

Branching of Automorphic Fundamental Solutions

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ABSTRACT. Automorphic fundamental solutions and, more generally, solutions of automorphic differential equations play a key role in the Diaconu–Garrett–Goldfeld prescription for spectral identities involving moments of L-functions [4; 5; 6] as well as other applications, including an explicit formula relating the number of lattice points in a symmetric space to the automorphic spectrum [2]. In this paper we discuss two cases in which the automorphic fundamental solution exhibits branching: pathwise meromorphic continuations may differ by a term involving an Eisenstein series.

1. Introduction

Solutions of automorphic differential equations underlie the Diaconu–Garrett–Goldfeld prescription for spectral identities involving second moments for arbitrary Rankin–Selberg integral representations of L-functions [6]. This prescription is a vast generalization of the constructions of moment identities in their earlier papers, from which they extracted subconvex bounds for GL_2 automorphic L-functions [7; 8; 4; 5]. Essential to their prescription is a Poincaré series, whose data was originally constructed in imitation of Good’s kernel [9]; characterizing the Poincaré series as the solution to an automorphic differential equation allows generalization from GL_2 to higher rank. The automorphic spectral expansion of such a Poincaré series is heuristically immediate and can be legitimized using automorphic Sobolev theory, developed in [2]. In general, explicit geometric expressions for solutions of automorphic differential equations are very difficult to obtain; see [3] for some examples, including the automorphic fundamental solution that is used in the lattice-point counting application in [2] and is suitable for constructing moment identities for $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ Rankin–Selberg L-functions. Superficially, the spectral expansion of the automorphic fundamental solution appears to be invariant under a transformation of an auxiliary complex parameter w , but a closer look reveals, in certain cases, *branching* in w , eliminating the possibility of a straightforward functional equation.

Let G be a semisimple Lie group, K its maximal compact subgroup, and Γ a discrete subgroup. Consider the solution of the following differential equation on the arithmetic quotient $X = \Gamma \backslash G / K$:

$$(\Delta - \lambda_w)^v u_w = \delta_{z_o},$$

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where the Laplacian Δ is the image of the Casimir operator for \mathfrak{g} , λ_w a complex parameter, ν an integer, and $\delta_{z_o} = \delta_{\Gamma \cdot 1 \cdot K}$ the Dirac delta distribution at the basepoint in $\Gamma \backslash G/K$. In rank one, we parameterize λ_w as $\lambda_w = w(w - 1)$, and in higher rank, $\lambda_w = w^2 - |\rho|^2$, where ρ is the half-sum of positive roots. We recall the following results from [2]. Global automorphic Sobolev theory ensures the existence of a solution u_w , unique in global automorphic Sobolev spaces. The solution has a transparent automorphic spectral expansion, converging in a global automorphic Sobolev space for $\text{Re}(w)$ sufficiently large. Further, a global Sobolev embedding theorem ensures that, by choosing ν sufficiently large, we may ensure that the spectral expansion converges uniformly pointwise or in any C^k -topology that we wish.

Interestingly, the fundamental solution may exhibit *branching* in the complex variable w : meromorphic continuations along different w -paths in the complex plane may differ by a term of moderate growth. In particular, the resulting function may lie outside of global automorphic Sobolev spaces. We discuss two such cases below.

Branching of fundamental solutions on symmetric spaces has been discussed by Mazzeo and collaborators in several papers (see, e.g., [12]) and by Strohmaier [14]. See [1] for a discussion of automorphic Green functions on $\Gamma \backslash \mathfrak{H}$, where Γ is a Fuchsian group and [10], for automorphic Green functions with logarithmic singularities along modular divisors in a modular variety.

2. Branching of Hilbert–Maass Fundamental Solutions

Recall the general setup from Introduction, which invokes [2] for the existence and uniqueness of solutions to automorphic differential equations and convergence of their spectral expansions in global automorphic Sobolev spaces. For complete descriptions of automorphic spectral expansions, see [11; 13].

Let k be a totally real number field of degree $n > 1$ over \mathbb{Q} with \mathfrak{o} its ring of integers. For simplicity, suppose that \mathfrak{o} has narrow class number one, so that $SL_2(\mathfrak{o})$ is unicuspidal. Let $\sigma_1, \dots, \sigma_n$ be the Archimedean places of k , and let $SL_2(\mathfrak{o})$ act on \mathfrak{H}^n componentwise, as usual: for $\gamma \in SL_2(\mathfrak{o})$ and $z = (z_1, \dots, z_n) \in \mathfrak{H}^n$,

$$\gamma \cdot z = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \dots, \frac{a_n z_n + b_n}{c_n z_n + d_n} \right), \quad \text{where } \sigma_j(\gamma) = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}.$$

We construct the Laplacian Δ on $SL_2(\mathfrak{o}) \backslash \mathfrak{H}^n$ from the usual Laplacians on the factors:

$$\Delta = \frac{1}{n}(\Delta_1 + \dots + \Delta_n), \quad \text{where } \Delta_j = y_j^2 \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right).$$

This is a nonpositive symmetric operator. We parameterize the eigenvalues by $\lambda_w = w(w - 1)$. For λ_w to be nonpositive real, we need $w \in \frac{1}{2} + i\mathbb{R} \cup [0, 1]$.

