \left\{f; \| f \|_p = \left(\int_{-1}^{1} | f(x) |^p(1 - x^2)^{\lambda-1/2}dx\right) < \infty\right\} . \tag{4.8}$$

Since $\{C_k^\lambda\}$ is a Schauder basis in $X^{\lambda,p}$ if $p \in ((2\lambda + 1)/(\lambda + 1), (2\lambda + 1)/\lambda)$ (cf. [16]) and since $\{C_k^\lambda\}$ is a Cesàro basis in $X^{\lambda,p}$ if $p \in ((2\lambda + 1)/(\lambda + 2), (2\lambda + 1)/(\lambda - 1))$ for $\lambda \geqq 1$ and for all $p, 1 \leqq p < \infty$, if $0 \leqq \lambda < 1$ (cf. [2]), all the other properties required for $\{P_k\}$ are satisfied.

Let us mention that for ultraspherical polynomials there exists a strengthening of Theorem 3.1 analogously to the Marcinkiewicz result, due to Muckenhoupt-Stein [15]: $\alpha \in s$ is multiplier in $X^{\lambda,p}$, $(2\lambda+1)/(\lambda+1) < p < (2\lambda + 1)/\lambda$ if

$$| \alpha_k | \leqq M \quad (k \in \boldsymbol{P}), \sum_{k=2^N}^{2^{N+1}} | \alpha_k - \alpha_{k+1} | \leqq M^* \qquad (N \in \boldsymbol{P}) .$$

It would be interesting to know if these conditions are also sufficient in the Laguerre and Hermite series case, and how they may be related to the multiplier problem of general expansions of type (2.1) in case of strong convergence of the partial sums.

4.4 **Laguerre series.** Let $X = L^p(0, \infty)$, $1 \leqq p < \infty$, or $C[0, \infty)$ with e.g. $\| f \|_p = \left(\int_0^\infty | f(x) |^p dx\right)^{1/p}$ and consider the Laguerre polynomials of order $\alpha > -1$ defined by

$$L_k^{(\alpha)}(x) = [k!]^{-1}e^x x^{-\alpha}(d/dx)^k(e^{-x}x^{k+\alpha}) \qquad (k \in \boldsymbol{P}) .$$

Setting

$$\varphi_k^{(\alpha)}(x) = \left\{\Gamma(\alpha+1)\binom{k+\alpha}{k}\right\}^{-1/2} x^{\alpha/2}e^{-x/2}L_k^{(\alpha)}(x)$$

it is known that $\varphi_k^{(\alpha)}$ is an orthonormal system on $(0, \infty)$. Thus the projections

$$P_k^{(\alpha)}f(x) = \left[\int_0^\infty f(u)\varphi_k^{(\alpha)}(u)du\right]\varphi_k^{(\alpha)}(x)$$

are mutually orthogonal. Furthermore, Askey-Wainger [3] for $\alpha > 0$ and Muckenhoupt [14] for $\alpha > -1$ have shown that the partial sums are uniformly bounded and converge to f for $4/3 < p < 4$. Furthermore, Poiani [15a] has recently shown the uniform boundedness of the Cesàro mean operators for $1 \leqq p \leqq \infty$ if $\alpha > 0$, and $(1 + \alpha/2)^{-1} < p < -2/\alpha$ if $-1 < \alpha \leqq 0$. Hence

COROLLARY 4.4. *Let* $\{P_k^{(\alpha)}\}$ *be as above, and* $R_\kappa(n)$, $W_\kappa(n)$ *be given by* (4.1), (4.2) *respectively. Then*

i) $\| R_\kappa(n)f - f \| \sim \| W_\kappa(n)f - f \|$ $\quad (\kappa > 0)$,

ii) $\| R_{\kappa_2}(n)f - f \| \leqq D \| R_{\kappa_1}(n)f - f \|$ $\quad (\kappa_2 > \kappa_1 > 0)$

for all X-spaces in case $\alpha > 0$, *otherwise restricted as indicated above.*

A statement analogous to Corollary 4.4 may be formulated for the Hermite series case.

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