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THE SPLITTING AND DEFORMATIONS OF THE GENERALIZED GAUSS MAP OF COMPACT CMC SURFACES

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Abstract. We show that a non-conformal harmonic map from a Riemann surface into the Euclidean n-sphere can be considered as a component of minimal surfaces in higher dimensional spheres. In the same principle, we show that the generalized Gauss map of constant mean curvature surfaces in the 3-sphere globally splits into two non-conformal harmonic maps into the 2-sphere. Using this, we obtain examples of non-trivial harmonic map deformations for compact Riemann surfaces of arbitrary positive genus. In particular, we give a lower bound for the nullity (as harmonic maps) of the generalized Gauss map of compact CMC surfaces in the 3-sphere. Furthermore, we obtain an affirmative answer to Lawson's conjecture for superconformal minimal surfaces in $4m$-spheres.

1. Introduction. In this paper, we are interested in constant mean curvature surfaces including minimal surfaces in the 3-sphere, which we call CMC surfaces for short.

A harmonic map ϕ from a Riemann surface M into the Euclidean sphere S^n or into the complex projective space $\boldsymbol{C}P^n$ is associated with two important families of maps, the harmonic sequence $\{\phi_j\}$ and the associated S^1-family $\{\phi^\theta\}$. Using the latter, we construct a harmonic map $\tilde{\phi}$ into higher dimensional spheres or complex projective spaces, by taking direct product $S^n(c_1)\times\cdots\times S^n(c_k)\subset S^{k(n+1)-1}$ or $\boldsymbol{C}^{n+1}-\{0\}\times\cdots\times \boldsymbol{C}^{n+1}-\{0\}/\sim\ \subset \boldsymbol{C}P^{k(n+1)-1}$, and defining a map by $\tilde{\phi}=(1/\sqrt{c_1}\phi^{\theta_1},\dots,1/\sqrt{c_k}\phi^{\theta_k})=\bigoplus_{j=1}^k \phi^{\theta_j}/\sqrt{c_j}$ where $\sum 1/c_j=1$, or by $\tilde{\phi}=[(f^{\theta_1},\dots,f^{\theta_k})]$ using local sections f^{θ_j}'s of ϕ^{θ_j}'s (cf. [L2], [M]). In [M], we investigated superconformal harmonic maps in this method, while we now apply it to the CMC surface theory.

Choosing suitable θ_j's, we find a harmonic map $\tilde{\phi}$ having the isotropy dimension larger than that of ϕ (Theorem 3.4). An easy application of this yields conformal harmonic maps from a non-conformal harmonic map (Corollary 3.5). Even the simplest case implies an interesting result:

COROLLARY 3.2. *Let ϕ be a non-conformal harmonic map into S^2. Then $\tilde{\phi}=(\phi\oplus\phi^\pi)/\sqrt{2}$ is a minimal surface in S^5.*

This turns out to be the splitting of the bipolar surface in [L1] of a minimal

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surface in S^3 (see Theorem 5.3). In fact, because of $\mathrm{Gr}_2^+(\boldsymbol{R}^4)\cong S^2\times S^2$, the global splitting of the generalized Gauss map of a surface in $\boldsymbol{R}^4$ would be obvious (cf. [HO]). We clarify this splitting for CMC surfaces in S^3 by connecting directly the generarized Gauss map with the adapted secondary Gauss map (for the definition, see §4).

THEOREM 6.2. *For a* CMC-h *surface* $\psi: M\to S^3$, *there exists a pair of non-conformal harmonic maps* ϕ, $\phi^{2\theta}: M\to S^2$ *such that the generalized Gauss map* $\tilde{\psi}$ *of* ψ *splits into* $(\phi\oplus\phi^{2\theta})/\sqrt{2}$. *In fact,* ϕ *is the adapted secondary Gauss map of* ψ, *and* θ *is given by* $\cos^{-1}\sqrt{h^2/(h^2+1)}$. *Moreover,* $\tilde{\psi}$ *can be deformed into* ϕ *and/or* $\phi^{2\theta}$ *through harmonic maps* $\tilde{\phi}_s^{2\theta}=\cos s\phi\oplus\sin s\phi^{2\theta}$ *into* S^5.

The deformation of harmonic maps is important in investigating the moduli spaces. For harmonic maps from a compact Riemann surface with genus greater than one, nothing is known except some existence theorems (cf. [L1], [K1], [K2]). When we apply the theorem to the generalized Gauss map of Lawson's compact minimal surfaces in S^3, we obtain examples of non-trivial global deformations of harmonic maps from a compact Riemann surface of arbitrary positive genus. As an application, we show that the nullity (as harmonic maps) of the generalized Gauss map of CMC surfaces (of positive genus) in S^3 is at least 16. The classifying problem of surfaces having Gauss map with small Killing nullity would be interesting.

Recently, Aiyama and Akutagawa [AA] obtain Kenmotsu-Bryant type representation formula of CMC surfaces in S^3, using the framing matrix and the secondary Gauss map. After we obtain our theorem, we know that the first statement of Theorem 6.2 independently follows from their argument. However, our idea comes from the splitting of harmonic maps in various dimensional spheres as in Theorem 3.4.

Eventually, a global correspondence between CMC surfaces in $\boldsymbol{RP}^3$ and a pair of associated non-conformal harmonic maps into S^2 is obtained in [AAMU].

Another application of our argument is to show:

THEOREM 7.3. *A full superconformal minimal surface in* S^{4m} *cannot be isometric to a minimal surface in* S^3.

This generalizes the result by Sakaki [S] for minimal surfaces in S^4 and gives a partial answer to Lawson's conjecture [L2], together with the odd dimensional case given in [M, Corollary 6.6].

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2. Preliminaries. For details in this section, see [M, Part II]. We denote by $S^n(c)$ the n-dimensional Euclidean sphere of radius $1/\sqrt{c}$ and $S^n=S^n(1)$. Let $\phi: M\to S^n$ be a harmonic map from a Riemann surface M into S^n. Let U be a simply connected open domain of M with a complex parameter z, and put $\partial=\partial/\partial z$. Then we have

$$\langle\phi,\phi\rangle=1\,,\quad\langle\partial\phi,\phi\rangle=\langle\bar{\partial}\phi,\phi\rangle=0\,, \tag{2.1}$$

$$\partial\bar{\partial}\phi = -|\partial\phi|^2\phi\,, \tag{2.2}$$

where $\langle\ ,\ \rangle$ is the complex-linearly extended inner product. Moreover, defining

$$\begin{cases} \phi_0 = \phi \\ \phi_{j+1} = \partial\phi_j - \partial\log|\phi_j|^2\phi_j\,, \end{cases} \tag{2.3}$$

we obtain

$$\bar{\partial}\phi_j = -\frac{|\phi_j|^2}{|\phi_{j-1}|^2}\phi_{j-1}\,. \tag{2.4}$$

When we put $w_j = \log|\phi_j|$, the integrability condition $\partial\bar{\partial}\phi_j = \bar{\partial}\partial\phi_j$ is given by

$$2\partial\bar{\partial}w_j - e^{2(w_{j+1}-w_j)} + e^{2(w_j - w_{j-1})} = 0\,, \qquad j \in \boldsymbol{Z}\,, \tag{2.5}$$

which is known as the 2-dimensional affine Toda equations. Periodic solutions to this, (for instance, a solution to the sinh-Gordon equation (4.9), (7.1)), correspond to superconformal harmonic maps into odd-dimensional spheres (cf. [M, §6]).