Elements of global automorphic Sobolev spaces for $SL_2(\mathfrak{o}) \backslash \mathfrak{H}^n$ have spectral expansions, in terms of an orthonormal basis $\{F\}$ of spherical cusp forms, the

constant automorphic form 1, and the continuous family of Eisenstein series $E_{s,\chi}$ with s on the critical line and χ an unramified grossencharacter:

$$E_{s,\chi}(z) = \sum_{\gamma \in P \cap SL_2(\mathfrak{o}) \backslash SL_2(\mathfrak{o})} \prod_{j=1}^n (\text{Im}(\sigma_j(\gamma) \cdot z_j))^s \cdot \chi_j(\text{Im}(\sigma_j(\gamma) \cdot z_j)),$$

where, as usual, P is the standard parabolic subgroup of upper triangular matrices. Fix a basepoint $z_o \in \mathfrak{H}^n$. The automorphic delta distribution δ at z_o has a spectral expansion

$$\delta = \sum_F \overline{F}(z_o) \cdot F + \frac{1}{\langle 1, 1 \rangle} + \sum_{\chi} \frac{1}{4\pi i} \int_{1/2+i\mathbb{R}} E_{1-s,\overline{\chi}}(z_o) \cdot E_{s,\chi} ds,$$

converging in a negatively indexed automorphic Sobolev space. For $\text{Re}(w) > \frac{1}{2}$, there is a unique solution u_w to the automorphic differential equation $(\Delta - \lambda_w)u_w = \delta$, and its spectral expansion, converging in an automorphic Sobolev space, is

$$u_w = \sum_F \frac{\overline{F}(z_o) \cdot F}{\lambda_F - \lambda_w} + \frac{1}{(\lambda_1 - \lambda_w)\langle 1, 1 \rangle} + \sum_{\chi} \frac{1}{4\pi i} \int_{1/2+i\mathbb{R}} \frac{E_{1-s,\overline{\chi}}(z_o) \cdot E_{s,\chi}}{\lambda_{s,\chi} - \lambda_w} ds,$$

where λ_F , λ_1 , and $\lambda_{s,\chi}$ denote the Δ -eigenvalues of the Hilbert–Maass waveforms F , 1, and $E_{s,\chi}$ occurring in the spectral expansion. We emphasize that this is an equality of functions in an automorphic Sobolev space; it is not necessary to require pointwise equality. Since the eigenvalues corresponding to the cuspidal and residual spectrum are discrete, elementary estimates ensure the meromorphic continuation, in w , of the cuspidal and residual part of the spectral expansion, as a Sobolev-space-valued function. However, as we will show, the continuous part of the spectral expansion exhibits branching: for each nontrivial unramified grossencharacter χ , the corresponding integral has *two branch points* on the critical line.

NOTE. In contrast, when $n = 1$, that is, $k = \mathbb{Q}$, there is no branching since the continuous part of the spectral expansion of the fundamental solution,

$$u_w = \sum_F \frac{\overline{F}(z_o) \cdot F}{\lambda_F - \lambda_w} + \frac{1}{(\lambda_1 - \lambda_w)\langle 1, 1 \rangle} + \frac{1}{4\pi i} \int_{1/2+i\mathbb{R}} \frac{E_{1-s}(z_o) \cdot E_s}{\lambda_s - \lambda_w} ds,$$

does not involve a sum over grossencharacters. In Section 3.1 we prove an analogous fact for GL_3 .

Fix a grossencharacter χ . Take real parameters $t_{\chi} = (t_1, \dots, t_n)$ with $t_1 + \dots + t_n = 0$ such that

$$\chi(\alpha) = \sigma_1(\alpha)^{it_1} \dots \sigma_n(\alpha)^{it_n},$$

where $\alpha \in (k \otimes_{\mathbb{Q}} \mathbb{R})^{\times}$, and let

$$\|t_{\chi}\|^2 = \frac{1}{n} (|t_1|^2 + \dots + |t_n|^2).$$

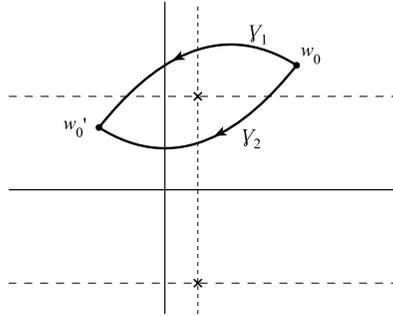


Figure 1 Pathwise meromorphic continuation along these two paths in the w -plane yields functions that differ by a term of moderate growth. The dotted vertical line is the critical line $\text{Re}(w) = \frac{1}{2}$. The dashed horizontal lines are $\text{Im}(w) = \pm \|t_\chi\|$

We will show that the χ th integral in the spectral expansion of u_w admits a *pathwise* meromorphic continuation to the complex plane with exactly two branch points $\frac{1}{2} \pm i \|t_\chi\|$ when χ is nontrivial.

Let $\mathcal{I}_\chi(w)$ denote the χ th integral in the spectral expansion of u_w as follows:

$$\mathcal{I}_\chi(w) = \int_{1/2+i\mathbb{R}} \frac{E_{1-s, \bar{\chi}}(z_0) E_{s, \chi}}{\lambda_{s, \chi} - \lambda_w} ds \quad \left(\text{Re}(w) > \frac{1}{2} \right).$$

This is a Sobolev-space-valued function of the complex parameter w , defined in a right half-plane.

Writing the eigenvalue in terms of s and t_χ ,

$$\lambda_{s, \chi} = \frac{1}{n}((s + it_1)(s + it_1 - 1) + \dots + (s + it_n)(s + it_n - 1)),$$

we can see that the integrand has poles when the following is satisfied.

$$\begin{aligned} \frac{1}{n}((s + it_1)(s + it_1 - 1) + \dots + (s + it_n)(s + it_n - 1)) &= w(w - 1), \\ \frac{1}{n} \left((s + it_1)(s + it_1 - 1) + \frac{1}{4} + \dots + (s + it_n)(s + it_n - 1) + \frac{1}{4} \right) \\ &= w(w - 1) + \frac{1}{4}, \\ \frac{1}{n} \left(\left(s - \frac{1}{2} + it_1 \right)^2 + \dots + \left(s - \frac{1}{2} + it_n \right)^2 \right) &= \left(w - \frac{1}{2} \right)^2. \end{aligned}$$

