A 3-MANIFOLD ADMITTING A UNIQUE PERIODIC PL MAP

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1. INTRODUCTION

In this paper, we show that a family $\{M(n)\}$ of closed, aspherical 3-manifolds has the property that each $M(n)$ admits a unique PL involution. These 3-manifolds are of special interest, since P. E. Conner and F. Raymond [3] have shown that very few finite groups can act effectively on them. In particular, Z_2 is the only group that can act effectively on $M(1)$. Thus we obtain the following result.

THEOREM 1. *The closed, aspherical* 3-*manifold* M(1) *admits exactly one periodic* PL *map* (*up to conjugation*).

Let T^2 denote the 2-dimensional torus, that is

$$\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C}: |z_1| = |z_2| = 1\}.$$

If n is a positive odd integer, let $\Phi(n)$ denote the homeomorphism $T^2 \to T^2$ defined by

$$\Phi(n)(z_1, z_2) = (z_1^{n-1} z_2, z_1^n z_2).$$

Let R^1 denote the real line, and let $M(n) = (T^2 \times R^1)/\Phi(n)$ be the torus bundle over the circle obtained from $T^2 \times R^1$ by identification of (z_1, z_2, t) with $(\Phi(n)(z_1, z_2), t+1)$.

Denote the points of $M(n)$ by $[z_1, z_2, t]$. Each $M(n)$ admits a standard involution h_0 defined by

$$h_0([z_1, z_2, t]) = [g(z_1, z_2), t],$$

where $g(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$.

Let h be a PL involution of $M(n)$. We obtain the uniqueness of involutions on $M(n)$ by actually constructing an equivalence between h and h_0. Our first step is to obtain an invariant torus fiber T that meets the fixed-point set Fix(h) of h in exactly four points. Then we split $M(n)$ along T to obtain $T \times [0, 1]$. The involution h defines a product involution $g \times 1$ on $T \times [0, 1]$. If we let ψ denote the homeomorphism repairing the cut made along T, we may view $M(n)$ as $T \times R^1/\psi$. The homeomorphism $\Phi(n)$ is isotopic to a conjugate of ψ, say $\alpha\psi\alpha^{-1}$, where $g\alpha = \alpha g$. In Section 2 we show that this isotopy can be realized by one that commutes with g at each level. We use this equivariant isotopy to define an equivalence between $\alpha h \alpha^{-1}$ and h_0.

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2. FIBER ISOTOPIES

Let g denote a fixed involution on the compact surface X such that $\dim(\mathrm{Fix}(g)) \leq 0$. If Φ is a homeomorphism of X that commutes with g, then $g \times 1$ defines an involution h_Φ on the space $X \times R^1/\Phi$. We are interested in determining when the assumption that Φ is isotopic to ψ implies that h_Φ is conjugate to h_ψ.

Let $p\colon X \to X/g$ be the projection to the orbit space of g. A homeomorphism Φ of X is *fiber-preserving* if $p(x) = p(x')$ implies $p\Phi(x) = p\Phi(x')$. A *fiber isotopy* H_t is an isotopy for which each H_t is fiber-preserving.

The existence of fiber isotopies is considered in a more general setting in [1] and [8]. It follows from [1] that if X is different from the torus and the 2-sphere, then fiber-preserving maps that are isotopic are fiber-isotopic. In the case where X is T^2 and $\mathrm{Fix}(g) = \emptyset$, the corresponding result follows from [8]. We take up the case where X is T^2 and $\mathrm{Fix}(g)$ is 0-dimensional. Let b denote a branch point of the projection $p\colon X \to X/g$, that is, a point in $\mathrm{Fix}(g)$.

LEMMA 1. *If α and α' are two homotopic simple closed curves in T^2 such that $b \in \alpha \cap \alpha'$, then α and α' are homotopic relative to b.*

This is established by moving α into general position with respect to α' by an isotopy fixing b and then using an induction argument on the number of points in $\alpha \cap \alpha'$.

