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Residual Intersections of Licci Ideals Are Glicci

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Dedicated to Mel Hochster for his groundbreaking work in commutative algebra

1. Introduction

This paper deals with the interplay of two generalizations of classical complete intersection linkage, namely Gorenstein linkage and residual intersection. Recall that two proper ideals I and J in a Cohen–Macaulay ring R are said to be *linked* or *linked with respect to* $\mathfrak{a}$, written $I \sim J$, if $J = \mathfrak{a} : I$ and $I = \mathfrak{a} : J$ for some complete intersection ideal $\mathfrak{a}$. The link is called *geometric* in case $I + J$ has height at least $g + 1$, where $g = \operatorname{ht} \mathfrak{a}$. As is well known, two linked ideals are automatically unmixed of height g, and conversely, whenever $\mathfrak{a} \subsetneq I$ are ideals of height g in a Gorenstein ring with $\mathfrak{a}$ a complete intersection and I unmixed, then $I \sim \mathfrak{a} : I$ is a link [32] (see also [15]). A sequence of two links $I \sim J \sim K$ is referred to as a *double link*. In a similar manner we define the (*even, odd*) *linkage class* of an ideal I, which is the set of all ideals obtained from I by a finite (even, odd) number of links. Finally, one says that an ideal is *licci* if it belongs to the *li*nkage *c*lass of a *c*omplete *i*ntersection. Licci ideals have been studied extensively. They are known to be perfect for instance, and hence define Cohen–Macaulay rings [32]. In addition they share more subtle homological properties of complete intersections: Licci ideals are strongly nonobstructed (provided R is Gorenstein) [5], are strongly Cohen–Macaulay [21], and the shifts in their homogeneous minimal free resolution grow "sufficiently fast" (provided R is a polynomial ring over a field and the ideal is homogeneous) [24].

These more refined properties have often been used to verify that a given ideal fails to be licci, but they also suggest that the classification provided by complete intersection linkage might be too fine for some purposes. Hence in recent years the emphasis has shifted to the more inclusive notion of *Gorenstein linkage,* where the complete intersection ideal $\mathfrak{a}$ in the definition of linkage is replaced by an unmixed Gorenstein ideal $\mathfrak{a}$, meaning an unmixed ideal such that $R/\mathfrak{a}$ is Gorenstein. The first systematic study of this notion can be found in [34]. Again one talks about *Gorenstein double linkage,* the (*even, odd*) *Gorenstein linkage class* of an ideal, and the property of being *glicci,* which means that an ideal belongs to the *G*orenstein *li*nkage *c*lass of a *c*omplete *i*ntersection. The same remarks as above apply to Gorenstein linkage, except that glicci ideals no longer have the more

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refined homological properties of licci ideals. In fact, Cohen–Macaulayness and unmixedness are the only known features of glicci ideals [32]. This has prompted the question of whether every unmixed ideal I in a regular ring R such that R/I is Cohen–Macaulay is glicci. An affirmative answer has been given for many classes of homogeneous ideals (see, e.g., [8; 9; 30; 16; 6; 7; 27; 12; 13]), including ideals of minors of maximal size [28] and, recently, of arbitrary size [14].

The origin of the present paper was the observation that ideals of maximal minors can be realized as residual intersections of perfect ideals of height 2, a result proved in [22]. As perfect ideals of height 2 are licci, we wish to extend the result that ideals of maximal minors are glicci to say that residual intersections of licci ideals are glicci, the main result of this paper.

Residual intersection is a generalization of complete intersection linkage in a different direction, maintaining a condition on the number of generators of the "linking" ideal $\mathfrak{a}$ but allowing the "linked" ideals I and J to have different heights, thus breaking the symmetry. Residual intersections are ubiquitous and play an important role, for instance, in intersection theory. Let I be a proper ideal of height g in a Noetherian ring R and let $s \geq g$ be an integer. A proper ideal J is called an s*-residual intersection* of I if J has height at least s and $J = \mathfrak{a} : I$ for some s-generated ideal $\mathfrak{a}$ contained in I [3]. The residual intersection is said to be *geometric* in case $I + J$ has height at least $s + 1$. When referring to an "s-residual intersection $J = \mathfrak{a} : I$ of I", we always imply that $\mathfrak{a}$ is an s-generated ideal contained in I. Whereas linkage or Gorenstein linkage in Gorenstein rings preserves the Cohen–Macaulay property [32], this is no longer true for residual intersection; nor are s-residual intersections necessarily unmixed, and they may not have the "expected" height s. However, these properties hold under suitable assumptions [22; 20; 26; 36], one of which is the condition G_s: An ideal I in a Noetherian ring R satisfies G_s if the number of generators $\mu(I_p)$ is at most $\dim R_p$ for every prime $p \in V(I)$ with $\dim R_p \leq s - 1$ [3].

Our main result says that in a local Gorenstein ring R with infinite residue field, every residual intersection of a licci ideal is strictly glicci. We call an ideal I *strictly glicci* if, for every $t \geq 0$ and every R-sequence $\underline{y} = y_1, \dots, y_t$ regular on R/I, the ideal $I\bar{R}$ is glicci in the ring $\bar{R} = R/(\underline{y})$. Whereas the licci property is known to pass from I to $I\bar{R}$ [24; 35], it is an open question whether glicci ideals are necessarily strictly glicci. This a priori stronger property is helpful in showing that classes of ideals are glicci, as it allows us to deform to the "generic case".

The proof of the main theorem proceeds as follows. Let J be an s-residual intersection of a licci ideal I in a local Gorenstein ring R with infinite residue field, and let $\underline{y}$ be an R-sequence that is regular on R/J. We show that without changing the residual intersection J, one can modify I within its linkage class until I satisfies G_s and $\underline{y}$ forms a regular sequence modulo I (Theorem 3.4). Once the G_s property holds, we can replace J by any other s-residual intersection of I without leaving the even Gorenstein linkage class of J (Theorem 4.3). Finally, we prove that a suitable s-residual intersection J of I remains constant as we link I two steps closer to a complete intersection, preserving the G_s property of I (Theorem 3.5). The main theorem then follows by induction on the number of double links leading to a complete intersection. The double linkage we use is known as tight double linkage (in algebra) or basic double linkage (in geometry). This means that all but one element in the two regular sequences giving the double link coincide. If furthermore these elements are contained in a fixed subideal $\mathfrak{a} \subset I$, we talk about a tight double link of I along $\mathfrak{a}$ (Definition 2.1). This notion allows one to control the behavior of residual intersections under linkage of I. To assure that properties of the ideal I can only improve as it is linked closer to a complete intersection, we introduce a "canonical" version of tight double linkage, dubbed universal tight double linkage along a subideal (Definition 2.4). The construction, which requires a purely transcendental extension of the residue field, is modeled after the notion of universal linkage developed in [24] and [25]. The basic material about universal tight double linkage can be found in Section 2 of the present paper; Section 3 contains the proofs of Theorems 3.4 and 3.5, which make use of universal tight double linkage. In Section 4 we prove our main result, Theorem 4.6. Applications can be found in Section 5, where we prove that certain classes of ideals are glicci, including residual intersections of complete intersections (Corollary 5.1), of height-2 perfect ideals (Corollary 5.2), and of height-3 perfect Gorenstein ideals (Corollary 5.3); more generally, we consider specializations and deformations of such residual intersections.

In this paper, we mostly work in the local case. There are also the notions of *homogeneous linkage, homogeneous Gorenstein linkage,* and *homogeneous residual intersection* of homogeneous ideals in graded rings, where the "linking ideal" $\mathfrak{a}$ is required to be homogeneous. It includes the corresponding concepts for subschemes of projective space, used in projective geometry. Most of the work on Gorenstein linkage has been done in the projective setting. While quite often, statements about graded rings or homogeneous ideals are special cases of the corresponding local results, this is not true for linkage, because of the additional requirement on the "linking ideal". Indeed, since our technique of universal tight double linkage involves taking generic linear combinations of generators, it will in general fail to produce homogeneous links and does not imply that homogeneous s-residual intersections of homogeneously licci ideals are homogeneously glicci. Nevertheless, we do have affirmative results in the graded setting for the classes of ideals of Section 5 or if s exceeds the height g of I by at most 1 (Theorem 6.1). Prompted by the difficulties in passing between local and homogeneous linkage, we compare the two theories more broadly in Section 7. We provide an example showing that local and homogeneous linkage do not define the same equivalence relation (Example 7.5) and otherwise mainly highlight how poorly the interplay of the two theories is understood.

2. Universal Tight Double Linkage

DEFINITION 2.1. Let R be a Cohen–Macaulay ring and let $\mathfrak{a} \subset I$ be R-ideals with $\operatorname{ht} I = g > 0$. We say that a sequence of links $I \sim I' \sim I''$ is a *tight double link of I along* $\mathfrak{a}$ if there are elements $a_1, \dots, a_{g-1}$ in $\mathfrak{a}$ such that the first link is defined by a regular sequence $a_1, \dots, a_{g-1}, a$ and the second one by a regular sequence of the form $a_1, \dots, a_{g-1}, b$. When $\mathfrak{a}$ and I are homogeneous ideals and R is graded, we call the tight double link *homogeneous* if the elements a_i, a, b are homogeneous. A double link is said to be *minimal* in case R is local and $a_1, \dots, a_{g-1}$ form part of a minimal generating set of $\mathfrak{a}$. It is called *s-minimal,* for s an integer, if $\mathfrak{a}/(a_1, \dots, a_{g-1})$ can be generated by $s - g + 1$ elements. When $\mathfrak{a} = I$ we simply speak of a *tight double link of I.*

DEFINITION AND DISCUSSION 2.2. (1) With notation as in Definition 2.1 write $^-$ for images in the ring $\bar{R} = R/(a_1, \dots, a_{g-1})$. Since $\bar{a}$ and $\bar{b}$ are $\bar{R}$-regular elements, it follows that

$$\bar{b}\bar{I} = (\overline{ab}) : \overline{I'} = \bar{a}\overline{I''}.$$

In other words, $\bar{I} \cong \bar{a}^{-1}\bar{b}\bar{I} = \overline{I''}$ as $\bar{R}$-modules. Let $\mathfrak{a}''$ be the unique R-ideal containing $a_1, \dots, a_{g-1}$ so that

$$\bar{b}\bar{\mathfrak{a}} = \bar{a}\overline{\mathfrak{a}''}.$$

It is also the preimage in R of $\bar{a}^{-1}\bar{b}\bar{\mathfrak{a}} \subset \overline{I''}$, which shows that $\mathfrak{a}'' \subset I''$. Since $\bar{\mathfrak{a}} : \bar{I} = (\bar{a}^{-1}\bar{b}\bar{\mathfrak{a}}) : (\bar{a}^{-1}\bar{b}\bar{I}) = \overline{\mathfrak{a}''} : \overline{I''}$, we obtain

$$\mathfrak{a} : I = \mathfrak{a}'' : I''.$$

We call $\mathfrak{a}''$ the *ideal derived from* $\mathfrak{a}$.

Notice that $\bar{\mathfrak{a}} \cong \overline{\mathfrak{a}''}$. In particular, $\mu(\mathfrak{a}'') \le \mu(\mathfrak{a})$ if the tight double link along $\mathfrak{a}$ is minimal, and $\mathfrak{a}''$ is generated by s elements if it is s-minimal. Thus whenever the tight double link is s-minimal and $J = \mathfrak{a} : I$ is an s-residual intersection of I, then $J = \mathfrak{a}'' : I''$ is an s-residual intersection of I''.

(2) Let R be a Cohen–Macaulay ring and let $\mathfrak{a} \subset I$ be R-ideals with $\operatorname{ht} I = g > 0$. A *sequence of n tight double links of I along* $\mathfrak{a}$ is a sequence of tight double links $I = I_0 \sim I_1 \sim I_2 \sim \cdots \sim I_{2n-2} \sim I_{2n-1} \sim I_{2n}$ together with ideals $\mathfrak{a}_{2i} \subset I_{2i}$ such that $\mathfrak{a}_0 = \mathfrak{a}$, and for $1 \le i \le n$, $I_{2i-2} \sim I_{2i-1} \sim I_{2i}$ is a tight double link of I_{2i-2} along $\mathfrak{a}_{2i-2}$ and $\mathfrak{a}_{2i}$ is the ideal derived from $\mathfrak{a}_{2i-2}$. We call $\mathfrak{a}_{2i}$ the ith *ideal derived from* $\mathfrak{a}$. Notice that $\mathfrak{a} : I = \mathfrak{a}_{2n} : I_{2n}$. In case all the tight double links occurring in the sequence are minimal, we obtain $\mu(\mathfrak{a}_{2n}) \le \mu(\mathfrak{a})$. If they are s-minimal, then $\mathfrak{a}_{2n}$ is s-generated, and if in addition $J = \mathfrak{a} : I$ is an s-residual intersection of I, then $J = \mathfrak{a}_{2n} : I_{2n}$ is an s-residual intersection of I_{2n}.

Whenever $\mathfrak{a} = I$ we have $\mathfrak{a}_{2i} = I_{2i}$ for every i and we speak of a *sequence of n tight double links of I.*

REMARK 2.3. Quite generally, any sequence of double links can be replaced by a (possibly longer) sequence of tight double links. Indeed, if R is a local Gorenstein ring and I, K are two R-ideals of height $g > 0$ that are doubly linked, then there exists a sequence of g tight double links joining I and K; see, for instance, [3, proof of 1.1 and 1.2], [21, proof 1.11], or Lemma 4.2 in this paper.

We are now going to define the "generic" and "universal" versions of tight double links, modeled after the notions of generic and universal linkage that were introduced in [24].