Since $\sum t_j = 0$ and $\frac{1}{n} \sum t_j^2 = \|t_\chi\|^2$, we have

$$\left(s - \frac{1}{2} \right)^2 - \|t_\chi\|^2 = \left(w - \frac{1}{2} \right)^2.$$

Thus, the integrand has poles at

$$s = \frac{1}{2} \pm \sqrt{\left(w - \frac{1}{2}\right)^2 + \|t_\chi\|^2}.$$

THEOREM 1. *Let χ be a nontrivial grossencharacter, and $\mathcal{I}_\chi(w)$ be the χ th integral in the automorphic spectral decomposition of the fundamental solution u_w , as defined above. Let γ_1 and γ_2 be w -paths in \mathbb{C} , each originating at a point w_0 in the right half-plane $\text{Re}(w) > \frac{1}{2}$, crossing the critical line once, and terminating at a point w'_0 in the left half-plane $\text{Re}(w) < \frac{1}{2}$, with γ_1 crossing the critical line at a height greater in magnitude than $\|t_\chi\|$ and γ_2 crossing at a height less in magnitude than $\|t_\chi\|$. Then pathwise meromorphic continuations $\mathcal{I}_{\chi,\gamma_1}(w)$ and $\mathcal{I}_{\chi,\gamma_2}(w)$ of $\mathcal{I}_\chi(w)$ along the paths γ_1 and γ_2 , respectively, differ by a term of moderate growth, namely by*

$$\mathcal{I}_{\chi,\gamma_1}(w) - \mathcal{I}_{\chi,\gamma_2}(w) = \frac{4\pi i \cdot E_{1-s(\chi,w),\bar{\chi}}(z_0) \cdot E_{s(\chi,w),\chi}}{1 - 2s(\chi,w)},$$

where $s(\chi, w)$ is defined as follows. For fixed w in $\text{Re}(s) > \frac{1}{2}$, $s(\chi, w)$ is the pole of the integrand of $\mathcal{I}_\chi(w)$ in $\text{Re}(s) > \frac{1}{2}$. As w crosses the critical line, $s(\chi, w)$ is defined by analytic continuation.

Proof. We meromorphically continue $\mathcal{I}_\chi(w)$ along two different paths. For fixed w to the right of the critical line, let $s(\chi, w)$ denote the pole of the integrand lying to the right of the critical line. Since the numerator of the integrand is invariant under $s \rightarrow 1 - s$, we regularize as follows:

$$\begin{aligned} \mathcal{I}_\chi(w) &= \int_{1/2+i\mathbb{R}} \frac{E_{1-s,\bar{\chi}}(z_0)E_{s,\chi} - E_{1-s(\chi,w),\bar{\chi}}(z_0)E_{s(\chi,w),\chi}}{\lambda_{s,\chi} - \lambda_w} ds \\ &\quad + E_{1-s(\chi,w),\bar{\chi}}(z_0)E_{s(\chi,w),\chi} \cdot \int_{1/2+i\mathbb{R}} \frac{ds}{\lambda_{s,\chi} - \lambda_w} \quad \left(\text{Re}(w) > \frac{1}{2}\right). \end{aligned}$$

By design the integrand of the first integral on the right side is continuous. The second integral can be evaluated by residues:

$$\begin{aligned} \int_{1/2+i\mathbb{R}} \frac{ds}{\lambda_{s,\chi} - \lambda_w} &= 2\pi i \times \underset{s=1-s(\chi,w)}{\text{Res}} \frac{1}{(s - s(\chi, w))(s - (1 - s(\chi, w)))} \\ &= \frac{2\pi i}{1 - 2s(\chi, w)} \quad \left(\text{Re}(w) > \frac{1}{2}\right). \end{aligned}$$

Thus, we can rewrite the integral in the following way:

$$\begin{aligned} \mathcal{I}_\chi(w) &= \int_{1/2+i\mathbb{R}} \frac{E_{1-s,\bar{\chi}}(z_0)E_{s,\chi} - E_{1-s(\chi,w),\bar{\chi}}(z_0)E_{s(\chi,w),\chi}}{\lambda_{s,\chi} - \lambda_w} ds \\ &\quad + E_{1-s(\chi,w),\bar{\chi}}(z_0)E_{s(\chi,w),\chi} \cdot \frac{2\pi i}{1 - 2s(\chi, w)} \quad \left(\text{Re}(w) > \frac{1}{2}\right). \end{aligned}$$

We will see that when we move w across the critical line, with imaginary part of greater magnitude than $\|t_\chi\|^2$, we pick up an Eisenstein series. Note that since Eisenstein series do not lie in any finite-index automorphic Sobolev space, it is not

trivial to determine where (in what function space) the (pathwise) meromorphic continuation lies.

Consider, for a moment, the case where $\chi = 1$, the trivial character. Then $s(\chi, w) = s(1, w) = \frac{1}{2} + \sqrt{(w - \frac{1}{2})^2} = w$ since $\text{Re}(w) > \frac{1}{2}$ and $s(\chi, w)$ is defined to be to the right of the critical line. Thus, the integral corresponding to the trivial grossencharacter is as follows:

$$\begin{aligned} \mathcal{I}_1(w) &= \int_{1/2+i\mathbb{R}} \frac{E_{1-s,1}(z_o)E_{s,1} - E_{1-w,1}(z_o)E_{w,1}}{\lambda_{s,1} - \lambda_w} ds \\ &\quad + E_{1-w,1}(z_o)E_{w,1} \cdot \frac{2\pi i}{1 - 2w} \quad \left(\text{Re}(w) > \frac{1}{2} \right). \end{aligned}$$