Because of the reality of ϕ, we get inductively:

$$\phi_{-j} = (-1)^j \frac{\bar{\phi}_j}{|\phi_j|^2}\,, \qquad j \in \boldsymbol{Z}\,. \tag{2.6}$$

The quadratic differential $\varphi_1 dz^2 = \langle\phi_1, \phi_1\rangle dz^2$ is holomorphic by (2.1), and is called the (first) Hopf differential. The isotropy dimension r of ϕ is defined by

$$\varphi_i = \langle\phi_i, \phi_i\rangle \equiv 0\,, \qquad \text{for} \quad 1 \le i \le r\,, \quad \text{and} \quad \langle\phi_{r+1}, \phi_{r+1}\rangle \not\equiv 0\,.$$

Then, $\varphi_{r+1}dz^{2(r+1)}$ is a holomorphic differential by (2.3) and (2.4), and is called the $(r+1)$*-st Hopf differential.* Note that ϕ is conformal if $r \ge 1$, and recall that a full map ϕ is superminimal if $r = \infty$, and superconformal if $r = m-1$, when $n = 2m$ or $2m-1$.

3. Construction of minimal surfaces from a non-conformal harmonic map.

FACT 3.1 (cf. [M, Theorem 10.1]). *Let $\phi\colon U \to S^{2m}$ be a full superconformal harmonic map. Then $g = (\phi \oplus \phi^{\pi})/\sqrt{2}\, : U \to S^{4m+1}$ is a harmonic map whose isotropy dimension is $2m-1$.*

A non-conformal harmonic map into S^2 is superconformal, hence we get immediately:

COROLLARY 3.2. *From a non-conformal harmonic map $\phi\colon M \to S^2$, we obtain an S^1-family of minimal surfaces $\tilde{\phi}^{\theta} = (\phi^{\theta} \oplus \phi^{\theta+\pi})/\sqrt{2}\, : U \to S^5$, $\theta \in [0, 2\pi)$, of isotropy dimension* 1.

To obtain more general results, we show:

PROPOSITION 3.3. *Let $\phi\colon M \to S^n$ be a non-superminimal harmonic map of isotropy dimension r with the $(r+1)$-st Hopf differential φ. Let U be a contractible domain of M.*

Then the associated S^1-family consists of harmonic maps $\phi^\theta\colon U\to S^n$ of isotropy dimension r with the $(r+1)$-st Hopf differential φ^θ, satisfying

$$|\phi_j^\theta|=|\phi_j|\,,\qquad j\in\boldsymbol{Z}\,, \tag{3.1}$$

$$\varphi^\theta=e^{i\theta}\varphi\,. \tag{3.2}$$

If this is shown, we obtain:

THEOREM 3.4. *Let $\phi\colon M\to S^n$ be a non-superminimal harmonic map of isotropy dimension r and let $\{\phi^\theta\}$ be the S^1-family of harmonic maps of isotropy dimension r. Then for any $k\geq 2$,*

$$\tilde\phi=\frac{1}{\sqrt{k}}\bigoplus_{l=1}^{k}\phi^l\colon U\to S^{k(n+1)-1}$$

$$\phi^l=\phi^{\theta_l}\,,\qquad \theta_l=2\pi l/k\,,$$

is a harmonic map of isotropy dimension at least $r+1$.

COROLLARY 3.5. *From a non-conformal harmonic map $\phi\colon M\to S^n$, we obtain an S^1-family of minimal surfaces*

$$\tilde\phi^\theta=\frac{1}{\sqrt{k}}\bigoplus_{l=1}^{k}\phi^l\colon U\to S^{k(n+1)-1}$$

for any $k\geq 2$, where $\phi^l=\phi^{\theta_l}$, $\theta_l=\theta+2\pi l/k$, $\theta\in[0,2\pi)$.

REMARK. (1) A non-conformal harmonic map into a sphere is thus characterized as a component of a minimal surface of higher-dimensional spheres.

(2) The image of $\tilde\phi$ lies in $S^n(k)\times\cdots\times S^n(k)\subset S^{k(n+1)-1}$, but is not necessarily full.

(3) A similar argument implies that we can construct harmonic maps into complex projective space, having larger isotropy dimension than the original one.

PROOF OF THEOREM 3.4. By Proposition 3.3, we have

$$\langle\phi_j^\theta,\phi_j^\theta\rangle=0\,,\qquad j=1,\dots,r$$

and $\varphi^\theta dz^{2(r+1)}=e^{i\theta}\varphi dz^{2(r+1)}$. Since each ϕ^θ satisfies the harmonic map equation

$$\partial\bar\partial\phi^\theta=-|\phi_1^\theta|^2\phi^\theta\,,$$

and since $|\phi_1^\theta|^2=e^{2w_1}$ does not depend on θ, $\tilde\phi$ satisfies the harmonic map equation. Moreover,

$$\tilde\phi_j=\frac{1}{\sqrt{k}}\bigoplus_{l=1}^{k}\phi_j^l$$

implies

$$\langle \tilde{\phi}_j, \tilde{\phi}_j \rangle = 0 , \qquad j = 1, \dots, r ,$$

$$\langle \tilde{\phi}_{r+1}, \tilde{\phi}_{r+1} \rangle = \frac{1}{k} \sum_{j=1}^{k} e^{i\theta_j} \varphi = 0 ,$$

which means that the isotropy dimension of $\tilde{\phi}$ is not less than $r+1$. $\square$

Proposition 3.3 might be well-known, but we show the proof for completeness.

PROOF OF PROPOSITION 3.3. Let $\pi : SO(n+1) \to S^n$ be the orthonormal frame bundle of S^n, and take a framing of ϕ by orthonormalizing $(\phi_0, \Re\phi_1, \Im\phi_1, \dots, \Re\phi_m, \varepsilon\Im\phi_m)$, where $n+1 = 2m+\varepsilon$, $\varepsilon = 0$ or 1, and $\Re\phi_j$ ($\Im\phi_j$, respectively) denotes the real part (the imaginary part, respectively) of ϕ_j. Extending $SO(n+1)$ to $SO(n+1)^{\boldsymbol{C}}$ and orthonormalizing $(\phi_0, \phi_1, \phi_{-1}, \phi_2, \phi_{-2}, \dots, \phi_m, \varepsilon\phi_{-m})$, we obtain the $SO(n+1)^{\boldsymbol{C}}$ framing $\Phi = (u_0, u_1, \dots, u_n)$. Recall that any $2r+2$ consecutive maps in the harmonic sequence are mutually orthogonal (cf. [BW, Theorem 2.4]). When $r=0$, putting $\varphi = \langle \phi_1, \phi_1 \rangle$, we have

$$u_0 = \phi_0 , \quad u_1 = \frac{\phi_1}{|\phi_1|} , \quad u_2 = \frac{|\phi_1|^2}{\sqrt{|\phi_1|^4 - |\varphi|^2}} \left(-\frac{\bar{\phi}_1}{|\phi_1|} + \frac{\bar{\varphi}_1}{|\phi_1|^2} u_1 \right), \dots .$$

Thus we get $\partial u_0 = |\phi_1| u_1$, and $\langle \partial u_1, \bar{u}_0 \rangle = -\langle u_1, \partial u_0 \rangle = -\varphi/|\phi_1|$, and hence $\mathfrak{so}(n+1)^{\boldsymbol{C}}$-valued 1-form $\alpha = \Phi^{-1} d\Phi = A dz + B d\bar{z}$, $B = -{}^t A$ is given by

$$A = \begin{pmatrix} 0 & -\varphi/r_1 & * & \cdots & * \\ r_1 & & & & \\ 0 & & * & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix} , \tag{3.3}$$

where $r_j = e^{w_j - w_{j-1}}$. Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$ be the symmetric decomposition of $\mathfrak{g} = \mathfrak{so}(n+1)$ for S^n, where $\mathfrak{p}$ is given by

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & \xi \\ {}^t\xi & 0 \end{pmatrix} \middle| \, {}^t\xi \in \boldsymbol{R}^n \right\} .$$