Definition and Discussion 2.4. (1) Let R be a Gorenstein ring, let $\mathfrak{a} \subset I$ be proper R-ideals with $1 + \operatorname{ht} \mathfrak{a} \geq \operatorname{ht} I = g > 0$, and assume that I is unmixed. Choose generating sequences $a_1, \dots, a_s$ and $f_1, \dots, f_\ell$ of the ideals $\mathfrak{a}$ and I, respectively, and let x_{ij} and y_k be variables, where $1 \leq i \leq g-1$, $1 \leq j \leq s$, and $1 \leq k \leq \ell$. In the ring $R' = R[\{x_{ij}, y_k\}]$ consider the elements $\alpha_i = \sum_{j=1}^{s} x_{ij} a_j$ of $\mathfrak{a}R'$ and $\alpha = \sum_{k=1}^{\ell} y_k f_k$ of IR'. Together they form an R'-regular sequence. Define the R'-ideal $I' = (\alpha_1, \dots, \alpha_{g-1}, \alpha) : IR'$. For a generating sequence $h_1, \dots, h_m$ of I' and new variables $z_1, \dots, z_m$ we consider the element $\beta = \sum_{k=1}^{m} z_k h_k$ in the ring $R'' = R'[z_1, \dots, z_m]$. Now $\alpha_1, \dots, \alpha_{g-1}, \beta$ form an R''-regular sequence contained in $I'R''$. Finally, we define the R''-ideal $I'' = (\alpha_1, \dots, \alpha_{g-1}, \beta) : I'R''$.

(2) The ideal I'' so defined is a tight double link of IR'' along $\mathfrak{a}R''$. Thus we may consider the first derived ideal $\mathfrak{a}''$. We write $T_0(I; \mathfrak{a}) = I$ and $T_1(I; \mathfrak{a}) = I''$, and for $n > 1$ we define inductively $T_n(I; \mathfrak{a}) = T_{n-1}(I''; \mathfrak{a}'')$. We call $T_n(I; \mathfrak{a})$ an nth *generic tight double link of I along* $\mathfrak{a}$. This ideal is defined in a polynomial ring over R in a finite set of variables, which we usually denote by a single capital letter, say X. Notice that $T_n(I; \mathfrak{a})$ is an nth tight double link of $IR[X]$ along $\mathfrak{a}R[X]$. If $\mathfrak{a} = I$ we simply write $T_n(I) = T_n(I; \mathfrak{a})$ and talk about an nth *generic tight double link of I*.

(3) Now assume in addition that $(R, \mathfrak{m})$ is local. If $\mathfrak{a} \subset I$ are as above, we can perform the same construction as in (1), replacing the rings R' and R'' by their localizations $R'_{\mathfrak{m}R'}$ and $R''_{\mathfrak{m}R''}$. We denote the resulting ideal by $T^1(I; \mathfrak{a})$. It is either the unit ideal or else a tight double link of $IR''_{\mathfrak{m}R''}$ along $\mathfrak{a}R''_{\mathfrak{m}R''}$. We also allow I to be the unit ideal and $\mathfrak{a}$ to be any ideal, in which case we set $T^1(I; \mathfrak{a}) = R$. In either case we can repeat the construction and inductively define an nth *universal tight double link of I along* $\mathfrak{a}$, which we denote by $T^n(I; \mathfrak{a})$. This ideal is defined in a ring of the form $R(Z) = R[Z]_{\mathfrak{m}R[Z]}$; it is either the unit ideal or else an nth minimal tight double link of $IR(Z) \neq R(Z)$ along $\mathfrak{a}R(Z)$. Whenever $\mathfrak{a} = I$ we write $T^n(I) = T^n(I; \mathfrak{a})$ and call this ideal an nth *universal tight double link of I*.

In order to work with these notions we need the following definitions from [24].

Definition 2.5. Let (R, I) and (S, J) be pairs, where R, S are Noetherian algebras over some ring and I, J are ideals of R and S, respectively. For items (c) and (e) assume that R and S are local.

(a) We say that (R, I) and (S, J) are *isomorphic* and write $(R, I) \cong (S, J)$ if there is an isomorphism of algebras $\phi\colon R \to S$ with $\phi(I) = J$.
(b) We say that (R, I) and (S, J) are *equivalent* and write $(R, I) \equiv (S, J)$ if there are finite sets of variables X over R and Z over S such that $(R[X], IR[X]) \cong (S[Z], JS[Z])$.
(c) We say that (R, I) and (S, J) are *generically equivalent* and write $(R, I) \approx (S, J)$ if there are finite sets of variables X over R and Z over S such that $(R(X), IR(X)) \cong (S(Z), JS(Z))$.
(d) We say that (S, J) is a *deformation* of (R, I), or (R, I) is a *specialization* of (S, J), if there is a sequence of elements $\underline{y} = y_1, \dots, y_t$ in S that is regular on S and on S/J such that $(S/(\underline{y}), (J, \underline{y})/(\underline{y})) \cong (R, I)$.

(e) We say that (S, J) is *essentially a deformation* of (R, I) if there is a finite sequence of pairs of local algebras and ideals (S_i, J_i), $0 \le i \le n$, such that $(S_0, J_0) = (R, I)$, $(S_n, J_n) = (S, J)$, and for every $0 \le i \le n-1$ one of the following conditions holds:
  (i) (S_{i+1}, J_{i+1}) is a deformation of (S_i, J_i);
  (ii) $(S_{i+1}, J_{i+1}) = ((S_i)_p, (J_i)_p)$ for some prime ideal p of S_i;
  (iii) $(S_{i+1}, J_{i+1}) \approx (S_i, J_i)$.

A priori, the definitions of generic and universal tight double linkage depend on the generating sequences chosen at the various steps of the construction. The next result shows that this dependence can be neglected.

REMARK 2.6. (a) In the setting of 2.4(2), the pair $(R[X], T_n(I; \mathfrak{a}))$ is uniquely determined up to equivalence, where all rings are considered as R-algebras.

(b) In the setting of 2.4(3), the pair $(R(Z), T^n(I; \mathfrak{a}))$ is uniquely determined up to generic equivalence, where all rings are considered as R-algebras.

We do not include proofs of these facts because they proceed along the lines of [24, proof of 2.11]. The same applies to the next result, which is an analogue of [24, 2.13].

REMARK 2.7. (a) In the setting of 2.4(2), let $T_n(I; \mathfrak{a}) \subset R[X]$ and let p be a prime ideal of R containing I. One has $(R_p[X], T_n(I; \mathfrak{a})R_p[X]) \equiv (R_p[X], T_n(I_p; \mathfrak{a}_p))$, considering all rings as R_p-algebras.

(b) In the setting of 2.4(2), assume that $(R, \mathfrak{m})$ is local and let $T_n(I; \mathfrak{a}) \subset R[X]$ and $T^n(I; \mathfrak{a}) \subset R(Z)$. One has $(R[X]_{\mathfrak{m}R[X]}, T_n(I; \mathfrak{a})_{\mathfrak{m}R[X]}) \approx (R(Z), T^n(I; \mathfrak{a}))$, considering all rings as R-algebras.

(c) In the setting of 2.4(3), let $T^1(I; \mathfrak{a}) \subset R(Z)$. One has $T^1(I; \mathfrak{a}) = R(Z)$ if and only if either $I = R$ or else I is a complete intersection and $I/\mathfrak{a}$ is cyclic.

The next three results address the important question of how specialization affects linkage and tight double linkage.

PROPOSITION 2.8. *Let R be a local Gorenstein ring, let $I \sim J$ be a Gorenstein link in R defined by a Gorenstein ideal $\mathfrak{b}$ of height g, and assume that R/I is Cohen–Macaulay. Let $y_1, \dots, y_t$ be an R-regular sequence and write $^-$ for images in the ring $\bar{R} = R/(y_1, \dots, y_t)$. If* $\operatorname{ht} \bar{\mathfrak{b}} \ge g$ *then $\bar{I} \sim \bar{J}$ is a Gorenstein link in $\bar{R}$ defined by the Gorenstein ideal $\bar{\mathfrak{b}}$, and $y_1, \dots, y_t$ form a regular sequence on $R/\mathfrak{b}$, on R/I, and on R/J.*

Proof. Clearly $y_1, \dots, y_t$ form a regular sequence on $R/\mathfrak{b}$, because $\operatorname{ht} \bar{\mathfrak{b}} \ge \operatorname{ht} \mathfrak{b}$ and R as well as $R/\mathfrak{b}$ are Cohen–Macaulay. Now, replacing R by $R/\mathfrak{b}$ we may assume that $\mathfrak{b} = 0$, and inducting on t we may further suppose that $t = 1$. But then the assertion is a special case of [23, 2.12]. □

COROLLARY 2.9. *Let R be a local Gorenstein ring, let $\mathfrak{a} \subset I$ be R-ideals with* $\operatorname{ht} I > 0$, *and assume that R/I is Cohen–Macaulay. Let $y_1, \dots, y_t$ be an R-regular* *sequence and write* $^-$ *for images in* $\bar{R} = R/(y_1, \ldots, y_t)$. *Let* $I = I_0 \sim I_1 \sim I_2 \sim \cdots \sim I_{2n}$ *be a sequence of tight double links of* I *along* $\mathfrak{a}$ *with derived ideals* $\mathfrak{a}_{2i}$. *If the images in* $\bar{R}$ *of the regular sequences defining these links are regular on* $\bar{R}$, *then they define a sequence* $\bar{I} = \overline{I_0} \sim \overline{I_1} \sim \overline{I_2} \sim \cdots \sim \overline{I_{2n}}$ *of tight double links of* $\bar{I}$ *along* $\bar{\mathfrak{a}}$ *with derived ideals* $\overline{\mathfrak{a}_{2i}}$, *and* $y_1, \ldots, y_t$ *form a regular sequence modulo* I_j *for every* j.

Proof. Notice that the rings R/I_j are all Cohen–Macaulay by [32, 1.3] or by Proposition 2.8. The assertion of the corollary now follows from Proposition 2.8 and the construction of the derived ideals $\mathfrak{a}_{2i}$ as described in Definition 2.2(1). □

COROLLARY 2.10. *Let* R *be a local Gorenstein ring, let* $\mathfrak{a} \subset I$ *be proper* R*-ideals with* $\operatorname{ht} I > 0$, *and assume that* R/I *is Cohen–Macaulay. Let* $y_1, \ldots, y_t$ *be a sequence of elements in* R *that is regular on* R *and on* R/I, *write* $^-$ *for images modulo the ideal generated by* $y_1, \ldots, y_t$ *in any appropriate ring, and suppose that* $1 + \operatorname{ht} \bar{\mathfrak{a}} \geq \operatorname{ht} \bar{I}$. *If* $T_n(I; \mathfrak{a}) \subset R[X]$ *is any* nth *generic tight double link of* I *along* $\mathfrak{a}$, *then* $\overline{T_n(I; \mathfrak{a})} \subset \overline{R[X]}$ *is an* nth *generic tight double link of* $\bar{I}$ *along* $\bar{\mathfrak{a}}$, *and* $y_1, \ldots, y_t$ *form a regular sequence modulo* $T_n(I; \mathfrak{a})$.

Proof. By induction on n it suffices to prove that the assertion holds for $n = 1$ and that the first derived ideal $\mathfrak{a}'' \subset T_1(I; \mathfrak{a})$ still satisfies the inequality $1 + \operatorname{ht} \overline{\mathfrak{a}''} \geq \operatorname{ht} \overline{T_1(I; \mathfrak{a})}$. The Cohen–Macaulay assumption is preserved as we pass from I to $T_1(I; \mathfrak{a})$, again by [32, 1.3] or by Proposition 2.8. Thus, consider the generic tight double link $IR[X] \sim I' \sim I'' = T_1(I; \mathfrak{a})$ along $\mathfrak{a}$. As $1 + \operatorname{ht} \bar{\mathfrak{a}} \geq \operatorname{ht} \bar{I} = \operatorname{ht} \overline{I}$ it follows that the regular sequences defining these links remain regular on $\overline{R[X]}$. Hence, applying Proposition 2.9 to the localizations at the homogeneous maximal ideal of $R[X]$, we conclude that $\overline{T_1(I; \mathfrak{a})}$ is a first generic tight double link of $\bar{I}$ along $\bar{\mathfrak{a}}$ with derived ideal $\overline{\mathfrak{a}''}$ and that $y_1, \ldots, y_t$ form a regular sequence modulo $T_1(I; \mathfrak{a})$. The definition of derived ideals, Definition 2.2(1), shows that $\operatorname{ht} \overline{\mathfrak{a}''} \geq \operatorname{ht} \overline{T_1(I; \mathfrak{a})} - 1$. □

The above results about specialization, and in particular Corollary 2.9, play a crucial role in proving the following analogue of [24, 2.17].

THEOREM 2.11. *Let* $(R, \mathfrak{m})$ *be a local Gorenstein ring, let* $\mathfrak{a} \subset I$ *be* R*-ideals with* $\operatorname{ht} I > 0$, *and assume that* R/I *is Cohen–Macaulay. Let* $I = I_0 \sim I_1 \sim I_2 \sim \cdots \sim I_{2n}$ *be a sequence of tight double links of* I *along* $\mathfrak{a}$. *For* $1 \leq i \leq n$ *consider successive* ith *generic and universal tight double links of* I *along* $\mathfrak{a}$*—namely,* $T_i(I; \mathfrak{a}) \subset R_i$ *and* $T^i(I; \mathfrak{a}) \subset R^i$, *where* R_i *is a polynomial ring over* R_{i-1} *and* R^i *is obtained from* R^{i-1} *by a purely transcendental extension of the residue field. Write* $R_n = R[X]$ *and* $R^n = R(Z)$, *and consider all rings as* R*-algebras.*

(a) *There exists a prime ideal* Q *of* $R[X]$ *containing* $\mathfrak{m}$ *such that* $(R[X]_Q, T_i(I; \mathfrak{a})_Q)$ *is a deformation of* (R, I_{2i}) *for every* i.
(b) $(R(Z), T^i(I; \mathfrak{a}))$ *is essentially a deformation of* (R, I_{2i}) *and of* (R, I) *for every* i.