We move w across the critical line and reverse the regularization:

$$\begin{aligned} \mathcal{I}_1(w) &= \int_{1/2+i\mathbb{R}} \frac{E_{1-s,1}(z_o)E_{s,1}}{\lambda_{s,1} - \lambda_w} ds \\ &\quad - E_{1-w,1}(z_o)E_{w,1} \left(\int_{1/2+i\mathbb{R}} \frac{1}{\lambda_{s,1} - \lambda_w} ds - \frac{2\pi i}{1 - 2w} \right) \quad \left(\text{Re}(w) < \frac{1}{2} \right). \end{aligned}$$

Now $s = w$ is the pole to the left of the critical line, and we evaluate the singular integral by residue calculus:

$$\begin{aligned} \int_{1/2+i\mathbb{R}} \frac{ds}{\lambda_{s,\chi} - \lambda_w} &= 2\pi i \times \text{Res}_{s=w} \frac{1}{(s-w)(s-(1-w))} \\ &= \frac{2\pi i}{2w - 1} \quad \left(\text{Re}(w) < \frac{1}{2} \right). \end{aligned}$$

Thus, we have the following expression for the integral corresponding to $\chi = 1$:

$$\begin{aligned} \mathcal{I}_1(w) &= \int_{1/2+i\mathbb{R}} \frac{E_{1-s,1}(z_o)E_{s,1}}{\lambda_{s,1} - \lambda_w} ds \\ &\quad + E_{1-w,1}(z_o)E_{w,1} \cdot \frac{4\pi i}{1 - 2w} \quad \left(\text{Re}(w) < \frac{1}{2} \right). \end{aligned}$$

From this we see that the pathwise meromorphic continuation has an additional term when w is left of the critical line.

The pathwise meromorphic continuation also picks up an additional term in the case where χ is nontrivial, provided that w crosses the critical line sufficiently far away from the real axis. As w crosses the critical line, with imaginary part greater in magnitude than $\|t_\chi\|$, the radicand $(w - \frac{1}{2})^2 + \|t_\chi\|^2$ in the expression for $s(\chi, w)$ moves around the branch point of the square root, the origin. Thus, the analytic continuation of $s(\chi, w)$ along this path is given as follows:

$$s(\chi, w) = \frac{1}{2} - \sqrt{\left(w - \frac{1}{2}\right)^2 + \|t_\chi\|^2} \quad \left(\text{Re}(w) < \frac{1}{2} \right).$$

As before, regularizing, evaluating the singular integral by residues, reversing the regularization, and again evaluating the singular integral, we obtain the meromorphic continuation of the χ th integral along a path γ_1 where w crosses the critical line above a height of $\|t_\chi\|$ or below a height of $-\|t_\chi\|$:

$$\begin{aligned} \mathcal{I}_{\chi, \gamma_1}(w) &= \int_{1/2+i\mathbb{R}} \frac{E_{1-s, \bar{\chi}}(z_o) E_{s, \chi}}{\lambda_{s, \chi} - \lambda_w} ds \\ &\quad + E_{1-s(\chi, w), \bar{\chi}}(z_o) E_{s(\chi, w), \chi} \cdot \frac{4\pi i}{1 - 2s(\chi, w)} \left(\operatorname{Re}(w) < \frac{1}{2} \right). \end{aligned}$$

Meromorphically continuing along a path γ_2 in which w crosses the critical line with imaginary part within a distance of $\|t_\chi\|$ of the real axis does *not* result in the additional Eisenstein series term because, in this case, the radicand $(w - \frac{1}{2})^2 + \|t_\chi\|^2$ in the expression for $s(\chi, w)$ stays strictly in the right half-plane and thus does *not* travel around the origin. Thus, branching is evident: pathwise meromorphic continuations of $\mathcal{I}_\chi(w)$ depend nontrivially on the path, the branch points being $w = \frac{1}{2} \pm i\|t_\chi\|$. \square

2.1. Additional Details: Branching of $s(\chi, w)$

To understand better the different pathwise meromorphic continuations of $\mathcal{I}_\chi(w)$, the χ th integral in the spectral expansion of the automorphic fundamental solution, we explicitly parameterize w -paths crossing the critical line and show how the height of the crossing affects the radicand in the expression for the poles of the integrand.

For χ nontrivial, we can parameterize w as follows:

$$w = \left(\sigma + \frac{1}{2} \right) + (\alpha \|t_\chi\|)i$$

with $\alpha \neq 0$. To describe w crossing the critical line on a horizontal path, we fix α , and let σ range from positive to negative values. In terms of this parameterization, the radicand in the expression for the poles is

$$\left(w - \frac{1}{2} \right)^2 + \|t_\chi\|^2 = (\sigma^2 + (1 - \alpha^2)\|t_\chi\|^2) + (2\sigma\alpha\|t_\chi\|)i.$$

Let x denote the real part of the radicand and y the imaginary part.

$$\begin{aligned} x &= \sigma^2 + (1 - \alpha^2)\|t_\chi\|^2, \\ y &= 2\sigma\alpha\|t_\chi\|. \end{aligned}$$

Eliminating the parameter σ , we can see that the curve is a right-facing parabola:

$$x = \frac{1}{4\alpha^2\|t_\chi\|^2} (y^2 + 4\alpha^2(1 - \alpha^2)\|t_\chi\|^4).$$

The direction that the radicand travels along this curve as σ varies will depend on the sign of α , that is, it depends on whether w is crossing below or above the real axis. What is critical is whether the radicand travels around the origin, the branch point of the square root. When $|\alpha| < 1$ (i.e., w crosses the critical line between

$\frac{1}{2} \pm i \|t_\chi\|$), the radicand does not travel around the origin, but when $|\alpha| > 1$, the radicand does travel around the origin.