Let $\alpha = \alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{h}} + \alpha''_{\mathfrak{p}}$ be the decomposition of α into the $\mathfrak{p}^{(1,0)}$, $\mathfrak{h}$ and $\mathfrak{p}^{(0,1)}$ components, respectively. Then the extended framing Φ_λ is given by integrating

$$\alpha_\lambda = \lambda^{-1} \alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{h}} + \lambda \alpha''_{\mathfrak{p}} \quad \lambda \in S^1 , \tag{3.4}$$

where

$$\lambda^{-1} \alpha'_{\mathfrak{p}} + \lambda \alpha''_{\mathfrak{p}} = \begin{pmatrix} 0 & -\lambda r_1 d\bar{z} - (\lambda^{-1} \varphi / r_1) dz & * & \cdots & * \\ \lambda^{-1} r_1 dz + (\lambda \bar{\varphi} / r_1) d\bar{z} & & & & \\ * & & & & \\ \vdots & & & & \\ * & & & & \end{pmatrix} .$$

Using

$$U=\begin{pmatrix}\lambda^{-1} & 0\\ 0 & I\end{pmatrix}\in U(n+1)\,,$$

we obtain

$$\mathrm{Ad}\,U(\lambda^{-1}\alpha_{\mathfrak{p}}'+\lambda\alpha_{\mathfrak{p}}'')=\begin{pmatrix} 0 & -r_1d\bar z-(\lambda^{-2}\varphi/r_1)dz & * & \cdots & * \\ r_1dz+(\lambda^2\bar\varphi/r_1)d\bar z & & & & \\ * & & & & \\ \vdots & & & & \\ * & & & & \end{pmatrix}$$

or

$$\mathrm{Ad}\,U(A_\lambda)=\begin{pmatrix} 0 & -\lambda^{-2}\varphi/r_1 & * & \cdots & * \\ r_1 & & & & \\ 0 & & & * & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}. \tag{3.5}$$

By this, the harmonic map $\phi^\lambda=\pi\circ\Phi_\lambda$ satisfies $|\phi_1^\lambda|=r_1=|\phi_1|$. Since all $|\phi_j^\lambda|^2$ are determined from two consecutive ones by (2.5), and we have $|\phi_0^\lambda|^2=1$, we obtain (3.1). Comparing (3.3) and (3.5), we know that ϕ^λ is non-conformal and has the Hopf differential $\lambda^{-2}\varphi$. Thus putting $\lambda^{-2}=e^{i\theta}$, $\phi^\theta=\phi^\lambda$ satisfies (3.1) and (3.2).

When $r\geq 1$, ϕ is lifted up to a unique primitive map ψ into the flag manifold $F^r(S^n)=SO(n+1)/(SO(2)\times\cdots SO(2)\times SO(n-2r))$ (cf. [B, Theorem 3.2]), by $\psi=(\psi_1\subset\psi_2\subset\cdots\subset\psi_r)$ where

$$\psi_j(z)=\mathrm{span}_{\boldsymbol{C}}\{\phi_i(z),\,1\leq i\leq j\}\subset(T_{\phi(z)}S^n)^{\boldsymbol{C}}\,,\qquad z\in U\,,\quad j=1,\dots,r\,.$$

A primitive map exists in an S^1-family ψ^θ (cf. [BP, 3.3, p. 247]), and by [BP, Theorem 3.7], using the projection $\varpi\colon F^r(S^n)\to S^n$, we obtain an S^1-family of harmonic maps $\phi^\theta=\varpi\circ\psi^\theta$, which, by construction, has isotropy dimension r. We show that ϕ^θ satisfies (3.1) and (3.2). The $SO(n+1)^{\boldsymbol{C}}$ framing $\Psi=(u_0,u_1,\dots,u_n)$ satisfies

$$u_{2j-1}=\frac{\phi_j}{|\phi_j|}\,,\quad u_{2j}=\frac{\phi_{-j}}{|\phi_{-j}|}=(-1)^j\frac{\bar\phi_j}{|\phi_j|}\,,\quad 1\leq j\leq r\,,\quad u_{2r+1}=\frac{\phi_{r+1}}{|\phi_{r+1}|}\,. \tag{3.6}$$

CLAIM. *We have*

$$\partial u_{2j-1}=r_{j+1}u_{2j+1}+\partial w_ju_{2j-1}\,,\qquad j=1,\dots,r\,, \tag{3.7}$$

$$\partial u_{2j}=r_ju_{2(j-1)}-\partial w_ju_{2j}\,,\qquad j=1,\dots,r\,, \tag{3.8}$$

$$\langle\partial u_{2r+i},\bar u_k\rangle=0\qquad \textit{for}\quad 1\leq i\leq n-2r\,,\quad 0\leq k\leq 2r-1\,,$$

$$\langle \partial u_{2r+1}, \bar{u}_{2r} \rangle = (-1)^{r+1} \frac{\varphi}{|\phi_r||\phi_{r+1}|}.$$

Indeed, the first two are easily obtained from (3.6) using (2.3) and (2.4). Note that $u_{2j} = (-1)^j \bar{u}_{2j-1}$, $j = 1, \ldots, r$. For $1 \le k = 2j-1 \le 2r-1$, we have

$$\begin{aligned}
\langle \partial u_{2r+i}, \bar{u}_{2j-1} \rangle &= \langle \partial u_{2r+i}, (-1)^j u_{2j} \rangle = (-1)^{j+1} \langle u_{2r+i}, \partial u_{2j} \rangle \\
&= (-1)^{j+1} \langle u_{2r+i}, r_j u_{2(j-1)} - \partial w_j u_{2j} \rangle \\
&= \langle u_{2r+i}, r_j \bar{u}_{2j-3} + \partial w_j \bar{u}_{2j-1} \rangle = 0
\end{aligned}$$

by (3.8), and for $0 \le k = 2j \le 2(r-1)$,

$$\begin{aligned}
\langle \partial u_{2r+i}, \bar{u}_{2j} \rangle &= \langle \partial u_{2r+i}, (-1)^j u_{2j-1} \rangle = (-1)^{j+1} \langle u_{2r+i}, \partial u_{2j-1} \rangle \\
&= (-1)^{j+1} \langle u_{2r+i}, r_{j+1} u_{2j+1} + \partial w_j u_{2j-1} \rangle \\
&= \langle u_{2r+i}, r_{j+1} \bar{u}_{2(j+1)} - \partial w_j \bar{u}_{2j} \rangle = 0
\end{aligned}$$

by (3.7). Finally,

$$\begin{aligned}
\langle \partial u_{2r+1}, \bar{u}_{2r} \rangle &= (-1)^{r+1} \langle u_{2r+1}, r_{r+1} u_{2r+1} + \partial w_r u_{2r-1} \rangle \\
&= (-1)^{r+1} \frac{\varphi}{|\phi_r||\phi_{r+1}|},
\end{aligned}$$

and we obtain the claim.

Put $\Psi^{-1} d\Psi = A dz + B d\bar{z}$, $B = -{}^t\bar{A}$. Then we get

$$A = \begin{pmatrix}
0 & M_1 & 0 & \cdots & \cdots & \cdots & 0 \\
N_0 & K_1 & \ddots & \ddots & & & \vdots \\
0 & N_1 & \ddots & M_j & \ddots & & \vdots \\
\vdots & \ddots & \ddots & K_j & \ddots & \ddots & \vdots \\
\vdots & & \ddots & N_j & \ddots & M_r & 0 \\
\vdots & & & \ddots & \ddots & K_r & M_{r+1} \\
0 & \cdots & \cdots & \cdots & 0 & N_r & K_{r+1}
\end{pmatrix},$$

where

$$M_1 = (0 \quad r_1), \quad M_j = \begin{pmatrix} 0 & 0 \\ 0 & r_j \end{pmatrix}, \qquad j = 2, \ldots, r$$

$$M_{r+1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ (-1)^{r+1} \varphi / |\phi_r||\phi_{r+1}| & * & \cdots & * \end{pmatrix},$$

$$N_0=\begin{pmatrix} r_1 \\ 0 \end{pmatrix},\quad N_r=\begin{pmatrix} r_{r+1} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix},\quad N_j=\begin{pmatrix} r_{j+1} & 0 \\ 0 & 0 \end{pmatrix},\qquad j=1,\dots,r-1$$

$$K_j=\begin{pmatrix} \partial w_j & 0 \\ 0 & -\partial w_j \end{pmatrix},\qquad j=1,\dots,r\,,$$

and K_{r+1} is an $(n-2r)\times(n-2r)$ matrix, N_r is an $(n-2r)\times 2$ matrix.