Most ideal-theoretic properties can only improve under modifications that are essentially a deformation; see, for instance, [24, 2.3]. Thus part (b) of this theorem shows that the universal sequence of tight double links of an ideal along a subideal is indeed the "optimal model" for any sequence of such links. This general idea is further implemented in Lemmas 2.13 and 2.15, which will be needed in our later proofs. First, however, we have to prove an analogue for generic tight double links of a property already known for generic links.

LEMMA 2.12. *Let R be a local Gorenstein ring, let I be an R-ideal with* $\operatorname{ht} I > 0$, *and assume that R/I is Cohen–Macaulay. If I satisfies G_r for some integer r, then any n*th *generic tight double link $T_n(I)$ of I satisfies G_r.*

Proof. We may assume that $n = 1$. Set $g = \operatorname{ht} I$. Let $I' \subset R'$ be as in Definition 2.4(1) with $\mathfrak{a} = I$, so that $T_1(I) \subset R'[Z]$ is a link of $I'R'[Z]$. Let $Q \in V(T_1(I))$ be such that $\dim R'[Z]_Q \le r - 1$. We need to prove that $\mu(T_1(I)_Q) \le \dim R'[Z]_Q$. We may assume that $I'R'[Z] \subset Q$ because otherwise $T_1(I)_Q$ is a complete intersection.

Since $I'R'[Z]$ and $T_1(I)$ are linked, one has

$$\mu(T_1(I)_Q) \le r((R'[Z]/I'R'[Z])_Q) + g.$$

As I satisfies G_r, [26, 2.5] implies that the latter is bounded above by

$$\dim(R'[Z]/I'R'[Z])_Q + g = \dim R'[Z]_Q. \qquad \square$$

LEMMA 2.13. *Let R be a local Gorenstein ring, and let $\mathfrak{a} \subset I$ be R-ideals with* $\operatorname{ht} I = g > 0$ *and* $\operatorname{ht}(\mathfrak{a} : I) \ge r \ge g$. *Assume that R/I is Cohen–Macaulay and that I can be linked to an ideal satisfying G_r by a sequence of n tight double links. Then any rn*th *universal tight double link $T^{rn}(I; \mathfrak{a})$ of I along $\mathfrak{a}$ satisfies G_r.*

Proof. Write $\mathfrak{m}$ for the maximal ideal of R. By induction on i we prove that any inth universal tight double link $T^{in}(I; \mathfrak{a}) \subset R(Y)$ of I along $\mathfrak{a}$ satisfies G_i for $0 \le i \le r$. The case $i = 0$ is trivial, so we assume that $i > 0$. Let $L = T^{(i-1)n}(I; \mathfrak{a}) \subset R(X)$ be an $(i-1)n$th universal tight double link of I along $\mathfrak{a}$, and write $\mathfrak{b} = \mathfrak{a}_{2(i-1)n}$ for the $(i-1)n$th ideal derived from $\mathfrak{a}$. Notice that $\mathfrak{b} : L = (\mathfrak{a} : I)R(X)$ by Discussion 2.2(2), which gives $\operatorname{ht}(\mathfrak{b} : L) \ge r$. Now set $(R', m') = (R(X), \mathfrak{m}R(X))$ and let $T_n(L; \mathfrak{b}) \subset R'[U]$ be an nth generic tight double link of L along $\mathfrak{b}$. According to Remarks 2.6(b) and 2.7(b) we have

$$(R(Y), T^{in}(I; \mathfrak{a})) \approx (R'[U]_{\mathfrak{m}'R'[U]}, T_n(L; \mathfrak{b})_{\mathfrak{m}'R'[U]}).$$

Because generic equivalence does not affect the G_i property, we may assume that these generic equivalences are equalities:

$$(R(Y), T^{in}(I; \mathfrak{a})) = (R'[U]_{\mathfrak{m}'R'[U]}, T_n(L; \mathfrak{b})_{\mathfrak{m}'R'[U]}).$$

Now let Q be a prime ideal of $R'[U]$ with $T_n(L; \mathfrak{b}) \subset Q \subset \mathfrak{m}'R'[U]$ and $\dim R'[U]_Q \le i - 1$. Our goal is to show that the minimal number of generators of $T_n(L; \mathfrak{b})_Q \subset R'[U]_Q$ is at most $\dim R'[U]_Q$. Write p for the contraction of Q to R'. Notice that $\dim R'_p \le \dim R'[U]_Q \le i - 1$, which gives in particular that $\mathfrak{b}_p = L_p$ as $\mathrm{ht}(\mathfrak{b} : L) \geq r \geq i$. Now Remark 2.7(a), Remark 2.6(a), and the equality $\mathfrak{b}_p = L_p$ imply that

$$(R'_p[U], T_n(L; \mathfrak{b})R'_p[U]) \equiv (R'_p[U], T_n(L_p; \mathfrak{b}_p)) \equiv (R'_p[V], T_n(L_p)),$$

where $T_n(L_p) \subset R'_p[V]$ is an nth generic tight double link of L_p. Under this equivalence, the prime ideal $QR'_p[U]$ gives rise to a prime ideal P of $R'_p[V]$ via a sequence of extensions to polynomial rings, applications of R'_p-algebra isomorphisms, and contractions. Notice that $\dim R'_p[V]_P \leq \dim R'[U]_Q$ and that $P \cap R'_p = pR'_p$. Furthermore, by faithful flatness it suffices to prove that the minimal number of generators of the ideal $T_n(L_p)_P \subset R'_p[V]_P$ is at most $\dim R'_p[V]_P$ or, equivalently, that this ideal satisfies G_i.

We first consider the case where $\dim R'_p \leq i-2$. By our induction hypothesis, the ideal $L_p \subset R'_p$ satisfies G_{i-1} and hence G_i because $\dim R'_p \leq i-2$. Now Lemma 2.12 implies that $T_n(L_p)$ satisfies G_i, and therefore so does $T_n(L_p)_P$.

Next we treat the case where $\dim R'_p = i-1$. Now $\dim R'_p = \dim R'_p[V]_P$, which gives $P = pR'_p[V]$. Thus $T_n(L_p)_P \subset R'_p[V]_P$ is an nth universal tight double link of L_p according to Remark 2.7(b). By Theorem 2.11(b), the pair (R'_p, L_p) is essentially a deformation of (R, I). Our assumption says that I can be linked to an ideal K satisfying G_r by a sequence of n tight double links. Using [23, 2.12] one can lift this sequence of links to obtain a sequence of n tight double links from L_p to an R'_p-ideal K', where either (R'_p, K') is essentially a deformation of (R, K) or else K' is a complete intersection. Again by Theorem 2.11(b), the pair $(R'_p[V]_P, T_n(L_p)_P) = (R'_p(V), T^n(L_p))$ is essentially a deformation of (R'_p, K') and hence essentially a deformation of (R, K) unless K' is a complete intersection. As the property G_r can be expressed in terms of lower bounds on heights of Fitting ideals, it is preserved by any operation that is essentially a deformation in a local Cohen–Macaulay ring. Consequently, $T^n(L_p)_P$ satisfies G_r as well. □

Lemma 2.14. *Let R be a local Gorenstein ring and let $\mathfrak{a} \subset I$ be proper R-ideals with* $\mathrm{ht}\, I = g > 0$ *and* $\mathrm{ht}(\mathfrak{a} : I) \geq g+1$. *Let y be an R-regular element. If I is generically a complete intersection, then y is regular modulo $T_1(I; \mathfrak{a})$.*

Proof. Consider a generic tight double link $T_1(I; \mathfrak{a}) \subset R[X]$. Let P be any associated prime of the ideal $T_1(I; \mathfrak{a})$ and let p be its contraction to R. Notice that $\dim R_p \leq \dim R[X]_P = g$. Thus $\mathfrak{a}_p = I_p$ because $\mathrm{ht}(\mathfrak{a} : I) > g$. It suffices to show that y is a nonzerodivisor modulo $T_1(I; \mathfrak{a})R_p[X]$.

Suppose that $I \subset p$. In this case $g \leq \dim R_p \leq \dim R[X]_P = g$, showing that $P = pR[X]$. Now Remarks 2.7(a) and (b) imply that $T_1(I; \mathfrak{a})_P$ is a first universal tight double link of I_p along $\mathfrak{a}_p$. Such a link is the unit ideal according to Remark 2.7(c), because I_p is a complete intersection by our assumption and $\mathfrak{a}_p = I_p$. This contradicts the fact that P is an associated prime. Therefore $I \not\subset p$. But then, since $\mathfrak{a}_p = I_p$ is the unit ideal, one easily sees that $(R[X]_p, T_1(I; \mathfrak{a})R_p[X]) \equiv (R_p[Z_1, \ldots, Z_g], (Z_1, \ldots, Z_g))$, where $Z_1, \ldots, Z_g$ are variables and all rings are considered as R_p-algebras. The assertion of the lemma now follows because $y \in R$ is a nonzerodivisor modulo $(Z_1, \ldots, Z_g)$. □

LEMMA 2.15. *Let R be a local Gorenstein ring, $\mathfrak{a} \subset I$ ideals in R with* $\operatorname{ht} I = g > 0$, *and $y_1, \dots, y_t$ a sequence of $t \geq 0$ elements that is R-regular. Assume R/I is Cohen–Macaulay. If $t > 0$ further suppose* $\operatorname{ht}(\mathfrak{a} : I, y_1, \dots, y_{t-1}) \geq g + t$ *and I is in the even linkage class of an ideal K such that $y_1, \dots, y_{t-1}$ form a regular sequence on R/K and the ideal $(K, y_1, \dots, y_{t-1})/(y_1, \dots, y_{t-1})$ is generically a complete intersection. Then either $T^n(I; \mathfrak{a})$ is the unit ideal for $n \gg 0$ or else $y_1, \dots, y_t$ form a regular sequence modulo $T^n(I; \mathfrak{a})$ for $n \gg 0$.*

Proof. By Definition 2.4(3) we know that if $T^n(I; \mathfrak{a})$ is the unit ideal for some n, then this holds for every $n \gg 0$. Thus we may assume that none of these ideals is the unit ideal. We proceed by induction on $t \geq 0$. Since the assertion is trivial for $t = 0$ and our assumptions are preserved as t is decreased, we may suppose that $t > 0$ and the claim holds for $t - 1$.

Thus $y_1, \dots, y_{t-1}$ form a regular sequence modulo $T^n(I; \mathfrak{a})$ for large n. We write $I' = T^n(I; \mathfrak{a}) \subset R' = R(X)$ for this universal tight double link, and we let $\bar{}$ denote reduction modulo the ideal generated by $y_1, \dots, y_{t-1}$ in any appropriate ring. Notice that R'/I' is Cohen–Macaulay. The ideal KR' is in the even linkage class of I' and hence, by Remark 2.3, can be obtained from I' by a sequence of, say m, tight double links. Now Theorem 2.11(b) shows that for any universal tight double link $T^m(I') \subset R'(Y)$ of I', the pair $(R'(Y), T^m(I'))$ is essentially a deformation of (R', KR'), where all rings are considered as R'-algebras. Since $y_1, \dots, y_{t-1}$ form a regular sequence on R' and on R'/KR', the pair $(\overline{R'(Y)}, \overline{T^m(I')})$ is still essentially a deformation of $(\overline{R'}, \overline{KR'})$. Thus the property of being generically a complete intersection passes from $\bar{K}$ to $\overline{T^m(I')}$. On the other hand, since $y_1, \dots, y_{t-1}$ form a regular sequence modulo I', Corollary 2.10 together with Remarks 2.7(b) and 2.6(b) gives $(\overline{R'(Y)}, \overline{T^m(I')}) \approx (\overline{R'}(Y), T^m(\overline{I'}))$. Thus $T^m(\overline{I'})$ is generically a complete intersection and a proper ideal. This shows that $\overline{I'}\,\overline{R'}(Y)$ can be linked, by a sequence of m tight double links, to the generic complete intersection $T^m(\overline{I'})$. In fact, since $I'R'(Y)$ is also an nth universal tight double link of I along $\mathfrak{a}$, we may replace $I'R'(Y)$ by I'; then $\overline{I'}$ can be linked to a generic complete intersection by $m > 0$ tight double links.

Now consider $T^{n+(g+1)m}(I; \mathfrak{a}) = T^{(g+1)m}(I'; \mathfrak{a}') \subset R'(Z)$, where $\mathfrak{a}'$ is the nth derived ideal of $\mathfrak{a}$. Recall that $\mathfrak{a}' : I' = \mathfrak{a}R' : IR'$ according to Discussion 2.2(2). We have $\operatorname{ht} \overline{I'} = g$ because $y_1, \dots, y_{t-1}$ form a regular sequence on R/I', and $\operatorname{ht}(\overline{\mathfrak{a}'} : \overline{I'}) \geq \operatorname{ht} \overline{\mathfrak{a}' : I'} = \operatorname{ht} \overline{\mathfrak{a} : I} \geq g + 1$ by our assumption. Since $\operatorname{ht}(\overline{\mathfrak{a}'} : \overline{I'}) \geq g - 1$ and $y_1, \dots, y_{t-1}$ is a regular sequence on R'/I', another application of Corollary 2.10 shows that

$$(\overline{R'(Z)}, \overline{T^{n+(g+1)m}(I; \mathfrak{a})}) = (\overline{R'(Z)}, \overline{T^{(g+1)m}(I'; \mathfrak{a}')}) \approx (\overline{R'}(Z), T^{(g+1)m}(\overline{I'}; \overline{\mathfrak{a}'}))$$

and that $y_1, \dots, y_{t-1}$ form a regular sequence modulo

$$T^{n+(g+1)m}(I; \mathfrak{a}) = T^{(g+1)m}(I'; \mathfrak{a}').$$

By the foregoing, however, $\overline{I'}$ is linked to a generic complete intersection by a sequence of m tight double links. As $\operatorname{ht}(\overline{\mathfrak{a}'} : \overline{I'}) \geq g + 1$, Lemma 2.13 then implies that $\overline{T^{(g+1)m}(\overline{I'}; \overline{\mathfrak{a}'})}$ is generically a complete intersection, and hence $\overline{T^{n+(g+1)m}(I; \mathfrak{a})}$ is.