When χ is trivial, parameterize w as $w = (\sigma + \frac{1}{2}) + it_o$. Fixing $t_o \neq 0$ and letting σ range from positive to negative values describe w crossing the critical line along a horizontal curve of height t_o . In terms of these parameters, the radicand is

$$\left(w - \frac{1}{2}\right)^2 + \|t_\chi\|^2 = (\sigma^2 - t_o^2) + (2\sigma t_o)i.$$

Again denoting the real and imaginary parts of the radicand by x and y , respectively, we can see that the curve along which the radicand travels is

$$x = \frac{1}{4t_o^2}(y - 2t_o^2)(y + 2t_o^2).$$

This is a right-facing parabola, going around the origin.

The poles of the integrand of the χ th term of the spectral expansion of u_w are at

$$s = \frac{1}{2} \pm \sqrt{\left(w - \frac{1}{2}\right)^2 + \|t_\chi\|^2}.$$

For fixed w to the right of the critical line, we let $s(\chi, w)$ denote the pole to the right of the critical line. We may choose a branch of the square root such that the following holds:

$$s(\chi, w) = \frac{1}{2} + \sqrt{\left(w - \frac{1}{2}\right)^2 + \|t_\chi\|^2} \quad \left(\operatorname{Re}(w) > \frac{1}{2}\right).$$

As w crosses the critical line, we analytically continue $s(\chi, w)$, and if the radicand travels around the origin, we have a sign change:

$$s(\chi, w) = \begin{cases} \frac{1}{2} + \sqrt{\left(w - \frac{1}{2}\right)^2 + \|t_\chi\|^2} & (w\text{-path crosses sufficiently close to the real axis}), \\ \frac{1}{2} - \sqrt{\left(w - \frac{1}{2}\right)^2 + \|t_\chi\|^2} & (w\text{-path crosses sufficiently far from the real axis}). \end{cases}$$

3. Branching of GL_3 Automorphic Fundamental Solution

Let $G = SL_3(\mathbb{R})$, $K = SO(3)$, and $\Gamma = SL_3(\mathbb{Z})$. For simplicity, we consider spherical automorphic forms.

Functions in global automorphic Sobolev spaces have spectral expansions, in terms of a spectral family of automorphic forms, consisting of cusp forms, Eisenstein series, and residues of Eisenstein series. For $GL_3(\mathbb{R})$, it suffices to take an orthonormal basis $\{F\}$ of spherical cusp forms, the continuous family of minimal parabolic Eisenstein series $E_\chi^{1,1,1}$ where $\chi = \exp(\mu)$, for some $\mu \in \rho + i\mathfrak{a}^*$, and the family of $P^{2,1}$ -Eisenstein series, $E_{f,s}^{2,1}$, with cuspidal data f in an orthonormal basis of GL_2 cusp forms and complex parameter $s \in \frac{1}{2} + i\mathbb{R}$, along with the

constant automorphic form (residue of minimal parabolic Eisenstein series). In particular, for Φ in a GL_3 automorphic Sobolev space,

$$\begin{aligned} \Phi &= \sum_{\text{cfm } F} \langle F, \Phi \rangle \cdot F + \frac{\langle \Phi, 1 \rangle}{\langle 1, 1 \rangle} + \frac{1}{|W|} \int_{\rho+i\mathfrak{a}^*} \langle \Phi, E_{\chi_\mu}^{1,1,1} \rangle \cdot E_{\chi_\mu}^{1,1,1} d\mu \\ &\quad + \sum_{GL_2 \text{ cfms } f} \int_{1/2+i\mathbb{R}} \langle \Phi, E_{f,s}^{2,1} \rangle \cdot E_{f,s}^{2,1} ds, \end{aligned}$$

where convergence is in a global Sobolev topology. From now on, we drop the superscripts denoting the relevant parabolic for the Eisenstein series.

Fix a basepoint $x_0 \in G/K \approx \mathbb{H}^3$. Then the automorphic delta distribution δ at x_0 has the spectral expansion

$$\begin{aligned} \delta &= \sum_{\text{cfm } F} \overline{F}(x_0) \cdot F + \frac{1}{\langle 1, 1 \rangle} + \frac{1}{|W|} \int_{\rho+i\mathfrak{a}^*} E_{\bar{\chi}_\mu}(x_0) \cdot E_{\chi_\mu} d\mu \\ &\quad + \sum_{GL_2 \text{ cfms } f} \int_{1/2+i\mathbb{R}} E_{\bar{f},1-s}(x_0) \cdot E_{f,s} ds, \end{aligned}$$

converging in a negatively indexed automorphic Sobolev space. For $\text{Re}(w) > \frac{1}{2}$, there is a unique solution u_w to the automorphic differential equation $(\Delta - \lambda_w)^\nu u_w = \delta$, and its spectral expansion, converging in an automorphic Sobolev space, is

$$\begin{aligned} u_w &= \sum_{\text{cfm } F} \frac{\overline{F}(x_0)}{(\lambda_F - \lambda_w)^\nu} \cdot F + \frac{1}{\langle 1, 1 \rangle (\lambda_1 - \lambda_w)^\nu} \\ &\quad + \frac{1}{|W|} \int_{\rho+i\mathfrak{a}^*} \frac{E_{\bar{\chi}_\mu}(x_0)}{(\lambda_\chi - \lambda_w)^\nu} \cdot E_{\chi_\mu} d\mu \\ &\quad + \sum_{GL_2 \text{ cfms } f} \int_{1/2+i\mathbb{R}} \frac{E_{\bar{f},1-s}(x_0)}{(\lambda_{f,s} - \lambda_w)^\nu} \cdot E_{f,s} ds, \end{aligned}$$

where $\lambda_F, \lambda_1, \lambda_\chi$, and $\lambda_{f,s}$ denote the Δ -eigenvalues of the waveforms $F, 1, E_\chi$, and $E_{f,s}$ occurring in the spectral expansion. For simplicity, we choose ν to be the smallest integer, namely $\nu = 2$, that will ensure (using an automorphic Sobolev embedding theorem) uniform pointwise convergence of the spectral expansion.