Let $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{k}$ be the homogeneous decomposition of $\mathfrak{so}(n+1)$ for $F^r(S^n)$, and decompose the $\mathfrak{g}^{\boldsymbol{C}}$-valued 1-form $\alpha=\Psi^{-1}d\Psi=\alpha'_{\mathfrak{m}}+\alpha_{\mathfrak{k}}+\alpha''_{\mathfrak{m}}$ into the $\mathfrak{m}^{(1,0)}$, $\mathfrak{k}$ and $\mathfrak{m}^{(0,1)}$ components, respectively. Then the $\mathfrak{k}$ component of A consists of $K_0,\dots,K_{r+1}$ and the rest is the $\mathfrak{m}$ component. Ψ_μ is given by integrating

$$\alpha_\mu=\mu^{-1}\alpha'_{\mathfrak{m}}+\alpha_{\mathfrak{k}}+\mu\alpha''_{\mathfrak{m}}\qquad \mu\in S^1\,,$$

which yields $\psi^\mu=\pi^r\circ\Psi_\mu$, where $\pi^r\colon SO(n+1)\to F^r(S^n)$ is the coset projection, and further $\phi^\mu=\varpi\circ\psi^\mu$. Let

$$U=\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & U_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & U_r & 0 \\ 0 & \cdots & \cdots & 0 & U_{r+1} \end{pmatrix}\in U(n+1)$$

where

$$U_j=\begin{pmatrix} \mu^j & 0 \\ 0 & \mu^{-j} \end{pmatrix},\quad j=1,\dots,r\,,\quad U_{r+1}=\begin{pmatrix} \mu^{r+1} & 0 \\ 0 & I_{n-2r-1} \end{pmatrix},\quad \mu\in S^1\,.$$

Then we get

$$\mathrm{Ad}\,U(A_\mu)=\begin{pmatrix} 0 & M_1 & 0 & \cdots & \cdots & \cdots & 0 \\ N_0 & K_1 & \ddots & \ddots & & & \vdots \\ 0 & N_1 & \ddots & M_j & \ddots & & \vdots \\ \vdots & \ddots & \ddots & K_j & \ddots & \ddots & \vdots \\ \vdots & & \ddots & N_j & \ddots & M_r & 0 \\ \vdots & & & \ddots & \ddots & K_r & M'_{r+1} \\ 0 & \cdots & \cdots & \cdots & 0 & N_r & K'_{r+1} \end{pmatrix},$$

where

$$M'_{r+1}=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ (-1)^{r+1}\mu^{-2(r+1)}\varphi/|\phi_r||\phi_{r+1}| & * & \cdots & * \end{pmatrix}.$$

Comparing M_{r+1} with M'_{r+1}, we obtain $|\phi^\mu_j|=|\phi_j|$, and ϕ^μ has the $(r+1)$-st Hopf differential $\mu^{-2(r+1)}\varphi$. Then for $e^{i\theta}=\mu^{-2(r+1)}$, $\phi^\theta=\phi^\mu$ satisfies (3.1) and (3.2). Finally comparing Φ_λ given by integrating (3.4) with Ψ_μ, we obtain Ad $U\Psi_\mu=$ Ad $V\Phi_\lambda$, where

$$V=\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & V_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & V_r & 0 \\ 0 & \cdots & \cdots & 0 & V_{r+1} \end{pmatrix} \in U(n+1)\,, \quad V_j=\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad j=1,\dots,r\,,$$

$$V_{r+1}=\begin{pmatrix} \lambda & 0 \\ 0 & I_{n-2r-1} \end{pmatrix},$$

and $\lambda=\mu^{r+1}\in S^1$. □

COROLLARY 3.6. *The S^1-family of harmonic maps obtained by projecting the S^1-family of primitive maps coincides with the S^1-family obtained from the extended framing* (3.4).

4. CMC-surface theory and the natural Lawson correspondence. A non-conformal harmonic map $\phi\colon M\to S^2$ is locally the Gauss map of a CMC surface in $\boldsymbol{R}^3$. More generally, by [L1, Theorem 8], we obtain S^1-families of isometric CMC-$\sqrt{H^2-c}$ surfaces $\{\psi^\theta_{c,H}\colon U\to S^3(c),\,\theta\in[0,2\pi)\}$ for any $H\neq 0$ and $c\leq H^2$ from ϕ (we do not treat the hyperbolic case $c<0$ here).

We briefly review this fact. We fix the orientations of M and $\boldsymbol{R}^4$, and use the star operator $*$ of $\boldsymbol{R}^4$ to identify $\boldsymbol{R}^4\cong\wedge^3\boldsymbol{R}^4$, $*\wedge^2\boldsymbol{R}^4\cong\wedge^2\boldsymbol{R}^4$. For an isometric immersion

$$\begin{cases} \psi_0\colon M\to\boldsymbol{R}^3 \\ \psi_c\colon M\to S^3(c)\,, \quad c>0\,, \end{cases}$$

with metric $ds^2=2F|dz|^2$, we define its unit normal vector by

$$\begin{cases} \psi_0^*=\dfrac{1}{iF}\,\partial\psi_0\wedge\bar\partial\psi_0 \\[2ex] \psi_c^*=\dfrac{\sqrt{c}}{iF}\,\psi_c\wedge\partial\psi_c\wedge\bar\partial\psi_c\,, \quad c>0\,. \end{cases}$$

The CMC-H_c surface equation for ψ_c where $H_c=\sqrt{H^2-c}$ is given by

$$\partial\bar\partial\psi_c+Fc\psi_c=H_cF\psi_c^*\,. \tag{4.1}$$

Define the quadratic differential $Q=\beta dz^2$ by

$$\beta=\langle\partial^2\psi_c,\psi_c^*\rangle=\frac{1}{4}(\beta_{11}-\beta_{22}-2i\beta_{12})\,, \tag{4.2}$$

where β_{ij} is the coefficients of the second fundamental form with respect to $z=x_1+ix_2$. We have

(4.3) $$\partial^2\psi_c-\frac{\partial F}{F}\partial\psi_c=\beta\psi_c^*$$

(4.4) $$\partial\psi_c^*=-H_c\partial\psi_c-\frac{\beta}{F}\bar{\partial}\psi_c$$

(4.5) $$|\partial\psi_c^*|^2=H_c^2F+\frac{|\beta|^2}{F}$$

(4.6) $$\langle\partial\psi_c^*,\partial\psi_c^*\rangle=2H_c\beta$$

(4.7) $$\partial\bar{\partial}\psi_c^*=-\left(H_c^2F+\frac{|\beta|^2}{F}\right)\psi_c^*+H_ccF\psi_c\,.$$

Note that when $H_cc=0$, ψ_c^* is harmonic by (4.5) and (4.7). As is well-known, β is holomorphic for any c by (4.1), (4.2) and (4.4). Now, taking an oriented framing

$$\begin{cases}\Phi_0=(\partial\psi_0,\bar{\partial}\psi_0,\psi_0^*)\\ \Phi_c=(\psi_c,\partial\psi_c,\bar{\partial}\psi_c,\psi_c^*)\,, \quad c>0\,,\end{cases}$$

consider the system of ordinary differential equations

(4.8) $$\begin{cases}\partial\Phi_c=\Phi_cA_c\\ \bar{\partial}\Phi_c=\Phi_cB_c\,.\end{cases}$$