Changing notation once more we set $I'' = T^{n+(g+1)m}(I; \mathfrak{a}) \subset R'' = R'(Z)$, recalling that $\overline{I''}$ is generically a complete intersection and $y_1, \dots, y_{t-1}$ form a regular sequence on R''/I''. Writing $\mathfrak{a}''$ for the $(g+1)m$th derived ideal of $\mathfrak{a}'$, we have $\operatorname{ht}(\overline{\mathfrak{a}''} : \overline{I''}) \geq \operatorname{ht} \overline{\mathfrak{a}'' : I''} = \operatorname{ht} \overline{\mathfrak{a} : I} \geq g + 1$. Consider the universal tight double link $T^{n+(g+1)m+1}(I; \mathfrak{a}) = T^1(I''; \mathfrak{a}'') \subset R''(U)$. By Corollary 2.10, $(\overline{R''(U)}, \overline{T^1(I''; \mathfrak{a}'')}) \approx (\overline{R''}(U), T^1(\overline{I''}; \overline{\mathfrak{a}''}))$ and $y_1, \dots, y_{t-1}$ form a regular sequence modulo the ideal $T^{n+(g+1)m+1}(I; \mathfrak{a}) = T^1(I''; \mathfrak{a}'')$. Therefore it remains to prove that $y = y_t$ is regular modulo the ideal $\overline{T^{n+(g+1)m+1}(I; \mathfrak{a})} = \overline{T^1(I''; \mathfrak{a}'')}$ or, equivalently, modulo $T^1(\overline{I''}; \overline{\mathfrak{a}''})$. But this follows from Lemma 2.14 because $\operatorname{ht}(\overline{\mathfrak{a}''} : \overline{I''}) \geq g + 1$ and $\overline{I''}$ is generically a complete intersection. Thus $y_1, \dots, y_t$ do indeed form a regular sequence modulo $T^{n+(g+1)m+1}(I; \mathfrak{a})$. Finally, by Corollary 2.10 this property passes to subsequent universal tight double links. □

3. Applying Universal Tight Double Linkage

In this section we use universal tight double linkage to prove two crucial technical results, which allow us to "prepare" a licci ideal I along its even linkage class without changing a given residual intersection of I. Through the first result, Theorem 3.4, we achieve that I satisfies G_s and a given sequence of elements becomes regular modulo I; the second result, Theorem 3.5, guarantees that the G_s property persists as I is linked two steps closer to a complete intersection. However, in order to apply universal tight double linkage, we must first prove that these links—which are defined in a suitable ring extension of R—descend to a sequence of links in R proper. This is the purpose of the next three lemmas. The corresponding statements for universal linkage can be found in [25, 2.1–2.5].

Lemma 3.1. *Let $(R, \mathfrak{m})$ be an equidimensional and catenary Noetherian local ring with infinite residue field k, let $S = R[x_1, \dots, x_m]$ be a polynomial ring, and let J be an S-ideal with $g = \operatorname{ht} J_{\mathfrak{m}S}$. If $(\underline{\lambda}) = (\lambda_1, \dots, \lambda_m)$ is a vector in R^m, write $(\overline{\underline{\lambda}})$ for its image in k^m and $J(\underline{\lambda})$ for the image of J under the evaluation map sending x_i to λ_i. There exists a dense open subset U of k^m such that* $\operatorname{ht} J(\underline{\lambda}) \geq g$ *whenever $(\overline{\underline{\lambda}}) \in U$.*

Proof. Write $d = \dim R$. We first consider the case where $J_{\mathfrak{m}S}$ contains $\mathfrak{m}^e$ for some integer $e \geq 0$. There exists a polynomial $f \in S \setminus \mathfrak{m}S$ such that $f\mathfrak{m}^e S \subset J$. Let $\bar{f}$ be the image of f in $k[x_1, \dots, x_m]$ and notice that $\bar{f} \neq 0$. Thus $U = D(\bar{f})$ is a dense open subset of k^m. If $(\underline{\lambda}) \in R^m$ with $(\overline{\underline{\lambda}}) \in U$, then $f(\underline{\lambda})$ is a unit in R, and the containment $f(\underline{\lambda})\mathfrak{m}^e \subset J(\underline{\lambda})$ gives that $\mathfrak{m}^e \subset J(\underline{\lambda})$. If $g = \infty$ we may take $e = 0$ and obtain $\operatorname{ht} J(\underline{\lambda}) = \infty$. Otherwise, $g = d$ and $\operatorname{ht} J(\underline{\lambda}) \geq d$.

If $J_{\mathfrak{m}S}$ does not contain $\mathfrak{m}^e$ for some $e \geq 0$, then $J \subset \mathfrak{m}S$ and $g < d$. Thus there is a sequence of $d - g$ elements $\underline{y} = y_1, \dots, y_{d-g}$ in $\mathfrak{m}$ such that $\mathfrak{m}S$ is a minimal prime of the S-ideal $(J, \underline{y})S$. Hence by the above, there exists a dense open subset U of k^m such that $\operatorname{ht}(J(\underline{\lambda}), \underline{y}) \geq d$ whenever $(\overline{\underline{\lambda}}) \in U$. Since $\underline{y}$ is a sequence of $d-g$ elements in $\mathfrak{m}$ and since the ring R is local, equidimensional, and catenary, we conclude that $\operatorname{ht} J(\underline{\lambda}) \geq g$. □

The next lemma is a standard fact that can be found in [25, 2.3] for instance.

Lemma 3.2. *Let $(R,\mathfrak{m})$ be a Noetherian local ring with infinite residue field k, $S = R[x_1,\dots,x_m]$ a polynomial ring, and M a finitely generated S-module. If $(\underline{\lambda}) = (\lambda_1,\dots,\lambda_m)$ is a vector in R^m, write $(\overline{\underline{\lambda}})$ for its image in k^m. Then there exists a dense open subset U of k^m such that $\mu(M \otimes_S S_{(\mathfrak{m},\{x_i-\lambda_i\})}) \leq \mu(M \otimes_S S_{\mathfrak{m}S})$ whenever $(\overline{\underline{\lambda}}) \in U$.*

Lemma 3.3. *Let $(R,\mathfrak{m})$ be a local Gorenstein ring with infinite residue field k, let $\mathfrak{a} \subset I$ be proper R-ideals with $1 + \operatorname{ht}\mathfrak{a} \geq \operatorname{ht} I > 0$ and $\mu(\mathfrak{a}) \leq s$, and assume that R/I is Cohen–Macaulay. For $1 \leq i \leq n$ consider successive ith generic tight double links $T_i(I;\mathfrak{a}) \subset R_i$ of I along $\mathfrak{a}$. Write $S = R_n = R[x_1,\dots,x_m]$ and suppose that $T_n(I;\mathfrak{a})_{\mathfrak{m}S} \neq S_{\mathfrak{m}S}$. Let $IS = H_0 \sim H_1 \sim H_2 \sim \cdots \sim H_{2n}$ be the sequence of tight double links of IS along $\mathfrak{a}S$ obtained by extending the tight double links $T_i(I;\mathfrak{a})$ to S, and write $\mathcal{A}_{2i}$ for the derived ideals and $\mathcal{C}_j$ for the complete intersections defining these links. If $(\underline{\lambda}) = (\lambda_1,\dots,\lambda_m)$ is a vector in R^m, write $(\overline{\underline{\lambda}})$ for its image in k^m and set $I_j = H_j(\underline{\lambda})$, $\mathfrak{a}_{2i} = \mathcal{A}_{2i}(\underline{\lambda})$, and $\mathfrak{c}_j = \mathcal{C}_j(\underline{\lambda})$.*

Then there exists a dense open subset U of k^m such that $I = I_0 \sim I_1 \sim I_2 \sim \cdots \sim I_{2n}$ is a sequence of s-minimal tight double links of I along $\mathfrak{a}$ whenever $(\overline{\underline{\lambda}}) \in U$. The derived ideals are $\mathfrak{a}_{2i}$, and the links are defined by the complete intersections $\mathfrak{c}_j$. One can further achieve that $\mu(I_{2i}) \leq \mu(T_i(I;\mathfrak{a})_{\mathfrak{m}R_i})$ for every i and that I_{2i} satisfies G_r for some i and r if $T_i(I;\mathfrak{a})_{\mathfrak{m}R_i}$ does.

Proof. From Remark 2.7(b) we know that $T_i(I;\mathfrak{a})_{\mathfrak{m}R_i}$ is an ith universal tight double link of I along $\mathfrak{a}$. Hence Definition 2.4(3) gives $T_i(I;\mathfrak{a})S_{\mathfrak{m}S} \neq S_{\mathfrak{m}S}$ for $1 \leq i \leq n$ because $T_n(I;\mathfrak{a})_{\mathfrak{m}S} \neq S_{\mathfrak{m}S}$. Thus, upon localizing at the prime ideal $\mathfrak{m}S$, the sequence $IS = H_0 \sim H_1 \sim H_2 \sim \cdots \sim H_{2n}$ remains a sequence of tight double links of the extension of I along the extension of $\mathfrak{a}$; in fact, it becomes minimal and hence s-minimal according to Discussions 2.4(3) and 2.2(2). In particular, the sequence remains a sequence of tight double links of the extension of I along the extension of $\mathfrak{a}$ if we merely localize at maximal S-ideals of the form $\mathcal{M}_{\underline{\lambda}} = (\mathfrak{m}, \{x_\ell - \lambda_\ell\})$.

Write $g = \operatorname{ht} I$. According to Lemma 3.1, there exists a dense open subset U of k^m such that $\operatorname{ht}\mathfrak{c}_j \geq g$ for $1 \leq j \leq 2n$ whenever $(\overline{\underline{\lambda}}) \in U$. Hence the R-ideals $\mathfrak{c}_j$ are complete intersections because they are g generated and proper. Applying Corollary 2.9 to the links $IS = H_0 \sim H_1 \sim H_2 \sim \cdots \sim H_{2n}$ localized at $\mathcal{M}_{\underline{\lambda}}$, we see that $I = I_0 \sim I_1 \sim I_2 \sim \cdots \sim I_{2n}$ is a sequence of tight double links of I along $\mathfrak{a}$ defined by the complete intersections $\mathfrak{c}_j$ and with derived ideals $\mathfrak{a}_{2i}$. By Lemma 3.2 we may assume that this sequence of links is s-minimal. From the same lemma we also obtain $\mu(I_{2i}) \leq \mu(T_i(I;\mathfrak{a})S_{\mathfrak{m}S})$.

To prove the last assertion of the lemma, assume that $T_i(I;\mathfrak{a})_{\mathfrak{m}R_i}$ satisfies G_r. For $1 \leq \ell \leq r-1$, let F_ℓ denote the ℓth Fitting ideal of $T_i(I;\mathfrak{a})S$ considered as an

S-module. The G_r condition means that $\operatorname{ht}(F_\ell)_{\mathfrak{m}S} \geq \ell + 1$ for every ℓ. Applying Lemma 3.1 to the finitely many ideals F_ℓ reveals that, in addition, $\operatorname{ht} F_\ell(\underline{\lambda}) \geq \ell + 1$ whenever $(\overline{\underline{\lambda}})$ belongs to a possibly smaller dense open subset U of k^m. Since $F_\ell(\underline{\lambda})$ is contained in the ℓth Fitting ideal $\operatorname{Fitt}_\ell(I_{2i})$ of I_{2i}, it follows that $\operatorname{ht}\operatorname{Fitt}_\ell(I_{2i}) \geq \ell+1$ in the range $1 \leq \ell \leq r-1$. Therefore I_{2i} satisfies G_r as well. $\square$

THEOREM 3.4. *Let R be a local Gorenstein ring with infinite residue field, I an R-ideal of positive height, J an s-residual intersection of I, and $y_1, \ldots, y_t$ a sequence of $t \geq 0$ elements in R that is regular on R and on R/J. Assume that R/I is Cohen–Macaulay and that I is in the even linkage class of an ideal K satisfying G_r for some $r \leq s$. If $t > 0$, further suppose that $y_1, \ldots, y_{t-1}$ form a regular sequence on R/K and that the ideal $(K, y_1, \ldots, y_{t-1})/(y_1, \ldots, y_{t-1})$ is generically a complete intersection. Then there exists an R-ideal I'' in the even linkage class of I such that J is an s-residual intersection of I'', the ideal I'' satisfies G_r, and the sequence $y_1, \ldots, y_t$ is regular on R/I''.*

Proof. Write $g = \operatorname{ht} I$. If $s = g$ then I and J are directly linked. Since

$$\operatorname{ht}(J, y_1, \ldots, y_t) = g + t,$$

we can find a proper subideal $(b_1, \ldots, b_g) \subsetneq J$ such that $b_1, \ldots, b_g, y_1, \ldots, y_t$ form an R-regular sequence. Now $I'' = (b_1, \ldots, b_g) : J$ has the desired properties. Indeed, this ideal is directly linked to J and hence doubly linked to I, and the sequence $y_1, \ldots, y_t$ is regular modulo I'' by Proposition 2.8. Furthermore, the property G_r is vacuous because $r \leq s = g$. Thus we may assume from now on that $s \geq g+1$. In particular, $\operatorname{ht}(J, y_1, \ldots, y_t) \geq s+t \geq g+1+t$ and $\operatorname{ht}(J, y_1, \ldots, y_{t-1}) \geq g+t$ for $t > 0$.