LEMMA 2. *Let Φ be a homeomorphism of $X = T^2$ such that $\Phi(b) = b$. If Φ is homotopic to 1_X, then Φ is isotopic to 1_X by an isotopy H that fixes b.*

Proof. We construct a homotopy $H\colon X \times [0, 1] \to X$ from Φ to 1_X as follows. On the ends we let $H_0 = \Phi$ and $H_1 = 1_X$. Choose two simple closed curves α and β in X such that $\alpha \cap \beta = b$ and $X - (\alpha \cup \beta)$ is an open 2-cell. Since α and β are homotopic to $\Phi(\alpha)$ and $\Phi(\beta)$, respectively, by homotopies fixing b, we can define H on $(\alpha \cup \beta) \times [0, 1]$, keeping b fixed. Now we can extend our map over the remaining open 3-cell. Hence H is a homotopy from Φ to 1_X that fixes b. It follows from [4] that we may assume H to be an isotopy fixing b.

LEMMA 3. *Let Φ be a fiber-preserving homeomorphism of $X = T^2$ such that $\Phi(b) = b$. If Φ is isotopic to 1_X, then Φ is fiber-isotopic to 1_X.*

Proof. The isotopy H from Lemma 2 can be adjusted slightly so that there exists an invariant closed-star neighborhood D about b; that is, $H_t(D) = D$ for $0 \leq t \leq 1$. Let

$$Y = T^2 - \mathrm{Int}(D) \qquad \text{and} \qquad G = H \mid Y \times [0, 1].$$

It follows from [1] that there exists a fiber isotopy G' from $\Phi \mid Y$ to 1_Y. By working on one simplex at a time, we can extend G' over $D \times [0, 1]$ to obtain a fiber isotopy between Φ and 1_X.

THEOREM 2. *Let Φ and ψ be isotopic homeomorphisms of X that commute with g. Then h_Φ is conjugate to h_ψ if either X is not the torus or 2-sphere, or X is the torus T^2 and g is fixed-point-free. If X is T^2 and $\mathrm{Fix}(g)$ is 0-dimensional, and if in addition Φ and ψ agree at one point of $\mathrm{Fix}(g)$, then h_Φ is conjugate to h_ψ.*

Proof. It follows from the discussion above that the given isotopic homeomorphisms Φ and ψ are fiber-isotopic. Thus there exists a fiber isotopy H_t from 1_X to $\psi\Phi^{-1}$. Define the homeomorphism $f\colon X \times R^1/\Phi \to X \times R^1/\psi$ by $f([x, t]) = [H_t(x), t]$. Now observe that $h_\Phi = f h_\psi f^{-1}$.

Remark. Consider the case where X is T^2 and $g(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$. Let Φ be isotopic to ψ, and let both homeomorphisms commute with g. We can always choose $r(z_1, z_2) = (\lambda^a z_1, \lambda^b z_2)$, where λ is a primitive second root of unity, such that the isotopic homeomorphisms Φ and ψr agree at the point (1, 1).

3. SOME PROPERTIES OF M(n)

In Section 4, we shall need some special properties of M(n) to show that M(n) admits only one involution. In the present section, we establish these properties.

First we consider the fundamental group of M(n). If we choose a presentation for $\pi_1(T^2) = \langle x, y: [x, y] = 1 \rangle$, then a presentation for $G_n = \pi_1(M(n))$ is given by

$$G_n = \langle x, y, t: [x, y] = 1,\ txt^{-1} = x^{n-1}y^n,\ tyt^{-1} = xy \rangle .$$

LEMMA 4. *The group* G_n *has a trivial center.*

Proof. Since $\Phi(n)_*$ has infinite order and fixes no element of $\pi_1(T^2)$, this follows from Section 3 of [7].

LEMMA 5. *The subgroup* K_n *of* G_n *generated by the elements* $\{x, y\}$ *is invariant under any automorphism of* G_n.