From (4.1), (4.3) and (4.4), we easily obtain

$$A_c=\begin{pmatrix}0&0&-cF&0\\1&\partial F/F&0&-H_c\\0&0&0&-\beta/F\\0&\beta&H_cF&0\end{pmatrix},\quad B_c=\begin{pmatrix}0&-cF&0&0\\0&0&0&-\bar{\beta}/F\\1&0&\bar{\partial}F/F&-H_c\\0&H_cF&\bar{\beta}&0\end{pmatrix},$$

where we ignore the first column and row when $c=0$. The integrability condition of (4.8) is $\bar{\partial}A_c-\partial B_c-[A_c,B_c]=0$ which turns out to be

(4.9) $$2\partial\bar{\partial}w+(c+H_c^2)e^{2w}-|\beta|^2e^{-2w}=0\,,$$

where we put $F=|\partial\psi_c|^2=e^{2w}$. When either one of $H=H_0$ and β does not vanish identically, Φ_0 can be rewritten as a framing of a harmonic map $\phi:=\psi_0^*: M\to S^2$ by (4.4). This means that when we are given a non-conformal harmonic map $\phi: M\to S^2$ with a real number H^2 and a holomorphic function β satisfying $\langle\partial\phi,\partial\phi\rangle=2H\beta$, and if we define F by

(4.10) $$|\partial\phi|^2=H^2F+\frac{|\beta|^2}{F}\,,$$

$w=(\log F)/2$ must be a solution of

$$2\partial\bar{\partial}w+H^2e^{2w}-|\beta|^2e^{-2w}=0\,. \tag{4.9'}$$

Then putting $H^2=H_c^2+c$ and $\beta^\theta=e^{i\theta}\beta$, we obtain an S^1-family of CMC-$\sqrt{H^2-c}$ surfaces $\psi_{c,H}^\theta$ in $S^3(c)$, $c\in[0, H^2]$, $\theta\in[0, 2\pi)$, having the metric $ds^2=2F|dz|^2$ and the differential $Q^\theta=\beta^\theta dz^2$. We call $\psi_{c,H}^\theta$ the associated CMC-$\sqrt{H^2-c}$ surfaces of ϕ. (Note: F is chosen in two ways. The corresponding CMC surfaces form a Bonnet pair.)

REMARK. By the homothety $x\mapsto\lambda x$ in $\boldsymbol{R}^4$, the mean curvature of a surface changes $h\to h/\lambda$. Thus a different choice H' instead of H yields a CMC-$\sqrt{(H')^2-c'}$ surface $\psi_{c',H'}^\theta$ in $S^3(c')$ which is homothetic to a CMC-$\sqrt{H^2-c}$ surface $\psi_{c,H}^\theta$ in $S^3(c)$, where $c/c'=(H/H')^2$.

We do not treat the case where ϕ is holomorphic or anti-holomorphic, which occurs when $H\equiv 0$, hence for the moment, we put $H^2=1$ and $\psi_c^\theta=\psi_{c,H}^\theta$. The associated surfaces $\{\psi_c^\theta\}$ have two parameters c and θ. We define a one-parameter subset $\{\psi_c^\sigma, \sigma=\cos^{-1}\sqrt{1-c}\}$ consisting of surfaces naturally corresponding to each other in the following sense. When $\psi_0\colon U\to\boldsymbol{R}^3$ is a CMC-1 surface having the second fundamental form (β_{ij}), we define the naturally corresponding minimal surface in S^3 by

$$\psi_1^{\pi/2}\colon U\to S^3\,.$$

Then, the differential $Q^{\pi/2}$ is given by $i\beta=(2\beta_{12}+i(\beta_{11}-\beta_{22}))/4$, so that $\psi_1^{\pi/2}$ has the second fundamental form

$$(\beta_{ij}^1)=\begin{pmatrix} \beta_{12} & -(\beta_{11}-\beta_{22})/2 \\ -(\beta_{11}-\beta_{22})/2 & -\beta_{12} \end{pmatrix}.$$

Similarly, we define the naturally corresponding CMC-$\sqrt{1-c}$ surface in $S^3(c)$ by

$$\psi_c^\sigma\colon U\to S^3(c)\,,\quad \sigma=\cos^{-1}\sqrt{1-c}\,,$$

which has the second fundamental form

$$(\beta_{ij}^c)=\cos\sigma\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{pmatrix}+\sin\sigma\begin{pmatrix} \beta_{12} & -(\beta_{11}-\beta_{22})/2 \\ -(\beta_{11}-\beta_{22})/2 & -\beta_{12} \end{pmatrix},$$

of which the mean curvature is given by

$$\frac{1}{4F}\operatorname{Tr}(\beta_{ij}^c)=\cos\sigma=\sqrt{1-c}\,.$$

We say elements in $\{\psi_c^\sigma, \sigma=\cos^{-1}\sqrt{1-c}\}$ are in natural Lawson correspondence. In this paper, we call ϕ the *adapted* secondary Gauss map of ψ_c^σ for each c, i.e. the Gauss map ϕ of ψ_0^0 is called the adapted secondary Gauss map of ψ_c^σ for $0\le c\le 1$.

5. A local behavior. Put $\psi_c=\psi_c^0$ for simplicity. When $c>0$, we define the generalized Gauss map of ψ_c by

$$\tilde{\psi}_c=\sqrt{c}\,\psi_c\wedge\psi_c^*: M\to S^5 .$$

Since $*\tilde{\psi}_c=(\partial\psi_c\wedge\bar{\partial}\psi_c)/iF$, we may put

$$\tilde{\psi}_0=\phi: M\to S^2 .$$

The map $\tilde{\psi}_c$ is a map into the oriented Grassmannian $\mathrm{Gr}_2^+(\boldsymbol{R}^4)\cong S^2(2)\times S^2(2)$ (cf. [HO]), but we consider it as a map into S^5 because:

LEMMA 5.1. *$\tilde{\psi}_c=\sqrt{c}\,\psi_c\wedge\psi_c^*: U\to S^5$ is a harmonic map satisfying*

$$|\partial\tilde{\psi}_c|^2=F+\frac{|\beta|^2}{F} \tag{5.1}$$

$$\partial\bar{\partial}\tilde{\psi}_c=-\left(F+\frac{|\beta|^2}{F}\right)\tilde{\psi}_c \tag{5.2}$$

$$\langle\partial\tilde{\psi}_c,\partial\tilde{\psi}_c\rangle=2H_c\beta\,,\quad H_c=\sqrt{1-c} \tag{5.3}$$

$$\langle\partial^2\tilde{\psi}_c,\partial^2\tilde{\psi}_c\rangle=4\beta^2(1-2c)-2H_c\beta\left(\frac{\partial F}{F}\right)^2 \tag{5.4}$$

$$\langle\partial^3\tilde{\psi}_c,\partial^3\tilde{\psi}_c\rangle=8H_c(1-4c)\beta^3+4(2c-1)\beta^2\left(\frac{\partial^2F}{F}+\partial^2\left(\frac{1}{F}\right)F\right)+2H_c\beta\partial^2F\partial^2\left(\frac{1}{F}\right), \tag{5.5}$$

where in (5.5), *we use coordinates so that β is constant.*

REMARK. (1) By (5.1)$\sim$(5.3), $\tilde{\psi}_1$ is regularly minimal with respect to the induced metric (the bipolar surface in [L1]), and $\tilde{\psi}_c$ is non-conformal harmonic for $0\leq c<1$. In this paper, we occasionally regard the generalized Gauss map as a harmonic map into S^5.

(2) For ψ_c^θ, (5.1)$\sim$(5.5) hold if we replace β by $e^{i\theta}\beta$.