We write the residual intersection J as $J = \mathfrak{a} : I$. According to Remark 2.3, the ideal K can be obtained from I by a sequence of, say n, tight double links. Hence Lemma 2.13 implies that an rnth universal tight double link $T^{rn}(I; \mathfrak{a}) \subset R(X)$ of I along $\mathfrak{a}$ satisfies G_r. We apply Lemma 2.15 to conclude that $y_1, \ldots, y_t$ form a regular sequence on $R(X, Z)/T^{rn+m}(I; \mathfrak{a})$ for some $m \gg 0$, provided $T^{rn+m}(I; \mathfrak{a}) \neq R(X, Z)$. But $(R(X, Z), T^{rn+m}(I; \mathfrak{a}))$ is essentially a deformation of $(R(X), T^{rn}(I; \mathfrak{a}))$ by Theorem 2.11(b). Hence the property G_r passes from $T^{rn}(I; \mathfrak{a})$ to $T^{rn+m}(I; \mathfrak{a})$. Changing notation sightly, we have now shown that for some $\ell \gg 0$ and $T^\ell(I; \mathfrak{a}) \subset R(X)$, either $T^\ell(I; \mathfrak{a})$ satisfies G_r and $y_1, \ldots, y_t$ form a regular sequence modulo $T^\ell(I; \mathfrak{a})$ or else $T^\ell(I; \mathfrak{a}) = R(X)$. In the latter case we replace ℓ by the largest integer so that $T^\ell(I; \mathfrak{a}) \neq R(X)$. In any case we write $\mathcal{A} = \mathcal{A}^{2\ell} \subset T^\ell(I; \mathfrak{a})$ for the ℓth derived ideal and recall that $JR(X) = \mathcal{A} : T^\ell(I; \mathfrak{a})$ by Discussions 2.4(3) and 2.2(2). Since $T^{\ell+1}(I; \mathfrak{a})$ is the unit ideal in the second case, Remark 2.7(c) shows that $T^\ell(I; \mathfrak{a})$ is generated by a regular sequence $\alpha_1, \ldots, \alpha_{g-1}, \alpha$ and that $\alpha_1, \ldots, \alpha_{g-1}$ form part of a minimal generating set of $\mathcal{A}$. Writing $\mathcal{A} = (\alpha_1, \ldots, \alpha_s)$ and letting β_i be elements of $R(X)$ with $\alpha_i \equiv \beta_i \alpha \bmod (\alpha_1, \ldots, \alpha_{g-1})$ for $g \leq i \leq s$, we see that $JR(X)$ is generated by the s elements $\alpha_1, \ldots, \alpha_{g-1}, \beta_g, \ldots, \beta_s$. Since $\operatorname{ht} JR(X) \geq s$, these elements form a regular sequence, and since $y_1, \dots, y_t$ are a regular sequence modulo $JR(X)$, it follows that $\alpha_1, \dots, \alpha_{g-1}, y_1, \dots, y_t$ form a regular sequence. To summarize, we are in one of two cases as follows.

Case 1: $T^\ell(I; \mathfrak{a})$ satisfies G_r, and $y_1, \dots, y_t$ form a regular sequence modulo $T^\ell(I; \mathfrak{a})$.

Case 2: $\alpha_1, \dots, \alpha_{g-1}, y_1, \dots, y_t$ form a regular sequence, $T^\ell(I; \mathfrak{a})/(\alpha_1, \dots, \alpha_{g-1})$ is cyclic, and $\mathcal{A}/(\alpha_1, \dots, \alpha_{g-1})$ is generated by $s - g + 1$ elements.

Write $\mathfrak{m}$ for the maximal ideal of R. By Remark 2.7(b) we may choose $T^\ell(I; \mathfrak{a})$ to be $T_\ell(I; \mathfrak{a})_{\mathfrak{m}R[X]}$ for some generic tight double link $T_\ell(I; \mathfrak{a}) \subset R[X] = R[x_1, \dots, x_m]$ with ℓth derived ideal $\mathcal{A}_{2\ell}$. We may further suppose that $\mathcal{A} = (\mathcal{A}_{2\ell})_{\mathfrak{m}R[X]}$. We wish to use Lemma 3.3 and the notation introduced there, writing I' for the R-ideal $I_{2\ell} = [T_\ell(I; \mathfrak{a})](\underline{\lambda})$. From the lemma we know that I' is obtained from I by a sequence of ℓ s-minimal tight double links along $\mathfrak{a}$ with ℓth derived ideal $\mathfrak{a}' = \mathcal{A}_{2\ell}(\underline{\lambda})$. In particular, $\mathfrak{a}' : I' = \mathfrak{a} : I = J$ is an s-residual intersection of I by Discussion 2.2(2).

Now assume we are in Case 1. According to Lemma 3.3, we may assume that I' satisfies G_r. Furthermore, the image of $T_\ell(I; \mathfrak{a})$ in the polynomial ring $(R/(y_1, \dots, y_t))[x_1, \dots, x_m]$ is an ideal that has height g after localizing at the extension of $\mathfrak{m}$. Thus, after passing to a possibly smaller dense open subset U of k^m, Lemma 3.1 shows that the image of I' in the ring $R/(y_1, \dots, y_t)$ has height at least g. Therefore $y_1, \dots, y_t$ form a regular sequence modulo I' because R/I' is Cohen–Macaulay and $\operatorname{ht} I' = g$. Taking I'' to be I', we have completed the proof in Case 1.

Next we consider Case 2. We may assume that the elements $\alpha_1, \dots, \alpha_{g-1}$ are in $\mathcal{A}_{2\ell}$; we write $a_1, \dots, a_{g-1}$ for their images in $\mathfrak{a}' = \mathcal{A}_{2\ell}(\underline{\lambda})$. Applying Lemma 3.2 to the $R[x_1, \dots, x_m]$-modules $T_\ell(I; \mathfrak{a})/(\alpha_1, \dots, \alpha_{g-1})$ and $\mathcal{A}_{2\ell}/(\alpha_1, \dots, \alpha_{g-1})$, we obtain a possibly smaller dense open subset U of k^m such that $I'/(a_1, \dots, a_{g-1})$ is cyclic and $\mathfrak{a}'/(a_1, \dots, a_{g-1})$ is $s - g + 1$ generated. By Lemma 3.1 we may further suppose that $a_1, \dots, a_{g-1}, y_1, \dots, y_t$ form an R-regular sequence. Since $\dim R \ge g + t$, there exists an element b such that $a_1, \dots, a_{g-1}, b, y_1, \dots, y_t$ are an R-regular sequence. Linking $I' = (a_1, \dots, a_{g-1}, a)$ with respect to the regular sequences $a_1, \dots, a_{g-1}, ab$ and $a_1, \dots, a_{g-1}, b^2$ defines an s-minimal tight double link of I' along $\mathfrak{a}'$ that produces an ideal I'' and a derived ideal $\mathfrak{a}''$. Clearly $I'' = (a_1, \dots, a_{g-1}, b)$ is a complete intersection and $y_1, \dots, y_t$ form a regular sequence modulo I''. Furthermore, $\mathfrak{a}'' : I'' = \mathfrak{a}' : I' = J$ is an s-residual intersection of I'' according to Discussion 2.2(2). This finishes the proof in Case 2. □

THEOREM 3.5. *Let R be a local Gorenstein ring with infinite residue field, I an R-ideal, and s an integer with $\operatorname{ht} I \le s \le \dim R$. Assume that I can be linked to a complete intersection by a sequence of n tight double links and that I satisfies G_s. Then there exist an s-residual intersection $J = \mathfrak{a} : I$ of I and a sequence of at most n tight double links to a complete intersection such that the first double link $I \sim I' \sim I''$ is an s-minimal tight double link of I along $\mathfrak{a}$ and I'' satisfies G_s.*

Proof. Write $\mathfrak{m}$ for the maximal ideal, k for the residue field of R, and K for a complete intersection obtained from I by a sequence of n tight double links. Set

$g = \operatorname{ht} I$ and $r = \min\{s, \mu(I) - 1\}$, and notice that $0 \le g - 1 \le r \le s \le \dim R$. Let $f_1, \dots, f_\ell$ be a generating sequence of I, and let $x_{\nu\mu}$ be variables for $1 \le \nu \le r$ and $1 \le \mu \le \ell$. In the ring $R' = k[\{x_{\nu\mu}\}]$ consider elements $\beta_\nu = \sum_{\mu=1}^{\ell} x_{\nu\mu} f_\mu$ and ideals $\mathcal{B} = (\beta_1, \dots, \beta_r)$ and $\mathcal{B} : IR'$. The elements $\beta_1, \dots, \beta_{g-1}$ can be used to define a generic tight double link of I, which we extend to a sequence of generic tight double links $T_i(I) \subset R_i$ as in Theorem 2.11. Finally, write

$$S = R_n[\{x_{\nu\mu} \mid \nu \ge g\}] = R[x_1, \dots, x_m].$$

Notice that $T_1(I)S$ is obtained from IS by a tight double link along the ideal $\mathcal{B}S$ and that $\mathcal{B}S/(\beta_1, \dots, \beta_{g-1})$ is $r - g + 1$ generated.

Write $\mathcal{J} = (\mathcal{B} : IR')S = \mathcal{B}S : IS$. According to [26, proof of 5.1] we have $\operatorname{ht} \mathcal{J} \ge r$ because I satisfies G_r. Hence $\operatorname{ht} \mathcal{J}_{\mathfrak{m}S} \ge r$. On the other hand, the ideals $T_i(I)_{\mathfrak{m}R_i}$ are ith universal tight double links of I by Remark 2.7(b). Thus Theorem 2.11(b) implies that $(S_{\mathfrak{m}S}, T_1(I)S_{\mathfrak{m}S})$ is essentially a deformation of (R, I) and that $(S_{\mathfrak{m}S}, T_n(I)S_{\mathfrak{m}S})$ is essentially a deformation of (R, K). Therefore $T_1(I)S_{\mathfrak{m}S}$ satisfies G_s, and $T_n(I)S_{\mathfrak{m}S}$ is either a complete intersection or the unit ideal. In the latter case, we replace n by the largest integer i for which $T_i(I)S_{\mathfrak{m}S} \ne S_{\mathfrak{m}S}$ and then observe that this ideal is a complete intersection according to Remark 2.7(c). Thus, in either case we may assume that $T_n(I)S_{\mathfrak{m}S}$ is a complete intersection. We can further suppose that $n > 0$.

If $(\underline{\lambda})$ is a vector in R^m, write $(\overline{\underline{\lambda}})$ for its image in k^m. We use Lemma 3.3 and the notation introduced there. If $(\overline{\underline{\lambda}})$ lies in a suitable dense open subset of k^m, then specializing the ideals $T_i(I)S$ via $x_j \mapsto \lambda_j$ yields a sequence of tight double links $I = I_0 \sim I_1 \sim I_2 \sim \cdots \sim I_{2n}$ in the ring R such that I_2 satisfies G_s and I_{2n} is a complete intersection. Observe that $I = I_0 \sim I' = I_1 \sim I'' = I_2$ is automatically an r-minimal tight double link along $\mathfrak{b} = \mathcal{B}(\underline{\lambda})$. On the other hand, in applying Lemma 3.1 to the S-ideal $\mathcal{J}$ we may also assume that the R-ideal $\mathcal{J}(\underline{\lambda})$ has height at least r. Since $\mathcal{J}(\underline{\lambda}) \subset \mathfrak{b} : I$, it follows that $\operatorname{ht}(\mathfrak{b} : I) \ge r$. Furthermore, $r < \mu(I)$; hence $\mathfrak{b} \ne I$ and therefore $\mathfrak{b} : I \ne R$. Thus if $r = s$ then $\mathfrak{b} : I$ is an s-residual intersection of I. In this case we may choose $\mathfrak{a}$ to be $\mathfrak{b}$ and the proof is complete. Otherwise, $r = \mu(I) - 1$ and, invoking Lemma 3.2, we may assume that $I/\mathfrak{b}$ is cyclic. Write $I = (\mathfrak{b}, b)$ and let $c_{r+1}, \dots, c_s$ be elements in $\mathfrak{m}$ such that the R-ideal $(\mathfrak{b} : I, c_{r+1}, \dots, c_s)$ has height at least s. Set $\mathfrak{a} = (\mathfrak{b}, c_{r+1}b, \dots, c_s b)$. Since $(\mathfrak{b} : I, c_{r+1}, \dots, c_s) \subset \mathfrak{a} : I$ we have $\operatorname{ht}(\mathfrak{a} : I) \ge s$, and since $\mathfrak{b} \ne I$ it follows that $\mathfrak{a} \ne I$; thus $\mathfrak{a} : I \ne R$. Now $\mathfrak{a} : I$ is an s-residual intersection of I. Finally, observe that $I = I_0 \sim I' = I_1 \sim I'' = I_2$ is an s-minimal tight double link along $\mathfrak{a}$ because $\mathfrak{a}/\mathfrak{b}$ is $s - r$ generated. $\square$

4. Residual Intersections of Licci Ideals: The Local Case

In this section we prove our main result, Theorem 4.6, which says that residual intersections of licci ideals are glicci. The proof makes use of a standard technique for producing Gorenstein links, which can be found in [18, 3.5] or [28, 5.10] for the graded case and in [27, 4.1] for the local case. We record the version of this fact that we will use and include a proof for the reader's convenience. Here and in the next three results we think of a local ring as being trivially graded; in particular, every element and every ideal of such a ring is homogeneous.