Since the eigenvalues corresponding to the cuspidal and residual spectrum are discrete, elementary estimates ensure the meromorphic continuation in w of the cuspidal and residual part of the spectral expansion, as a Sobolev-space-valued function. The part of the expansion corresponding to minimal parabolic Eisenstein series also admits a meromorphic continuation, as in the GL_2 case, but the part of the expansion corresponding to cuspidal data Eisenstein series exhibits branching: for each GL_2 cusp form f in the chosen orthonormal basis, the corresponding integral has *two branch points* on the critical line.

3.1. Pathwise Meromorphic Continuations of Minimal Parabolic Eisenstein Series Component

Let $\mathcal{I}_\chi(w)$ denote the most continuous part of the spectral expansion of the automorphic fundamental solution u_w :

$$\mathcal{I}_\chi(w) = \int_{\rho+i\mathfrak{a}^*} \frac{E_{\bar{\chi}\mu}(x_0)}{(\lambda_\chi - \lambda_w)^2} \cdot E_{\chi\mu} d\mu \quad (\text{Re}(w) > 0).$$

Then $\mathcal{I}_\chi(w)$ is a Sobolev-space-valued integral defined for $\text{Re}(w) > 0$. We show that $\mathcal{I}_\chi(w)$ does not exhibit branching in w .

Let $\mu = \rho + i\eta$, where $\eta \in \mathfrak{a}^*$. Let $\lambda_w = w^2 - \|\rho\|^2$, where w is a complex number with $\text{Re}(w) > 0$. Since the eigenvalue λ_χ is

$$\langle \mu, \mu \rangle - 2\langle \mu, \rho \rangle = -(\langle \eta, \eta \rangle + \langle \rho, \rho \rangle) = -(\|\eta\|^2 + \|\rho\|^2),$$

the denominator of the integrand is $-(\|\eta\|^2 + w^2)$. We rewrite the integral with these normalizations:

$$\int_{\rho+i\mathfrak{a}^*} \frac{E_{\bar{\chi}\mu}(x_0)}{(\lambda_\chi - \lambda_w)^2} \cdot E_{\chi\mu} d\mu = - \int_{\mathfrak{a}^*} \frac{E_{\rho-i\eta}(x_0)}{(\|\eta\|^2 + w^2)^2} \cdot E_{\rho+i\eta} d\eta.$$

Thus, the integrand is undefined when w is purely imaginary and η lies on the circle $\|\eta\| = |w|$ in $\mathfrak{a}^* \approx \mathbb{R}^2$. Let \mathcal{J}_w be the function-valued integral

$$\mathcal{J}_w = \int_{\|\eta\|=|w|} E_{\rho-i\eta}(x_0) E_{\rho+i\eta} d\eta.$$

Then we regularize:

$$\begin{aligned} \int_{\mathfrak{a}^*} \frac{E_{\rho-i\eta}(x_0)}{(\|\eta\| + w^2)^2} \cdot E_{\rho+i\eta} d\eta &= \int_{\mathfrak{a}^*} \frac{E_{\rho-i\eta}(x_0) \cdot E_{\rho+i\eta} - \mathcal{J}_w}{(\|\eta\|^2 + w^2)^2} d\eta \\ &\quad + \mathcal{J}_w \cdot \int_{\mathfrak{a}^*} \frac{1}{(\|\eta\|^2 + w^2)^2} d\eta. \end{aligned}$$

We evaluate the singular integral:

$$\int_{\mathfrak{a}^*} \frac{1}{(\|\eta\|^2 + w^2)^2} d\eta = \int_{\mathbb{R}^2} \frac{1}{(\|\eta\|^2 + w^2)^2} d\eta = 2\pi \cdot \int_0^\infty \frac{r dr}{(r^2 + w^2)^2} = \frac{\pi}{w^2}.$$

So we have the following:

$$\begin{aligned} \int_{\mathfrak{a}^*} \frac{E_{\rho-i\eta}(x_0)}{(\|\eta\| + w^2)^2} \cdot E_{\rho+i\eta} d\eta &= \int_{\mathfrak{a}^*} \frac{E_{\rho-i\eta}(x_0) \cdot E_{\rho+i\eta} - \mathcal{J}_w}{(\|\eta\|^2 + w^2)^2} d\eta \\ &\quad + \frac{\pi \mathcal{J}_w}{w^2} \quad (\text{Re}(w) > 0). \end{aligned}$$

Now we may move w across the imaginary axis and undo the regularization. Since the value of the singular integral is again π/w^2 , the extra terms cancel as follows:

$$\begin{aligned} &\int_{\mathfrak{a}^*} \frac{E_{\rho-i\eta}(x_0) \cdot E_{\rho+i\eta}}{(\|\eta\| + w^2)^2} d\eta \\ &= \int_{\mathfrak{a}^*} \frac{E_{\rho-i\eta}(x_0) \cdot E_{\rho+i\eta}}{(\|\eta\|^2 + w^2)^2} d\eta - \mathcal{J}_w \cdot \int_{\mathfrak{a}^*} \frac{1}{(\|\eta\|^2 + w^2)^2} d\eta \end{aligned}$$

$$\begin{aligned}
 & + \frac{\pi \mathcal{J}_w}{w^2} \quad (\operatorname{Re}(w) < 0) \\
 & = \int_{\mathfrak{a}^*} \frac{E_{\rho-i\eta}(x_0) \cdot E_{\rho+i\eta}}{(\|\eta\|^2 + w^2)^2} d\eta - \frac{\pi \mathcal{J}_w}{w^2} + \frac{\pi \mathcal{J}_w}{w^2} \quad (\operatorname{Re}(w) < 0).
 \end{aligned}$$

3.2. Pathwise Meromorphic Continuations of Cuspidal Eisenstein Series Component

Let f be a GL_2 cusp form occurring in the orthonormal basis chosen above, and let $s_f(s_f - 1)$ be its eigenvalue with $s_f \in [0, 1] \cup \frac{1}{2} + i\mathbb{R}$. If we let $s_f = \frac{1}{2} + it_f$, then $t_f \in -i[0, \frac{1}{2}] \cup \mathbb{R}$ with $t_f = -i/2$ corresponding to $s_f = 0, 1$. Let $\mathcal{I}_f(w)$ denote the integral corresponding to f in the spectral expansion of u_w :

$$\mathcal{I}_f(w) = \int_{1/2+i\mathbb{R}} \frac{E_{\bar{f},1-s}(x_0)E_{f,s}}{(\lambda_{f,s} - \lambda_w)^2} ds \quad \left(\operatorname{Re}(w) > \frac{1}{2}\right).$$

This is a Sobolev-space-valued integral defined in a right half-plane.