PROOF. When $c=0$, (5.1)$\sim$(5.3) follows from (4.5)$\sim$(4.7), while for (5.4) and (5.5), see the proof of Lemma 5.2. When $c>0$, put $\hat{\psi}_c=\psi_c\wedge\psi_c^*$. Using (4.1) and (4.3)$\sim$(4.7), we obtain,

$$\partial\hat{\psi}_c=\partial\psi_c\wedge\psi_c^*-H_c\psi_c\wedge\partial\psi_c-\frac{\beta}{F}\psi_c\wedge\bar{\partial}\psi_c \tag{5.6}$$

$$\begin{aligned}\partial^2\hat{\psi}_c=&\frac{\partial F}{F}\partial\psi_c\wedge\psi_c^*-\frac{2\beta}{F}\partial\psi_c\wedge\bar{\partial}\psi_c\\&-2H_c\beta\hat{\psi}_c-H_c\frac{\partial F}{F}\psi_c\wedge\partial\psi_c-\partial\left(\frac{\beta}{F}\right)\psi_c\wedge\bar{\partial}\psi_c\end{aligned} \tag{5.7}$$

$$(5.8)\qquad \partial^3\hat{\psi}_c=\left(\frac{\partial^2 F}{F}-4H_c\beta\right)\partial\psi_c\wedge\psi_c^*-\frac{2\beta^2}{F}\psi_c^*\wedge\bar{\partial}\psi_c$$
$$+\left(-2c\beta-H_c\frac{\partial^2 F}{F}+2H_c^2\beta\right)\psi_c\wedge\partial\psi_c+\left(\frac{2H_c}{F}\beta^2-\partial^2\left(\frac{\beta}{F}\right)\right)\psi_c\wedge\bar{\partial}\psi_c\,,$$

where in (5.8), we use coordinates so that β is constant. Thus noting that $\tilde{\psi}_c=\sqrt{c}\,\hat{\psi}_c$, we obtain (5.1)$\sim$(5.5).

Because of Corollary 3.2, it is natural to ask the relationship between $\tilde{\psi}_c$ and $\tilde{\phi}^\omega=(\phi\oplus\phi^\omega)/\sqrt{2}: U\to S^5$ for ϕ, ϕ^ω belonging to the S^1-family of the secondary Gauss map of ψ_c.

LEMMA 5.2. *Let $\phi: M\to S^2$ be the Gauss map of a* CMC-1 *surface $\psi: M\to \boldsymbol{R}^3$ with $F=|\partial\psi|^2$ and $\beta=\langle\partial^2\psi,\phi\rangle$. Then $\tilde{\phi}^\omega=(\phi\oplus\phi^\omega)/\sqrt{2}: U\to S^5$, $\omega\in[0,2\pi)$, is a harmonic map satisfying*

$$(5.9)\qquad |\partial\tilde{\phi}^\omega|^2=|\partial\phi|^2=F+\frac{|\beta|^2}{F}$$

$$(5.10)\qquad \partial\bar{\partial}\tilde{\phi}^\omega=-\left(F+\frac{|\beta|^2}{F}\right)\tilde{\phi}^\omega$$

$$(5.11)\qquad \langle\partial\tilde{\phi}^\omega,\partial\tilde{\phi}^\omega\rangle=\beta(1+e^{i\omega})$$

$$(5.12)\qquad \langle\partial^2\tilde{\phi}^\omega,\partial^2\tilde{\phi}^\omega\rangle=2\beta^2(1+e^{2i\omega})-\beta(1+e^{i\omega})\left(\frac{\partial F}{F}\right)^2$$

$$(5.13)\qquad \langle\partial^3\tilde{\phi}^\omega,\partial^3\tilde{\phi}^\omega\rangle=4\beta^3(1+e^{3i\omega})-2\beta^2(1+e^{2i\omega})\left(F\partial^2\left(\frac{1}{F}\right)+\frac{\partial^2 F}{F}\right)$$
$$+\beta\partial^2 F\partial^2\left(\frac{1}{F}\right)(1+e^{i\omega})\,,$$

where we use coordinates so that β is constant.

PROOF. Differentiating

$$(4.4')\qquad \partial\phi=-\partial\psi_0-\frac{\beta}{F}\bar{\partial}\psi_0$$

and using (4.1) and (4.3), we obtain

$$\partial^2\phi=-\frac{\partial F}{F}\partial\psi_0-2\beta\phi-\partial\left(\frac{\beta}{F}\right)\bar{\partial}\psi_0$$

$$\partial^3\phi=\left(-\frac{\partial^2 F}{F}+2\beta\right)\partial\psi_0-\left(\partial^2\left(\frac{\beta}{F}\right)-\frac{2\beta^2}{F}\right)\bar{\partial}\psi_0\,,$$

from which follows

$$\langle \partial^2\phi, \partial^2\phi\rangle = 2\left\{2\beta^2 - \beta\left(\frac{\partial F}{F}\right)^2\right\}$$

$$\langle \partial^3\phi, \partial^3\phi\rangle = 2\left\{4\beta^3 - 2\beta^2\left(F\partial^2\left(\frac{1}{F}\right) + \frac{\partial^2 F}{F}\right) + \beta(\partial^2 F)\left(\partial^2\left(\frac{1}{F}\right)\right)\right\}.$$

Then noting $\langle \partial\phi^\omega, \partial\phi^\omega\rangle = 2e^{i\omega}\beta$, we obtain the lemma.

THEOREM 5.3. *Let $\phi: U \to S^2$ be a non-conformal harmonic map with the Hopf differential $2\beta dz^2 = \langle \partial\phi, \partial\phi\rangle dz^2$. Take $\sigma \in [0, \pi/2]$ satisfying $\cos\sigma = \sqrt{1-c}$, and let $\psi_c^\sigma: U \to S^3(c)$ be an isometric CMC-$\sqrt{1-c}$ surface associated with ϕ having $Q^\sigma = e^{i\sigma}\beta dz^2$. Let $(\psi_c^\sigma)^*$ be the unit normal vector. Then the harmonic map*

$$\tilde{\psi}_c^\sigma = \sqrt{c}\,\psi_c^\sigma \wedge (\psi_c^\sigma)^*: U \to S^5$$

is congruent to the harmonic map

$$\tilde{\phi}^{2\sigma} = \frac{1}{\sqrt{2}}(\phi \oplus \phi^{2\sigma}): U \to S^5 .$$

PROOF. This follows from Bolton and Woodward's congruence theorem in [BW, Theorem 4.1] and from Lemmas 5.1 and 5.2. Indeed, by (5.1) and (5.9), and by the congruence theorem, it is sufficient to prove that

$$\langle \partial^j\tilde{\psi}_c^\sigma, \partial^j\tilde{\psi}_c^\sigma\rangle = \langle \partial^j\tilde{\phi}^{2\sigma}, \partial^j\tilde{\phi}^{2\sigma}\rangle \qquad \text{for} \quad j = 1, 2, 3 .$$

Noting Remark (2) after Lemma 5.1 and $2\cos\alpha e^{i\alpha} = 1 + e^{2i\alpha}$, we obtain from (5.3) and (5.11),

$$\langle \partial\tilde{\psi}_c^\sigma, \partial\tilde{\psi}_c^\sigma\rangle = 2H_c e^{i\sigma}\beta = 2\cos\sigma e^{i\sigma}\beta = (1 + e^{2i\sigma})\beta = \langle \partial\tilde{\phi}^{2\sigma}, \partial\tilde{\phi}^{2\sigma}\rangle . \tag{5.14}$$