PROPOSITION 4.1. *Let R be a Gorenstein ring that is either local or positively graded over a field, let H be a homogeneous R-ideal, and let $\bar{}$ denote images in the ring $\bar{R} = R/H$. Assume that $\bar{R}$ is Cohen–Macaulay and generically Gorenstein. Let J and K be unmixed homogeneous R-ideals containing H, of height one greater than the height of H. If some shifts of $\bar{J}$ and $\bar{K}$ are isomorphic as graded $\bar{R}$-modules, then J and K are linked in R by a homogeneous Gorenstein double link.*

Proof. Since the ring $\bar{R}$ is generically Gorenstein, it has a canonical ideal: a homogeneous ideal ω that is isomorphic to a shift of the graded canonical module of $\bar{R}$. Such an ideal necessarily has positive height. Multiplying ω by a homogeneous nonzerodivisor in $\bar{R}$, we may assume that $\omega \subsetneq \bar{J}$. Notice that $\bar{K} = x\bar{J}$ for some quotient x of homogeneous nonzerodivisors in $\bar{R}$. One has $x\omega \subsetneq x\bar{J} = \bar{K}$, and $x\omega$ is a canonical ideal as well. Furthermore, $\omega :_{\bar{R}} \bar{J} = x\omega :_{\bar{R}} \bar{K}$. Now consider preimages L and N of ω and $x\omega$ in R. Since ω and $x\omega$ are proper homogeneous canonical ideals, it follows that both $R/L \cong \bar{R}/\omega$ and $R/N \cong \bar{R}/x\omega$ are graded Gorenstein rings, necessarily of the same dimension as R/J and R/K. Furthermore, $L \subsetneq J$ and $N \subsetneq K$, the ideals J and K are unmixed, and $L :_R J = N :_R K$. Thus J and $L : J = N : K$ are indeed linked in R by a homogeneous Gorenstein link, and so are $L : J = N : K$ and K. □

The particular type of Gorenstein double link constructed in the proof of Proposition 4.1 is called *homogeneous basic Gorenstein double link*. If R is local, we simply talk about a *basic Gorenstein double link*.

The following lemma is essentially a standard result from basic element theory. For lack of a precise reference, however, we include a sketch of a proof. Particular care must be exerted in order to assure that the intermediate colon ideals cannot become unit ideals.

LEMMA 4.2. *Let k be an infinite field and R a Noetherian ring that is either local with residue field k or a standard graded k-algebra. Let I be a homogeneous R-ideal satisfying G_s, and let $\mathfrak{a} : I$ and $\mathfrak{b} : I$ be homogeneous s-residual intersections of I. Then there are homogeneous generating sequences $a_1, \dots, a_s$ of $\mathfrak{a}$ and $b_1, \dots, b_s$ of $\mathfrak{b}$ such that, for $\mathfrak{c}_i = (a_1, \dots, a_{i-1}, b_{i+1}, \dots, b_s)$ and $1 \le i \le s$, the ideals $\mathfrak{c}_i : I$ are geometric $(s-1)$-residual intersections (when $s \ge \operatorname{ht} I + 1$) and both $(\mathfrak{c}_i, a_i) : I$ and $(\mathfrak{c}_i, b_i) : I$ are s-residual intersections of I. These s-residual intersections are geometric if $\mathfrak{a} : I$ and $\mathfrak{b} : I$ are.*

Proof. We can assume that $s > 0$. We may even suppose that $\mathfrak{a} \ne 0$; for otherwise choose $a_1 = \dots = a_s = 0$ and $b_1, \dots, b_s$ any homogeneous generating sequence of $\mathfrak{b}$. Write $\mathfrak{m}$ for the homogeneous maximal ideal of R, and set $n = \mu(I)$ and $u = \mu(\mathfrak{a})$. Let $\delta_{s-u+1} \ge \dots \ge \delta_s$ be the generator degrees of $\mathfrak{a}$. Define $\delta_1 = \dots = \delta_{s-u} = 0$ if R is local and $\delta_1 = \dots = \delta_{s-u} = \delta_{s-u+1} + 1$ if R is graded. We first claim that there exist homogeneous elements $a_1, \dots, a_s$ of degrees $\delta_1, \dots, \delta_s$ generating $\mathfrak{a}$ such that, for $0 \le i \le s$, the following conditions are satisfied:

(i) $[I/(a_1, \dots, a_i)]_p = 0$ for $p \in \operatorname{Spec}(R)$ with $\dim R_p \le i - 1$;
(ii) $\mu([I/(a_1, \dots, a_i)]_p) \le \dim R_p - i$ for $p \in V(I)$ with $i \le \dim R_p \le s - 1$;
(iii) $\mu([I/(a_1, \dots, a_i)]_p) \le \dim R_p - i$ for $p \in V(I)$ with $\dim R_p = s$, provided the residual intersection $\mathfrak{a} : I$ is geometric;
(iv) $a_1, \dots, a_{s-n+1}$ are in $\mathfrak{m}I$ if R is local and $n \le s$.

We construct such elements $a_1, \dots, a_i$ by induction on i. The case $i = 0$ is obvious because I satisfies G_s. As for the induction step, assume that $i > 0$ and $a_1, \dots, a_{i-1}$ have been constructed. Consider the graded R-modules $M = [\mathfrak{a} \cap \mathfrak{m}I]/[(a_1, \dots, a_{i-1}) \cap \mathfrak{m}I]$ and $N = \mathfrak{a}/(a_1, \dots, a_{i-1})$ as well as the tensor products $M' = M \otimes (R/I)$ and $N' = N \otimes (R/I)$. Let $\mathcal{P} \subset \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ be the finite collection of homogeneous primes $p \ne \mathfrak{m}$ that are minimal primes in the supports of M, N or minimal primes of any Fitting ideals of M', N'. According to a graded version of a basic element lemma (see, e.g., [36, 1.3] and its proof), there exists a homogeneous element a_i of degree δ_i such that the following conditions hold for every $p \in \mathcal{P}$. For this we also recall that the kernel of the natural map $\mathfrak{a} \otimes k \to I \otimes k$ has dimension at least $u - n + 1$ as a k-vector space.

- If R is local and $i \le s - n + 1$, then $a_i \in \mathfrak{a} \cap \mathfrak{m}I$ and the image of a_i is a minimal generator of each of the modules $M_p, (M')_p$ that are not zero; if in addition $i \le u - n + 1$, then a_i is also a minimal generator of N.
- $a_i \in \mathfrak{a}$ and the image of a_i is a minimal generator of each of the modules $N_p, (N')_p$ that are not zero and if $i \ge s - u + 1$ then a_i is also a minimal generator of N when this module is not zero.

The elements $a_1, \dots, a_i$ so constructed indeed have the desired properties. To see this, notice that for each of the primes p occurring in items (i)–(iii), $I_p = \mathfrak{a}_p$. Furthermore, $s \le \dim R$ with $s < \dim R$ if the residual intersection $\mathfrak{a} : I$ is geometric. Thus none of the primes p in items (i)–(iii) is the maximal ideal $\mathfrak{m}$; in particular, $I_p = (\mathfrak{m}I)_p$. Finally, observe that the inequalities in items (ii) and (iii) for $i - 1$ are strict except possibly for $p \in \mathcal{P}$. For details we refer to the proof of [36, 1.4].

We now turn to the construction of the elements $b_1, \dots, b_s$. As before, we may assume that $\mathfrak{b} \ne 0$. Write $v = \mu(\mathfrak{b})$ and let $\varepsilon_1 \le \dots \le \varepsilon_v$ be the generator degrees of $\mathfrak{b}$. Define $\varepsilon_{v+1} = \dots = \varepsilon_s = 0$ in the local case and $\varepsilon_{v+1} = \dots = \varepsilon_s = \varepsilon_v + 1$ in the graded case. We argue that there exist homogeneous generators $b_1, \dots, b_s$ of $\mathfrak{b}$ having degrees $\varepsilon_1, \dots, \varepsilon_s$ such that the following statements hold for all non-negative integers i, j with $0 \le i + j \le s$:

- $[I/(a_1, \dots, a_i, b_{s-j+1}, \dots, b_s)]_p = 0$ for $p \in \operatorname{Spec}(R)$ with $\dim R_p \le i + j - 1$;
- $\mu([I/(a_1, \dots, a_i, b_{s-j+1}, \dots, b_s)]_p) \le \dim R_p - i - j$ for $p \in V(I)$ with $i + j \le \dim R_p \le s - 1$;
- $\mu([I/(a_1, \dots, a_i, b_{s-j+1}, \dots, b_s]_p) \le \dim R_p - i - j$ for $p \in V(I)$ with $\dim R_p = s$, provided the residual intersections $\mathfrak{a} : I$ and $\mathfrak{b} : I$ are geometric;
- $b_n, \dots, b_s$ are in $\mathfrak{m}I$ if R is local and $n \le s$.

One shows the existence of such elements $b_{s-j+1}, \dots, b_s$ by induction on j. The case $j = 0$ follows from the previous step of the proof. To obtain b_{s-j+1} in the induction step we argue as before, applying the basic element lemma to the modules $\mathfrak{b}/(b_{s-j+2}, \dots, b_s)$ as well as the modules

$$M_i = [(a_1, \dots, a_i, \mathfrak{b}) \cap \mathfrak{m}I]/[(a_1, \dots, a_i, b_{s-j+2}, \dots, b_s) \cap \mathfrak{m}I]$$

and

$$N_i = (a_1, \dots, a_i, \mathfrak{b})/(a_1, \dots, a_i, b_{s-j+2}, \dots, b_s)$$

and their tensor products with R/I, where $0 \le i \le s - j$.

Now the elements $a_1, \dots, a_s$ and $b_1, \dots, b_s$ have the properties asserted in the lemma. In particular we claim that $(a_1, \dots, a_i, b_{i+1}, \dots, b_s) \ne I$ for every i. This holds in the local case because $a_1, \dots, a_{s-n+1}$ and $b_n, \dots, b_s$ belong to $\mathfrak{m}I$ if $n \le s$. In the graded case, write $\underline{\beta} = \beta_{00}, \beta_{01}, \beta_{02}, \dots$ for the sequence of zeroth graded Betti numbers. As $\mathfrak{a} \subsetneq I$ and $\mathfrak{b} \subsetneq I$, we have $\underline{\beta}(\mathfrak{a}) < \underline{\beta}(I)$ and $\underline{\beta}(\mathfrak{b}) < \underline{\beta}(I)$ in the lexicographic order. Suppose that $(a_1, \dots, a_i, b_{i+1}, \dots, b_s) = I$ for some i. We may assume without loss of generality that $\delta_i \le \varepsilon_{i+1}$, in which case δ_i is the initial degree of I. Therefore $i \ge s - u + 1$. Since $a_1, \dots, a_{s-u}$ are in $\mathfrak{m}\mathfrak{a}$, we conclude that $(a_{s-u+1}, \dots, a_i, b_{i+1}, \dots, b_s) = I$ and that $a_{s-u+1}, \dots, a_s$ constitute a homogeneous minimal generating set of $\mathfrak{a}$. Again, since δ_i is the initial degree of I, we have $\delta_i = \delta_{i+1} = \cdots = \delta_s$. Also notice that $\delta_i \le \varepsilon_{i+1} \le \cdots \le \varepsilon_s$. Therefore $\underline{\beta}((a_{s-u+1}, \dots, a_i, b_{i+1}, \dots, b_s)) \le \underline{\beta}((a_{s-u+1}, \dots, a_i, a_{i+1}, \dots, a_s)) = \underline{\beta}(\mathfrak{a}) < \underline{\beta}(I)$, contradicting the equality $(a_{s-u+1}, \dots, a_i, b_{i+1}, \dots, b_s) = I$. □

The next theorem is another crucial ingredient in the proof of Theorem 4.6. It allows to pass from one s-residual intersection of I to another without leaving the even Gorenstein linkage class, provided I satisfies G_s and AN_{s-1}^-. Recall that an ideal I in a Cohen–Macaulay ring R has the *Artin–Nagata property* AN_r^- for an integer r if R/K is Cohen–Macaulay for every geometric i-residual intersection K of I and every integer $i \le r$. This property holds for instance when I is both licci and G_r (as follows from [21, 1.11] and [22, 3.1]) or, more generally, when I is licci (see [26, 5.3]).

THEOREM 4.3. *Let k be an infinite field and R a Gorenstein ring that is either local with residue field k or a standard graded k-algebra. Let I be a homogeneous R-ideal satisfying G_s and AN_{s-1}^-, and let J and K be two homogeneous s-residual intersections of I. Then J and K are obtained from each other by a sequence of s homogeneous basic Gorenstein double links.*

Proof. We write $J = \mathfrak{a} : I$ and $K = \mathfrak{b} : I$, and we retain the notation of Lemma 4.2. It suffices to prove that, for every $1 \le i \le s$, the homogeneous ideals $(\mathfrak{c}_i, a_i) : I$ and $(\mathfrak{c}_i, b_i) : I$ are obtained from each other by a homogeneous basic Gorenstein double link. We do so by invoking Proposition 4.1.