The cuspidal data Eisenstein series $E_{f,s}$ generates a principal series, induced from character on the minimal parabolic $P = MN$, which is determined by its action on M :

$$\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \mapsto |a_1|^{s_f+s} |a_2|^{-s_f+s} |a_3|^{-2s}.$$

As derived in the Appendix, the eigenvalue of Casimir on a minimal parabolic Eisenstein series E_χ is

$$\lambda_\chi = 2(s_1^2 + s_1s_2 + s_2^2 - 2s_1 - s_2), \quad \text{where } \chi \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = |a_1|^{s_1} |a_2|^{s_2} |a_3|^{s_3}.$$

Thus, the eigenvalue of Casimir on $E_{f,s}$ is

$$\lambda_{f,s} = 2(s_f(s_f - 1) + 3s(s - 1)).$$

Letting $\lambda_w = 6w(w - 1)$, we can see that the integrand has poles when the following is satisfied:

$$\begin{aligned}
 & 2(s_f(s_f - 1) + 3s(s - 1)) = 6w(w - 1), \\
 & \left(\left(s_f - \frac{1}{2}\right)^2 - \frac{1}{4}\right) + 3\left(\left(s - \frac{1}{2}\right)^2 - \frac{1}{4}\right) = 3\left(\left(w - \frac{1}{2}\right)^2 - \frac{1}{4}\right), \\
 & \left(s_f - \frac{1}{2}\right)^2 - \frac{1}{4} + 3\left(s - \frac{1}{2}\right)^2 = 3\left(w - \frac{1}{2}\right)^2, \\
 & \left(s - \frac{1}{2}\right)^2 = \left(w - \frac{1}{2}\right)^2 - \frac{1}{3}\left(\left(s_f - \frac{1}{2}\right)^2 - \frac{1}{4}\right), \\
 & \left(s - \frac{1}{2}\right)^2 = \left(w - \frac{1}{2}\right)^2 + \frac{1}{3}\left(t_f^2 + \frac{1}{4}\right).
 \end{aligned}$$

Thus, the integrand has poles at

$$s = \frac{1}{2} \pm \sqrt{\left(w - \frac{1}{2}\right)^2 + \frac{1}{3}\left(t_f^2 + \frac{1}{4}\right)}.$$

THEOREM 2. *Let f be a spherical cusp form in the chosen basis of GL_2 cusp forms, and $\mathcal{I}_f(w)$ be the corresponding integral in the automorphic spectral decomposition of the fundamental solution u_w , as defined before. Let γ_1 and γ_2 be w -paths in \mathbb{C} , each originating at a point w_0 in the right half-plane $\operatorname{Re}(w) > \frac{1}{2}$, crossing the critical line once, and terminating at a point w'_0 in the left half-plane $\operatorname{Re}(w) < \frac{1}{2}$, with γ_1 crossing the critical line at a height greater in magnitude than $\sqrt{\frac{1}{3}(t_f^2 + \frac{1}{4})}$ and γ_2 crossing at a height less in magnitude than $\sqrt{\frac{1}{3}(t_f^2 + \frac{1}{4})}$. Then pathwise meromorphic continuations $\mathcal{I}_{f,\gamma_1}(w)$ and $\mathcal{I}_{f,\gamma_2}(w)$ of $\mathcal{I}_f(w)$ along the paths γ_1 and γ_2 , respectively, differ by a term of moderate growth, namely by*

$$\mathcal{I}_{f,\gamma_1}(w) - \mathcal{I}_{f,\gamma_2}(w) = \frac{8\pi i \cdot E_{\bar{f},1-s(f,w)}(x_0) \cdot E_{f,s(f,w)}}{(1 - 2s(f, w))^3},$$

where $s(f, w)$ is defined as follows. For fixed w in $\operatorname{Re}(s) > \frac{1}{2}$, $s(f, w)$ is the pole of the integrand of $\mathcal{I}_f(w)$ in $\operatorname{Re}(s) > \frac{1}{2}$. As w crosses the critical line, $s(f, w)$ is defined by analytic continuation.

Proof. We meromorphically continue $\mathcal{I}_f(w)$ along two different paths. For fixed w to the right of the critical line, let $s(f, w)$ denote the pole of the integrand lying to the right of the critical line. Since the numerator of the integrand is invariant under $s \rightarrow 1 - s$, we regularize as follows:

$$\begin{aligned} \mathcal{I}_f(w) &= \int_{1/2+i\mathbb{R}} \frac{E_{\bar{f},1-s}(x_0)E_{f,s} - E_{\bar{f},1-s(f,w)}(x_0)E_{f,s(f,w)}}{(\lambda_{f,s} - \lambda_w)^2} ds \\ &\quad + E_{\bar{f},1-s(f,w)}(x_0)E_{f,s(f,w)} \cdot \int_{1/2+i\mathbb{R}} \frac{ds}{(\lambda_{f,s} - \lambda_w)^2} \quad \left(\operatorname{Re}(w) > \frac{1}{2}\right). \end{aligned}$$