From (5.4) and (5.12), using (5.14), we get

$$\begin{aligned}\langle \partial^2\tilde{\psi}_c^\sigma, \partial^2\tilde{\psi}_c^\sigma\rangle &= 4e^{2i\sigma}\cos 2\sigma\beta^2 - 2e^{i\sigma}H_c\beta\left(\frac{\partial F}{F}\right)^2 \\ &= 2(1 + e^{4i\sigma})\beta^2 - (1 + e^{2i\sigma})\beta\left(\frac{\partial F}{F}\right)^2 \\ &= \langle \partial^2\tilde{\phi}^{2\sigma}, \partial^2\tilde{\phi}^{2\sigma}\rangle .\end{aligned}$$

Similarly, from (5.5) and (5.13), we get

$$\langle \partial^3\tilde{\psi}_c^\sigma, \partial^3\tilde{\psi}_c^\sigma\rangle = 8\cos\sigma(1 - 4\sin^2\sigma)e^{3i\sigma}\beta^3 - 4\cos 2\sigma e^{2i\sigma}\beta^2\left(\frac{\partial^2 F}{F} + \partial^2\left(\frac{1}{F}\right)F\right)$$

$$+2\cos\sigma e^{i\sigma}\beta\partial^2 F\partial^2\left(\frac{1}{F}\right)$$

$$=4(1+e^{6i\sigma})\beta^3-2(1+e^{4i\sigma})\beta^2\left(\frac{\partial^2 F}{F}+\partial^2\left(\frac{1}{F}\right)F\right)+(1+e^{2i\sigma})\beta\partial^2 F\partial^2\left(\frac{1}{F}\right)$$

$$=\langle\partial^3\tilde{\phi}^{2\sigma},\partial^3\tilde{\phi}^{2\sigma}\rangle.$$

□

6. A global behavior. Let $\psi_c\colon M\to S^3(c)$ be a CMC-$\sqrt{1-c}$ surface. Then the Gauss map

$$\tilde{\psi}_c\colon M\to S^5$$

is defined globally, which is a harmonic map into S^5. Let ϕ be the adapted secondary Gauss map of ψ_c such that $\psi_c=\psi_c^\sigma$. By Theorem 5.3, we have a local congruence of $\tilde{\psi}_c^\sigma$ with $\tilde{\phi}^{2\sigma}=(\phi\oplus\phi^{2\sigma})/\sqrt{2}\colon U\to S^5$. In this section, we show the global congruence. By an isometry of $S^3(c)$, if necessary, we may assume that

$$\tilde{\psi}_c^\sigma|_U=\tilde{\phi}^{2\sigma}$$

in a coordinate neighborhood U of M. Then using this splitting, we define

$$\boldsymbol{R}^6=\boldsymbol{R}_1^3\oplus\boldsymbol{R}_2^3$$

so that

$$\phi\colon U\to S^2\subset\boldsymbol{R}_1^3,\quad \phi^\sigma\colon U\to S^2\subset\boldsymbol{R}_2^3.$$

Let π_i be the projection $\boldsymbol{R}^6\to\boldsymbol{R}_i^3$, $i=1, 2$, and define maps $\tilde{\psi}^i=\sqrt{2}\,\pi_i\tilde{\psi}_c^\sigma$, $i=1, 2$. Noting that $\tilde{\psi}^1=\phi$ and $\tilde{\psi}^2=\phi^{2\sigma}$ on U, we obtain:

PROPOSITION 6.1. *$\tilde{\psi}^1$ and $\tilde{\psi}^2$ are global non-conformal harmonic maps from M into S^2.*

This proposition is obvious from $\mathrm{Gr}_2^+(\boldsymbol{R}^4)\cong S^2(2)\times S^2(2)$.

PROOF. Note that the coordinate functions $(\psi^1,\dots,\psi^6)$ of $\tilde{\psi}_c^\sigma$ satisfy

$$\partial\bar{\partial}\psi^j=-|\partial\tilde{\psi}_c|^2\psi^j, \tag{6.1}$$

so are real analytic. Thus the same is true for coordinate functions of $\tilde{\psi}^1=(\psi^1,\psi^2,\psi^3)$ and $\tilde{\psi}^2=(\psi^4,\psi^5,\psi^6)$. Since

$$|\tilde{\psi}^1|^2=|\tilde{\psi}^2|^2\equiv 1\qquad\text{on}\quad U, \tag{6.2}$$

this holds all over M, and hence $\tilde{\psi}^i$ is a global map from M into the unit sphere S^2 of $\boldsymbol{R}_i^3$. In particular on U, we have

$$|\partial\tilde{\psi}^i|^2=|\partial\phi|^2=|\partial\phi^{2\sigma}|^2=|\partial\tilde{\psi}_c^\sigma|^2,\qquad i=1,2 \tag{6.3}$$

because of Theorem 5.3. By analyticity of $\tilde{\psi}^i$ again, (6.3) holds in any coordinate domains. This fact and (6.1) imply

$$\partial\bar{\partial}\tilde{\psi}^i = -|\partial\tilde{\psi}_c^\sigma|^2\tilde{\psi}^i = -|\partial\tilde{\psi}^i|^2\tilde{\psi}^i, \qquad \text{on} \quad M,$$

that is, $\tilde{\psi}^i: M \to S^2$, $i=1, 2$ are global harmonic maps from M into S^2. □

THEOREM 6.2. *For a* CMC-*h surface* $\psi: M \to S^3$, *there exists a pair of non-conformal harmonic maps* ϕ, $\phi^{2\theta}: M \to S^2$ *such that the generalized Gauss map* $\tilde{\psi}$ *of* ψ *splits into* $(\phi \oplus \phi^{2\theta})/\sqrt{2}$. *In fact*, ϕ *is the adapted secondary Gauss map of* ψ, *and* θ *is given by* $\cos^{-1}\sqrt{h^2/h^2+1}$. *Moreover*, $\tilde{\psi}$ *can be deformed into* ϕ *and/or* $\phi^{2\theta}$ *through harmonic maps* $\tilde{\phi}_s^{2\theta} = \cos s\phi \oplus \sin s\phi^{2\theta}$ *into* S^5.

PROOF. Put $H^2 = h^2 + 1$. Then by the Remark in §4, ψ is homothetic to a CMC-$\sqrt{1-c}$ surface ψ_c in $S^3(c)$, where $c = 1/H^2$. Since the generalized Gauss map of ψ_c coincides with that of ψ, we may consider ψ_c instead of ψ in the proof. Take the adapted secondary Gauss map ϕ of ψ_c such that $\psi_c = \psi_c^\sigma$, then $\theta = \sigma$ satisfies the first statement. We may prove the last part. Since

$$\partial\tilde{\phi}_s^{2\sigma} = \cos s\,\partial\phi \oplus \sin s\,\partial\phi^{2\sigma},$$

we obtain $|\partial\tilde{\phi}_s^{2\sigma}|^2 = \cos^2 s|\partial\phi|^2 + \sin^2 s|\partial\phi^{2\sigma}|^2 = |\partial\phi|^2$ and

$$\partial\bar{\partial}\tilde{\phi}_s^{2\sigma} = -|\partial\tilde{\phi}_s^{2\sigma}|^2\tilde{\phi}_s^{2\sigma},$$

which implies that $\tilde{\phi}_s^{2\sigma}$ is a harmonic map into S^5. Then the theorem follows from $\phi = \tilde{\phi}_0^{2\sigma}$, $\phi^{2\sigma} = \tilde{\phi}_{\pi/2}^{2\sigma}$ and $\tilde{\psi}_c^\sigma = \tilde{\phi}_{\pi/4}^{2\sigma}$. □

EXAMPLE. When $\psi: T^2 \to S^3$ is the Clifford torus, each of ϕ and ϕ^π degenerates to a map onto a geodesic of S^2. In this case, $\tilde{\psi}$ is congruent to ψ, and $\tilde{\psi} = (\phi \oplus \phi^\pi)/\sqrt{2}$: $T^2 \to S^1(2) \times S^1(2) \subset S^3$. The deformation $\tilde{\phi}_s^\pi$ is essentially the one in [Mu].