Write $\bar{}$ for images in the graded ring $\bar{R} = R/\mathfrak{c}_i : I$. The ideal I satisfies AN_{s-1}^- by our assumption and, by Lemma 4.2, $\mathfrak{c}_i : I$ is either a complete intersection or a geometric $(s-1)$-residual intersection of I. Therefore $\bar{R}$ is a Cohen–Macaulay ring. Furthermore, by [36, 1.7(a)] the ideal $\mathfrak{c}_i : I$ has height $s-1$ (see also [22, proof of 3.1]). Thus any of its associated primes p has height $s-1$ and hence cannot contain I, since $\mathfrak{c}_i : I$ is a geometric $(s-1)$-residual intersection of I when $s \geq \operatorname{ht} I + 1$. Therefore $(\mathfrak{c}_i : I)_p = (\mathfrak{c}_i)_p$ is a complete intersection, showing that $\bar{R}$ is generically Gorenstein. Again by Lemma 4.2, the ideals $(\mathfrak{c}_i, a_i) : I$ and $(\mathfrak{c}_i, b_i) : I$ are s-residual intersections of I. Since I satisfies G_s and AN^-_{s-1}, it follows from [36, 1.7] that $(\mathfrak{c}_i, a_i) : I$ and $(\mathfrak{c}_i, b_i) : I$ are unmixed of height s and that the homogeneous elements $\overline{a_i}$ and $\overline{b_i}$ are $\bar{R}$-regular with $\overline{(\mathfrak{c}_i, a_i) : I} = (\overline{a_i}) : \bar{I}$ and $\overline{(\mathfrak{c}_i, b_i) : I} = (\overline{b_i}) : \bar{I}$ (see also [22, proof of 3.1]). The last fact gives $\bar{R}$-isomorphisms that are homogeneous up to a degree shift, $\overline{(\mathfrak{c}_i, a_i) : I} \cong \overline{b_i}[\overline{(\mathfrak{c}_i, a_i) : I}] = (\overline{a_i b_i}) : \bar{I} = \overline{a_i}[\overline{(\mathfrak{c}_i, b_i) : I}] \cong \overline{(\mathfrak{c}_i, b_i) : I}$. Now an application of Proposition 4.1 shows that $(\mathfrak{c}_i, a_i) : I$ and $(\mathfrak{c}_i, b_i) : I$ are obtained from each other by a homogeneous basic Gorenstein double link. □

Corollary 4.4. *Let k be an infinite field and R a Gorenstein ring that is either local with residue field k or a standard graded k-algebra. Let I be a homogeneous complete intersection R-ideal, and let J be a homogeneous s-residual intersection of I. Then J can be linked to a complete intersection by a sequence of s homogeneous basic Gorenstein double links.*

Proof. According to [22, 3.1], the complete intersection I satisfies AN^-_{s-1}. Hence in light of Theorem 4.3 it suffices to show that I has a homogeneous s-residual intersection K that is a complete intersection. Write $g = \operatorname{ht} I$ and notice that $\dim R \geq s \geq g > 0$. Let $x_1, \ldots, x_g$ be a homogeneous R-regular sequence generating I and extend it to a homogeneous R-regular sequence $x_1, \ldots, x_s$. For $\mathfrak{b} = (x_1, \ldots, x_{g-1}) + x_g(x_g, \ldots, x_s)$, take $K = \mathfrak{b} : I$ and observe that $K = (x_1, \ldots, x_s)$. □

Proposition 4.5. *Let R be a local Gorenstein ring, I a licci R-ideal, and $J = \mathfrak{a} : I$ an s-residual intersection of I. Let $y_1, \ldots, y_t$ be a sequence of elements in R that is regular on R and on R/I, and write $^-$ for images in the ring $\bar{R} = R/(y_1, \ldots, y_t)$. Then $y_1, \ldots, y_t$ form a regular sequence on R/J if and only if* $\operatorname{ht}(\bar{\mathfrak{a}} : \bar{I}) \geq s$. *In this case, $\bar{J} = \bar{\mathfrak{a}} : \bar{I}$ is an s-residual intersection of $\bar{I}$.*

Proof. By [26, 5.3], the ring R/J is Cohen–Macaulay and $\operatorname{ht} J = s$. Thus $y_1, \ldots, y_t$ form a regular sequence on R/J if and only if $\operatorname{ht} \bar{J} \geq s$. On the other hand, $\operatorname{ht} \bar{J} = \operatorname{ht}(\bar{\mathfrak{a}} : \bar{I})$ according to [26, 4.1], because $y_1, \ldots, y_t$ form a regular sequence on R/I.

If $\operatorname{ht}(\bar{\mathfrak{a}} : \bar{I}) \geq s$ then $\bar{\mathfrak{a}} : \bar{I}$ is obviously an s-residual intersection of $\bar{I}$. It remains to show that $\bar{J} = \bar{\mathfrak{a}} : \bar{I}$. According to [26, proofs of 5.1 and 5.3], there exists a deformation $(\tilde{R}, \tilde{I})$ of (R, I) such that $\tilde{R}$ is a local Gorenstein ring, $\tilde{I}$ is a licci ideal satisfying G_{s+1}, and $\tilde{I}$ has a geometric s-residual intersection $\tilde{J} = \tilde{\mathfrak{a}} : \tilde{I}$ with $\tilde{\mathfrak{a}} R = \mathfrak{a}$. Since this residual intersection is geometric, applying [26, 4.7] via [26, 4.2(i) and 5.3] gives $\tilde{J} R = J$ and $\tilde{J} \bar{R} = (\tilde{\mathfrak{a}} \bar{R}) : (\tilde{I} \bar{R})$. But obviously $J \bar{R} = \bar{J}$ and $(\tilde{\mathfrak{a}} \bar{R}) : (\tilde{I} \bar{R}) = \bar{\mathfrak{a}} : \bar{I}$. □

We are now ready to prove our main result.

THEOREM 4.6. *In a local Gorenstein ring with infinite residue field, every residual intersection of a licci ideal is strictly glicci.*

Theorem 4.6 is a consequence of the next, more precise statement. A major source of technical complications in proving this statement stems from the fact that the sequence $y_1, \dots, y_t$ is *not* required to be regular modulo the ideal I. Allowing this level of generality, though, is essential for the applications in the next section—in particular for Example 5.4, where the sequence is contained in the ideal!

THEOREM 4.7. *Let R be a local Gorenstein ring with infinite residue field, I an R-ideal, and J a residual intersection of I. Let $y_1, \dots, y_t$ be a sequence of $t \geq 0$ elements that is regular on R and on R/J, and write $^-$ for images in the ring $\bar{R} = R/(y_1, \dots, y_t)$. If I is licci in R, then $\bar{J}$ can be linked to a complete intersection by a sequence of basic Gorenstein double links in $\bar{R}$.*

Proof. Assume that J is an s-residual intersection of I and write g for the height of I. Notice that $g > 0$. Because $\dim R \geq s + t \geq g + t$, there exists a complete intersection R-ideal H of height g such that $y_1, \dots, y_t$ form a regular sequence on R/H. Since H is directly linked to itself and in the linkage class of any other complete intersection of height g, it follows that I and H are in the same even linkage class. Thus, by Theorem 3.4, the ideal J is an s-residual intersection of a licci ideal I'' such that $y_1, \dots, y_t$ form a regular sequence on R/I''. Changing notation, we write $I = I''$. Now the $\bar{R}$-ideal $\bar{J}$ is again an s-residual intersection of $\bar{I}$ according to Proposition 4.5, and $\bar{I}$ is still licci by [35, 1.6]. Thus we may replace R by $\bar{R}$, and it then suffices to prove the claim for J in place of $\bar{J}$. After another application of Theorem 3.4 we may further assume that I satisfies G_s.

Thus, for I a licci R-ideal satisfying G_s, we need to show that any s-residual intersection J of I can be linked to a complete intersection by a sequence of basic Gorenstein double links. As I is in the even linkage class of a complete intersection, it can be linked to a complete intersection by a sequence of, say n, tight double links according to Remark 2.3. We proceed by induction on $n \geq 0$. For $n = 0$ the assertion follows from Corollary 4.4. If $n > 0$ we use Theorem 3.5. By that theorem, there exist an s-residual intersection $K = \mathfrak{b} : I$ of I and an s-minimal tight double link $I \sim I' \sim I''$ of I along $\mathfrak{b}$ such that I'' satisfies G_s and is linked to a complete intersection by a sequence of at most $n - 1$ tight double links. According to Discussion 2.2(1), the ideal K is also an s-residual intersection of I'', and it therefore has the asserted property by our induction hypothesis. On the other hand, since I is licci and G_s, [21, 1.11] and [22, 3.1] show that I satisfies AN^-_{s-1}. Thus, by Theorem 4.3, J and K are obtained from each other via a sequence of basic Gorenstein double links. $\square$

5. Examples

In this section we apply Theorem 4.7 to deduce that various classes of ideals are glicci. In fact, these ideals are strictly glicci and the Gorenstein links leading to a complete intersection can be chosen to be basic Gorenstein double links. The ideals we consider are obtained by specializing or deforming residual intersections of complete intersections, of perfect ideals of height 2, or of perfect Gorenstein ideals of height 3. We also study ideals defining associated graded rings of certain reduction ideals. In what follows, I_t always denotes the ideal generated by the $t \times t$ minors of a matrix.

COROLLARY 5.1. *Let R be a local Gorenstein ring with infinite residue field*; *let φ be a $1 \times g$ matrix and ψ a $g \times s$ matrix with entries in R, where $g \geq 1 \leq s$; and write $J = I_1(\varphi\psi) + I_g(\psi)$. If J is a proper ideal of height at least s, then J is strictly glicci.*

Proof. We may assume that $s \geq g$, for otherwise $J = I_1(\varphi\psi)$ is an s-generated ideal of height at least s and hence even a complete intersection. Write $\mathfrak{m}$ for the maximal ideal of R. Let $\Phi = (x_i)$ and $\Psi = (y_{ij})$ be matrices of variables having sizes $1 \times g$ and $g \times s$, respectively. Furthermore, write $\underline{y}$ for the sequence consisting of the entries of the two matrices $\Phi - \varphi$ and $\Psi - \psi$. In the localized polynomial ring $\tilde{R} = R[\{x_i\}, \{y_{ij}\}]_{(\mathfrak{m}, \underline{y})}$ consider the ideals $\tilde{I} = I_1(\Phi)$ and $\tilde{J} = I_1(\Phi\Psi) + I_g(\Psi)$. Notice that $\underline{y}$ forms a regular sequence on the local Gorenstein ring $\tilde{R}$ and that we may identify $R = \tilde{R}/(\underline{y})$. With this identification, one has $J = \tilde{J}R$.

If $\tilde{I} = \tilde{R}$ then $\tilde{J}$ is generated by s elements. In this case J is a complete intersection. We may therefore assume that $\tilde{I}$ is a proper ideal and thus a complete intersection. Now [26, proof of 3.4] shows that $\tilde{J}$ is an s-residual intersection of the complete intersection $\tilde{I}$, and then [22, 3.1] or [26, 5.3] implies that $\tilde{R}/\tilde{J}$ is Cohen–Macaulay with $\operatorname{ht} \tilde{J} = s$. As $\operatorname{ht} \tilde{J}R = \operatorname{ht} J \geq s = \operatorname{ht} \tilde{J}$, we deduce that $\underline{y}$ forms a regular sequence on $\tilde{R}/\tilde{J}$. By Theorem 4.7 the ideal $\tilde{J}$ is strictly glicci, hence so is J. □

COROLLARY 5.2. *Let R be a local Gorenstein ring with infinite residue field*; *let φ be a matrix with entries in R of size $m \times n$, where $1 \leq m \leq n$; and write $J = I_m(\varphi)$. If J is a proper ideal of height at least $n - m + 1$, then J is strictly glicci.*

Proof. We may assume that $m \geq 2$. Proceeding as in the previous proof, we write $\mathfrak{m}$ for the maximal ideal of R, $\Phi = (x_{ij})$ for an $m \times n$ matrix of variables, and $\underline{y}$ for the sequence consisting of the entries of $\Phi - \varphi$. In the ring $\tilde{R} = R[\{x_{ij}\}]_{(\mathfrak{m}, \underline{y})}$ we define the ideal $\tilde{J} = I_m(\Phi)$. Once again identifying $R = \tilde{R}/(\underline{y})$, we obtain $J = \tilde{J}R$.

Now let Ψ be the $m \times (m-1)$ matrix consisting of the first $m-1$ columns of Φ, and write $\tilde{I} = I_{m-1}(\Psi)$. If $\tilde{I} = \tilde{R}$, then $\tilde{J}$ is generated by $n - m + 1$ elements and hence J is a complete intersection. Thus we may assume that $\tilde{I} \neq \tilde{R}$, in which case this ideal is perfect of height 2 and hence licci according to [1; 2, 11] or [3, 3.2(b)]. Furthermore, [22, proof of 4.1] shows that $\tilde{J}$ is a residual intersection of $\tilde{I}$. Thus $\tilde{J}$ is strictly glicci by Theorem 4.7.

On the other hand, $\tilde{R}/\tilde{J}$ is Cohen–Macaulay with $\operatorname{ht} \tilde{J} = n - m + 1$, as is classically known by [10]. Since $\operatorname{ht} \tilde{J}R = \operatorname{ht} J \geq n - m + 1 = \operatorname{ht} \tilde{J}$ it now follows, again, that $\underline{y}$ forms a regular sequence on $\tilde{R}/\tilde{J}$. Thus $J = \tilde{J}R$ is strictly glicci as well. □

Corollary 5.3. *Let R be a local Gorenstein ring with infinite residue field, and let φ be an alternating $m \times m$ matrix with entries in R having a zero block of size $s \times s$ in its lower right-hand corner, where $m \geq s \geq 1$. Let J be the R-ideal generated by the Pfaffians of all sizes that involve the first $m - s$ rows and columns of φ. If J is a proper ideal of height at least s, then J is strictly glicci.*

Proof. As before, we "deform to the generic case"—replacing R, φ, J by their generic versions $\tilde{R}$, Φ, $\tilde{J}$ and making the identifications $R = \tilde{R}/(\underline{y})$ and $J = \tilde{J}R$.

In addition, define $t = m - s$ if $m - s$ is odd and $t = m - s + 1$ if $m - s$ is even. Notice that $1 \leq t \leq m$. Let Ψ be the $t \times t$ submatrix appearing in the upper left-hand corner of Φ, and write $\tilde{I}$ for the ideal generated by the $(t-1) \times (t-1)$ Pfaffians of this matrix Ψ.

From [29, 7.3(a)] we know that $\tilde{R}/\tilde{J}$ is Cohen–Macaulay with ht $\tilde{J} = s$. Hence $\underline{y}$ forms a regular sequence on $\tilde{R}/\tilde{J}$. Furthermore, [29, 8.9(b)] gives $\tilde{J} = \tilde{\mathcal{A}} : \tilde{I}$ for some s-generated ideal $\tilde{\mathcal{A}}$ contained in $\tilde{I}$. Thus, if $\tilde{I} = \tilde{R}$ then $\tilde{J}$ is generated by s elements, forcing J to be a complete intersection. We may therefore assume that $\tilde{I} \neq \tilde{R}$, in which case $\tilde{J}$ is an s-residual intersection of $\tilde{I}$. Furthermore, $\tilde{I}$ is a perfect Gorenstein ideal of height 3, as shown in [4, 2.1], and hence is licci by [38, proof of Theorem]. Once again appealing to Theorem 4.7, we deduce that $\tilde{J}$ and hence $J = \tilde{J}R$ are strictly glicci. □

Let R be a Noetherian local ring of dimension d with infinite residue field k, and let I be a proper R-ideal. An R-ideal H contained in I is called a *reduction* of I if $I^{n+1} = HI^n$ for some $n \geq 0$ or, equivalently, for $n \gg 0$. Any I admits a reduction generated by d elements—in fact, a reduction with minimal number of generators $\dim \operatorname{gr}_I(R) \otimes_R k \leq d$.