By design the integrand of the first integral on the right side is continuous. The second integral can be evaluated by residues:

$$\begin{aligned} \int_{1/2+i\mathbb{R}} \frac{ds}{(\lambda_{f,s} - \lambda_w)^2} &= 2\pi i \operatorname{Res}_{s=1-s(f,w)} \frac{1}{(s - s(f, w))^2(s - (1 - s(f, w)))^2} \\ &= \frac{4\pi i}{(2s(f, w) - 1)^3} \quad \left(\operatorname{Re}(w) > \frac{1}{2}\right). \end{aligned}$$

Thus, we can rewrite $\mathcal{I}_f(w)$:

$$\begin{aligned} \mathcal{I}_f(w) &= \int_{1/2+i\mathbb{R}} \frac{E_{\bar{f},1-s}(x_0)E_{f,s} - E_{\bar{f},1-s(f,w)}(x_0)E_{f,s(f,w)}}{(\lambda_{f,s} - \lambda_w)^2} ds \\ &\quad + E_{\bar{f},1-s(f,w)}(x_0)E_{f,s(f,w)} \cdot \frac{4\pi i}{(2s(f, w) - 1)^3} \quad \left(\operatorname{Re}(w) > \frac{1}{2}\right). \end{aligned}$$

As w crosses the critical line, with imaginary part greater in magnitude than $\sqrt{\frac{1}{3}(t_f^2 + \frac{1}{4})}$, the radicand $(w - \frac{1}{2})^2 + \frac{1}{3}(t_f^2 + \frac{1}{4})$ in the expression for $s(f, w)$ moves around the branch point of the square root, the origin. Thus, the analytic continuation of $s(f, w)$ along this path is as follows:

$$s(f, w) = \frac{1}{2} - \sqrt{\left(w - \frac{1}{2}\right)^2 + \frac{1}{3}\left(t_f^2 + \frac{1}{4}\right)} \quad \left(\operatorname{Re}(w) < \frac{1}{2}\right).$$

Regularizing, evaluating the singular integral by residues, reversing the regularization, and again evaluating the singular integral, we obtain the meromorphic continuation of $\mathcal{I}_f(w)$ along a path γ_1 where w crosses the critical line above a height of $\sqrt{\frac{1}{3}(t_f^2 + \frac{1}{4})}$ or below a height of $-\sqrt{\frac{1}{3}(t_f^2 + \frac{1}{4})}$:

$$\begin{aligned} \mathcal{I}_{f,\gamma_1}(w) &= \int_{1/2+i\mathbb{R}} \frac{E_{\bar{f},1-s}(x_0)E_{f,s}}{\lambda_{f,s} - \lambda_w} ds \\ &\quad + E_{\bar{f},1-s(f,w)}(x_0)E_{f,s(f,w)} \cdot \frac{8\pi i}{(1 - 2s(f, w))^3} \quad \left(\operatorname{Re}(w) < \frac{1}{2}\right). \end{aligned}$$

Meromorphically continuing along a path γ_2 in which w crosses the critical line with imaginary part within a distance of $\sqrt{\frac{1}{3}(t_f^2 + \frac{1}{4})}$ of the real axis does *not* result in the additional Eisenstein series term because, in this case, the radicand $(w - \frac{1}{2})^2 + \frac{1}{3}(t_f^2 + \frac{1}{4})$ in the expression for $s(f, w)$ stays strictly in the right half-plane and thus does *not* travel around the origin. Thus, branching is evident: pathwise meromorphic continuations of $\mathcal{I}_f(w)$ depend nontrivially on the path, the branch points being $\frac{1}{2} \pm i\sqrt{\frac{1}{3}(t_f^2 + \frac{1}{4})}$. □

Appendix: Eigenvalue of Casimir on Minimal Parabolic Eisenstein Series

The eigenvalue of Casimir on a minimal parabolic Eisenstein series with character e^μ , $\mu = \rho + i\eta$, where ρ is half the sum of positive roots and $\eta \in \mathfrak{a}^*$, is

$$\langle \mu, \mu \rangle - 2\langle \mu, \rho \rangle = -(\langle \eta, \eta \rangle + \langle \rho, \rho \rangle).$$

Consider $\mu \in \mathfrak{a}_{\mathbb{C}}^*$ as a linear combination of positive roots with coefficients s_α, s_β , and $s_{\alpha+\beta}$:

$$\mu = s_\alpha \alpha + s_\beta \beta + s_{\alpha+\beta}(\alpha + \beta) = (s_\alpha + s_{\alpha+\beta})\alpha + (s_\beta + s_{\alpha+\beta})\beta.$$

Then

$$e^\mu \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = \left| \frac{a_1}{a_2} \right|^{s_\alpha} \cdot \left| \frac{a_2}{a_3} \right|^{s_\beta} \cdot \left| \frac{a_1}{a_3} \right|^{s_{\alpha+\beta}} = \left| \frac{a_1}{a_2} \right|^{s_\alpha + s_{\alpha+\beta}} \cdot \left| \frac{a_2}{a_3} \right|^{s_\beta + s_{\alpha+\beta}},$$

and the eigenvalue is

$$2(s_\alpha^2 + s_\beta^2 - s_\alpha s_\beta + s_\alpha s_{\alpha+\beta} + s_\beta s_{\alpha+\beta} - s_\alpha - s_\beta - 2s_{\alpha+\beta}).$$

On the other hand, it is also common to parameterize the character as

$$e^{\mu} \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = |a_1|^{s_1} \cdot |a_2|^{s_2} \cdot |a_3|^{s_3}, \quad \text{where } s_1 + s_2 + s_3 = 0.$$

In this case, $\mu = s_1\alpha + (s_1 + s_2)\beta$, so the eigenvalue is

$$2(s_1^2 + s_1s_2 + s_2^2 - 2s_1 - s_2).$$

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