A deformation of a harmonic map $\tilde{\psi}: M \to S^5$ yields a Jacobi field along $\tilde{\psi}$. When M is compact, we call the dimension of the space of Jacobi fields the nullity of $\tilde{\psi}_c$, which is finite because Jacobi fields are solutions of an elliptic partial differential equation. Because the dimension of the Killing Jacobi fields is 15 and because we have another non-Killing Jacobi field by Theorem 6.2, we obtain:

COROLLARY 6.3. *The generalized Gauss map of a compact* CMC *surface of positive genus in* S^3, *has nullity* (*as harmonic maps*) *at least* 16.

REMARK. (1) When we define the Killing nullity to be the dimension of the fields given by the normal component of the Killing fields of S^5, the classification of CMC surfaces of which Gauss maps have small Killing nullity (= big homogeneity) would be interesting. The generalized Gauss map of the CMC surface $S^2(a)$, $a \geq 1$ has the smallest Killing nullity 3, and of $S^1(a) \times S^1(a/(a-1))$, $a > 1$ (parallel surfaces of the Clifford torus) has Killing nullity 4.

(2) When c and θ tend to 0 independently, $\tilde{\psi}_c^\theta$ tends to

$$\tilde{\phi}=\frac{1}{\sqrt{2}}(\phi\oplus\phi)\cong\phi: M\to S^2$$

and hence gives a *local* harmonic map deformation from $\tilde{\psi}_c^\theta$ to ϕ which is different from the global deformation through $\tilde{\phi}_s^{2\sigma}$.

(3) Examples of compact CMC-$\sqrt{1-c}$ surfaces in $S^3(c)$ are given in [L1] for $c=1$ and [K1], [K2] for $c=0$, but we do not know examples of $0<c<1$ except those of genus 0 and 1.

(4) We call a harmonic map reducible if it splits into harmonic maps into lower dimensional spheres (cf. [M]). Harmonic maps from a compact Riemann surface seem irreducible, but the splitting occurs in the bipolar surface case.

7. Lawson's conjecture.

LEMMA 7.1. *A minimal surface $\phi: M\to S^n$ is isometric to a minimal surface in S^3, if there exists a local coordinate z in which the induced metric is given by $ds^2=2e^{2w}|dz|^2$, where w is a solution of the* sinh-*Gordon equation*:

$$\partial\bar{\partial}w+\sinh 2w=0\,. \tag{7.1}$$

In this coordinate, $w_j=\log|\phi_j|$ satisfies $w_{2j}=0$ and $w=w_{4j+1}=-w_{4j+3}$, $j\in\boldsymbol{Z}$.

PROOF. This follows from $w_0\equiv 0$ and (4.9), where $c+H_c^2=1$ and we choose the parameter satisfying $\beta=1$.

Note that this is a special expression of the (spherical) Ricci condition (cf. (6.6), [M, §6]). A superminimal minimal surface fully lies in S^{2m} and satisfies $\phi_{m+1}\equiv 0$, hence we get immediately:

COROLLARY 7.2. *A superminimal minimal surface in S^{2m} cannot be isometric to a minimal surface in S^3.*

In [M, Lemma 9.4], we showed that a superconformal harmonic map into S^{2m} exists when $w_j=\log|\phi_j|$ satisfies

(1) $w_0=0$

(2) $2\partial\bar{\partial}w_j-e^{2(w_{j+1}-w_j)}+e^{2(w_j-w_{j-1})}=0,\ j=1,2,\dots,m-1$

(3) $2\partial\bar{\partial}w_m+r_m^2(1-G)-|s|^2=0$,

where $r_m=e^{w_m-w_{m-1}}$, $G=|\varphi_m|^2/|\phi_m|^4$, and $|s|^2=|\partial G|^2/4G(1-G)$, for any coordinate. Suppose that there exists a coordinate in which the induced metric satisfies (7.1). In this coordinate, when $m=2k$, (3) is rewritten as

$$e^{-2w_{m-1}}(1-|\varphi|^2)=|\partial|\varphi|^2|^2/4|\varphi|^2(1-|\varphi|^2)\,,\qquad \varphi=\varphi_m$$

so that

$$\mathrm{e}^{-2w_{m-1}}=|\partial\varphi|^2/4(1-|\varphi|^2)^2\,. \tag{7.2}$$

Since $\partial\varphi$ is holomorphic, we obtain

$$(7.3)\qquad 2\partial\bar{\partial}w_{m-1}=2\partial\bar{\partial}\log(1-|\varphi|^2)=-2|\partial\varphi|^2/(1-|\varphi|^2)^2=-8e^{-2w_{m-1}}.$$

On the other hand, by assumption and by Lemma 7.1, $w=\pm w_{m-1}$ satisfies (7.1), and we get $e^{2w}=3$ or $1/3$. This contradicts both (7.1) and (7.3). Hence we obtain:

THEOREM 7.3. *A full superconformal minimal surface in S^{4m} cannot be isometric to a minimal surface in S^3.*

Full minimal surfaces in S^4 are either superminimal or superconformal, thus we obtain:

COROLLARY 7.4 (cf. [S]). *Full minimal surfaces in S^4 cannot be isometric to a minimal surfaces in S^3.*

REFERENCES

[AA] R. AIYAMA AND K. AKUTAGAWA, Kenmotsu type representation formula for surfaces with prescribed mean curvature in $S^3(c)$ and adjusting it for CMC surfaces, preprint, Tsukuba Univ. and Sizuoka Univ. (1997).

[AAMU] R. AIYAMA, K. AKUTAGAWA, R. MIYAOKA AND M. UMEHARA, A global correspondence between CMC-surfaces in PR^3 and pairs of non-holomorphic harmonic maps into S^2, preprint, Tsukuba Univ., Sizuoka Univ., Tokyo Inst. Tech. and Osaka Univ. (1997).

[B] F. E. BURSTALL, Harmonic tori in spheres and complex projective spaces, J. Reine Angew. Math. 469 (1995), 149–177.

[BP] F. E. BURSTALL AND F. PEDIT, Harmonic maps via Adler-Kostant-Symes theory, Harmonic maps and integrable systems (Fordy and Wood, eds.), Vieweg, 1993, pp. 221–272.

[BW] J. BOLTON AND L. WOODWARD, Congruence theorems for harmonic maps from a Riemann surface into CP^n and S^n, J. London Math. Soc. 45 (1992), 363–376.

[HL] W. HSIANG AND B. LAWSON, Minimal submanifolds of low cohomogeneity, J. Differential Geom. 5 (1971), 1–38.

[HO] D. A. HOFFMAN AND R. OSSERMAN, The Gauss map of surfaces in R^3 and R^4, Proc. London Math. Soc. 50 (1985), 27–56.

[K1] N. KAPOULEAS, Compact constant mean curvature surfaces in Euclidean three-space, J. Differential Geom. 33 (1991), 638–715.

[K2] N. KAPOULEAS, Constant mean curvature surfaces constructed by fusing Wente tori, Invent. Math. 119 (1995), 443–518.

[L1] H. B. LAWSON, Complete minimal surfaces in S^3, Ann. of Math. 92 (1970), 335–374.

[L2] H. B. LAWSON, Some intrinsic characterizations of minimal surfaces, J. Analyse Math. 24 (1971), 151–161.

[M] R. MIYAOKA, The family of isometric superconformal harmonic maps and the affine Toda equations, J. Reine Angew. Math. 481 (1996), 1–25.

[Mu] M. MUKAI, The deformation of harmonic maps given by the Clifford tori, Kodai Math. J. 20 (1997), 252–268.

[S] M. SAKAKI, Minimal surfaces with the Ricci condition in 4-dimensional space forms, Proc. Amer. Math. Soc. 121 (1994), 573–577.

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