Corollary 5.4. *Let R be a local Gorenstein ring with infinite residue field, I a licci R-ideal satisfying G_s, and H a reduction of I generated by s elements. Consider the polynomial ring $S = R[x_1, \dots, x_s]$ with homogeneous maximal ideal $\mathcal{M}$, and let $\operatorname{gr}_H(R) \cong S/J$ be a homogeneous presentation of the associated graded ring of H. Then the $S_{\mathcal{M}}$-ideal $J_{\mathcal{M}}$ is strictly glicci.*

Proof. Let $R[Ht, t^{-1}]$ be the extended Rees algebra of H, considered as a graded R-subalgebra of the Laurent polynomial ring $R[t, t^{-1}]$. Notice that t^{-1} is a nonzerodivisor on this algebra and $\operatorname{gr}_H(R) \cong R[Ht, t^{-1}]/(t^{-1})$. Write $\tilde{S} = S[u] = R[x_1, \dots, x_s, u]$ for the polynomial ring in $s + 1$ variables, where u is given degree -1, and write $\widetilde{\mathcal{M}}$ for its maximal ideal $(\mathcal{M}, u)$. Mapping u to t^{-1}, one can lift the given presentation of the associated graded ring to a homogeneous presentation $R[Ht, t^{-1}] \cong \tilde{S}/\tilde{J}$ of the extended Rees algebra. Now $(S_{\mathcal{M}}, J_{\mathcal{M}})$ is a specialization of $(\tilde{S}_{\widetilde{\mathcal{M}}}, \tilde{J}_{\widetilde{\mathcal{M}}})$ in the sense of Definition 2.5(d).

The ideal I, being licci, is strongly Cohen–Macaulay by [21, 2.11]. Furthermore, it has the G_s property by assumption. So according to [19, proof of 5.1], the ideal I satisfies the depth conditions required in [37, 2.1]. The proof of the latter result, and in particular the equality [37, 2.2], show that $\tilde{J}_{\widetilde{\mathcal{M}}}$ is a residual intersection of the $\tilde{S}_{\tilde{\mathcal{M}}}$-ideal (I, u). As (I, u) is licci, Theorem 4.7 implies that $\tilde{J}_{\tilde{\mathcal{M}}}$ is strictly glicci, and hence so is $J_{\mathcal{M}}$. □

If, in the corollary, I satisfies G_{d+1} for $d = \dim R$, then one can take H to be I and so J becomes the ideal defining the associated graded ring of I itself. From [21, 1.11] and [19, 9.1] it is known that $S_{\mathcal{M}}/J_{\mathcal{M}}$ is Gorenstein in this case, but even so the glicci property of $J_{\mathcal{M}}$ is not obvious; see [7, 7.1].

6. Residual Intersections of Licci Ideals: The Graded Case

We now turn to homogeneous linkage and homogeneous residual intersection, which include the global versions of these constructions considered in projective geometry. Our results in the graded context are considerably weaker because we can no longer use the work of Section 3, which allowed us to "prepare" an ideal I along its linkage class without changing a given residual intersection. On the other hand, this preparatory work is not needed for the more structured examples in Section 5. Thus one obtains, for instance, a graded version of Corollary 5.2 (about ideals of maximal minors) that recovers the theorem of Kleppe, Migliore, Miró-Roig, Nagel, and Peterson [28, 3.6] mentioned in the Introduction. Other results pertaining to homogeneous Gorenstein linkage are Theorem 4.3 and Corollary 4.4. In this section we are able to treat the graded case of residual intersections whose height exceeds the height of I by at most 1.

THEOREM 6.1. *Let R be a standard graded Gorenstein algebra over an infinite field, I a homogeneous R-ideal of height g, and J a homogeneous $(g+1)$-residual intersection of I. If I can be linked to a complete intersection by a sequence of homogeneous geometric links, then J can be linked to a complete intersection by a sequence of homogeneous basic Gorenstein double links.*

Proof. We write $\mathfrak{m}$ for the homogeneous maximal ideal of R. Lemma 4.2, for instance, shows that I can be linked to a complete intersection by a sequence of, say n, homogeneous tight double links, with all links involved geometric. We prove the theorem by induction on $n \geq 0$. For $n = 0$ the assertion follows from Corollary 4.4. If $n > 0$ consider the first double link $I \sim I' \sim I''$ in the chosen sequence of tight double links. Let $b_1, \dots, b_g$ be a homogeneous regular sequence defining the link $I \sim I'$. Since this link is geometric, there exists a homogeneous element b_{g+1} in $\mathfrak{m}I$ that is regular on R/I'. Set $\mathfrak{b} = (b_1, \dots, b_{g+1})$ and $K = \mathfrak{b} : I$. Notice that the ideal K is proper and contains (I', b_{g+1}), which has height $g+1$. Therefore K is a homogeneous $(g+1)$-residual intersection of I. Since I admits a geometric link, it necessarily satisfies G_{g+1}, and since I is licci, it has the property AN_g^-. Now we can use Theorem 4.3 to deduce that J and K are linked by a sequence of homogeneous basic Gorenstein double links.

On the other hand, the definition of $\mathfrak{b}$ gives that $I \sim I' \sim I''$ is a homogeneous $(g+1)$-minimal tight double link of I along $\mathfrak{b}$. Hence, according to Discussion 2.2(1), the ideal $K = \mathfrak{b} : I$ is also a homogeneous $(g+1)$-residual intersection of I''. Thus our induction hypothesis implies that K can be linked to a complete intersection by a sequence of homogeneous basic Gorenstein double links. □

7. Local Linkage versus Homogeneous Linkage

Algebraic geometry has seen a fruitful interplay between local and global theories. A good example is the theory of local cohomology, which was modeled after sheaf cohomology on projective varieties and allows for a local duality theorem, a cohomological characterization of depth and dimension, and estimates on the arithmetic rank of ideals, for instance. When looking at linkage theory, we find a considerable literature about linkage of projective varieties, first defined using complete intersections and later generalized to Gorenstein linkage. There is also an extensive literature on linkage in local rings, in particular on properties of licci ideals. These two branches of study, the global or graded and the local, have progressed more or less independently. So we thought it timely to make a comparison of the two theories.

There are two ways to make such a comparison. Assume that two subschemes V and V' of $\mathbb{P}^n_k$ are linked by a complete intersection subscheme or, equivalently, that their saturated ideals I_V and $I_{V'}$ are homogenously linked. First, if P is any point of the intersection $V \cap V'$, then the ideals $\mathcal{I}_{V,P}$ and $\mathcal{I}_{V',P}$ in the local ring $\mathcal{O}_{\mathbb{P}^n,P}$ are also linked. Thus a global linkage gives rise to local linkages at each point of the intersection. Second, one can also consider the cones over V and V' in $\mathbb{A}^{n+1}_k$ and then localize at the origin. The localizations of the ideals I_V and $I_{V'}$ will be linked in the local ring $\mathcal{O}_{\mathbb{A}^{n+1},0}$ of $\mathbb{A}^{n+1}_k$ at the origin. In this case, the global linkage induces a local linkage of the cones at the vertex.

In what follows we assume that k is an algebraically closed field.

Example 7.1. The first comparison already allows for an interesting example. One knows from Rao's work that any curve (meaning a 1-dimensional generic complete intersection subscheme with no associated points) in $\mathbb{P}^3_k$ is in the same linkage class as a smooth irreducible curve [33, 2.6 and 2.8]. However, in $\mathbb{P}^4_k$ we can give an example of an integral curve that is not in the linkage class of a smooth curve. Indeed, take any integral curve $\mathcal{C} \subset \mathbb{P}^4_k$ having a 4-fold ordinary multiple point with linearly independent tangent directions. This can be achieved, for example, by a suitable morphism from $\mathbb{P}^1_k$ to $\mathbb{P}^4_k$ collapsing four distinct points into one. The curve $\mathcal{C}$ cannot be linked by any sequence of links to a smooth curve—in fact, not even to a curve that is locally a complete intersection—for otherwise $\mathcal{C}$ would have to be licci locally at its singularities. But this is not the case according to [24, 6.18], for instance.

We now come to our main questions, asking about a converse to the second local–global comparison.

Question 7.2. Let V and V' be subschemes of $\mathbb{P}^n_k$, and assume that the localizations of the ideals I_V and $I_{V'}$ belong to the same linkage class in the local ring $\mathcal{O}_{\mathbb{A}^{n+1},0}$. Does it follow that V and V' are in the same linkage class in $\mathbb{P}^n_k$?

Question 7.3. Let V be a subscheme of $\mathbb{P}^n_k$, and suppose that the localization of the ideal I_V is licci in $\mathcal{O}_{\mathbb{A}^{n+1},0}$. Does this imply that V is licci in $\mathbb{P}^n_k$?

Clearly Question 7.3 is a special case of Question 7.2. We will give a negative answer to Question 7.2, even in codimension 2. Our counterexample is based on a well-known theorem by Rao and a local analogue thereof. On the other hand, Question 7.3 has an affirmative answer in codimension 2, where the licci property is equivalent to arithmetic Cohen–Macaulayness; see [32, 3.2]. But this question remains open in codimension ≥ 3. Our method of answering Question 7.2 does not apply here because Rao's theorem is not available beyond codimension 2.

We begin by stating a local version of Rao's theorem. We only sketch the proof, since the result is essentially known.

THEOREM 7.4. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension 4 containing an infinite field, and let I and K be two unmixed R-ideals of height 2. Then I and K are in the same even linkage class if and only if the local cohomology modules $H^1_{\mathfrak{m}}(R/I)$ and $H^1_{\mathfrak{m}}(R/K)$ are R-isomorphic, and I and K are in the same odd linkage class if and only if the R-modules $H^1_{\mathfrak{m}}(R/I)$ and $H^1_{\mathfrak{m}}(R/K)$ are Matlis dual to each other.*

Proof. To prove the forward implication it suffices to show that, if I and K are directly linked, then $H^1_{\mathfrak{m}}(R/I)$ and $H^1_{\mathfrak{m}}(R/K)$ are Matlis dual to each other. But this follows from [34, proof of 3.3], for instance. Alternatively, let a, b be a regular sequence defining the link $I \sim K$ and let $\bar{}$ denote images in the ring $\bar{R} = R/(a)$. Now $\bar{I} = (\bar{b}) :_{\bar{R}} \bar{K} \cong \operatorname{Hom}_{\bar{R}}(\bar{K}, \bar{R})$. But then [15, 1.13] implies that $H^1_{\mathfrak{m}}(R/I) \cong H^2_{\bar{\mathfrak{m}}}(\bar{I})$ and $H^1_{\mathfrak{m}}(R/K) \cong H^2_{\bar{\mathfrak{m}}}(\bar{K})$ are Matlis dual to each other as modules over $\bar{R}$ and hence over R.

To show the reverse implication, we need only prove that if $H^1_{\mathfrak{m}}(R/I) \cong H^1_{\mathfrak{m}}(R/K)$ then I and K are in the same even linkage class. Let E and G be first syzygy modules of I and K, respectively. We have $H^3_{\mathfrak{m}}(E) \cong H^3_{\mathfrak{m}}(G)$. Now it follows as in [17, proof of 4.2] that E and G are stably isomorphic. Indeed, completing (using local duality) and then descending to R, one sees that $\operatorname{Ext}^1_R(E, R) \cong \operatorname{Ext}^1_R(G, R)$. On the other hand, since E and G have projective dimension ≤ 1, it follows that $\operatorname{Hom}_R(E, R)$ and $\operatorname{Hom}_R(G, R)$ are second syzygy modules of $\operatorname{Ext}^1_R(E, R) \cong \operatorname{Ext}^1_R(G, R)$. Therefore $\operatorname{Hom}_R(E, R)$ and $\operatorname{Hom}_R(G, R)$ are stably isomorphic, and hence the modules E and G are because they are reflexive.

Finally, since E and G are stably isomorphic, it follows (as shown in [31, 6.8]) that I and K are in the same even linkage class. □

We are now ready to give our counterexample to Question 7.2, two suitable curves in $\mathbb{P}^3_k$. The example is based on Rao's theorem, which states that two such curves are in the same linkage class if and only if their Rao modules are isomorphic up to shifts and k-duals [33, 2.3 and 2.8]. Here the *Rao module* of a curve $\mathcal{C} \subset \mathbb{P}^n_k$ is the graded module $\bigoplus_i H^1(\mathbb{P}^n_k, \mathcal{I}_{\mathcal{C}}(i))$ regarded as a module over the polynomial ring $k[x_0, \ldots, x_n]$.

EXAMPLE 7.5. Consider the two graded modules $M = k \oplus k$ and $M' = k \oplus k(-1)$ over the polynomial ring $S = k[x_0, \ldots, x_3]$. By [33, 2.6] there exist smooth irreducible curves $\mathcal{C}$ and $\mathcal{C}'$ in $\mathbb{P}^3_k$ whose Rao modules are suitable shifts of M and M', respectively. These two Rao modules cannot be isomorphic as graded S-modules, even after shifting or dualizing into k. Thus [33, 2.3] shows that $\mathcal{C}$ and $\mathcal{C}'$ belong to different linkage classes in $\mathbb{P}^3_k$.

On the other hand, writing I and I' for the defining ideals of the cones over $\mathcal{C}$ and $\mathcal{C}'$, localized at the vertex, we obtain two unmixed ideals of height 2 in the regular local ring $R = \mathcal{O}_{\mathbb{A}^4,0} = k[x_0,\dots,x_3]_{(x_0,\dots,x_3)}$. One has isomorphisms $H^1_{\mathfrak{m}}(R/I) \cong k \oplus k \cong H^1_{\mathfrak{m}}(R/I')$ as modules over the local ring $(R,\mathfrak{m})$. Thus Theorem 7.4 implies that indeed I and I' are in the same (even) linkage class in R.

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