Proof. Let $H_n = G_n/[G_n, G_n]$, and observe that H_n has the presentation

$$\langle y, t: [y, t] = 1,\ y^n = 1 \rangle \cong Z \oplus Z_n .$$

Since K_n is the kernel of the natural homomorphism $G_n \to H_n/\mathrm{Tor}(H_n)$, the subgroup K_n is invariant under each automorphism of G_n.

LEMMA 6. M(n) *has a unique two-sheeted covering space* $\tilde{M}(n)$.

Proof. Let $\tilde{M}(n) = T^2 \times R^1/\Phi(n)^2$, and define the two-sheeted covering projection p: $\tilde{M}(n) \to M(n)$ by $p([x, t]) = [x, 2t]$. To prove that $\tilde{M}(n)$ is unique, we recall that n is an odd integer. Since each homomorphism $G_n \to Z_2$ can be factored through H_n, it then follows that there exists a unique homomorphism onto Z_2. The two-sheeted covering spaces of M(n) are classified by the homomorphism $G_n \to Z_2$; therefore Lemma 6 follows.

For each element g of G_n, let C(g) denote its centralizer in G_n.

LEMMA 7. *If* $C(t^k)$ *contains a nontrivial element of* K_n, *then* $k = 0$.

Proof. Clearly, C(t) contains only the trivial element of K_n. Suppose that $C(t^k)$, for some $k > 1$, contains an element $x^s y^t$ of K_n. Then the automorphism $(\Phi(n)_*)^k$ fixes this element $x^s y^t$. Since the automorphism $\Phi(n)_*$ is represented by the matrix $\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$, the automorphism $(\Phi(n)_*)^k$ is represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where each entry is strictly positive and $d \geq 2$. By an easy calculation we can show that these conditions on the entries of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, together with the equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}$, imply that $s = t = 0$. Therefore $C(t^k)$, for $k \neq 0$, contains only the trivial element of K_n.

LEMMA 8. *Let* S *be a nonseparating, two-sided torus in* M(n) *or* $\tilde{M}(n)$. *Then the closure of the complement of a regular neighborhood* U(S) *of* S *is homeomorphic to* $S \times [0, 1]$.

Proof. Consider $S \subset M(n)$. Since S is nonseparating, S is incompressible in M(n), and $\pi_1(S)$ is contained in K_n. Let $p: T^2 \times R^1 \to M(n)$ be the covering space corresponding to the subgroup K_n. Then $p^{-1}(S) = \bigcup_n \tau^n(\tilde{S})$, where $p \mid \tilde{S}$ is a homeomorphism onto S, and where τ generates the group of covering transformations of this covering. It follows from [2] that $\tilde{S}$ and $\tau(\tilde{S})$ are parallel in $T^2 \times R^1$. Thus $\overline{M(n) - U(S)}$ is homeomorphic to $S \times [0, 1]$. The argument for the case $S \subset \tilde{M}(n)$ is similar.

LEMMA 9. *The* 3-*manifold* M(n) *does not contain* (a) *a two-sided, nonseparating Klein bottle, or* (b) *a one-sided torus or Klein bottle carrying a nontrivial element of* K_n.

Proof. (a) Suppose S is a nonseparating, two-sided Klein bottle in M(n). Consider the orientable double-covering $p: \tilde{M}(n) \to M(n)$. The torus $p^{-1}(S)$ is nonseparating and two-sided in $\tilde{M}(n)$. Therefore $p^{-1}(\overline{M(n) - U(S)})$ is homeomorphic to $T^2 \times [0, 1]$. It follows that $\overline{M(n) - U(S)}$ is homeomorphic to $S \times [0, 1]$ and that $\pi_1(S)$ is a subgroup of G_n contained in K_n. However, since a Klein bottle cannot cover the torus, it is not possible for $\pi_1(S)$ to be a subgroup of K_n.

(b) Suppose S is a one-sided torus or Klein bottle in M(n) that carries a nontrivial element $\{u\}$ of K_n. Observe that $\partial U(S) \times [0, 1]$ doublecovers U(S) and induces a double covering of M(n), namely $p: \tilde{M}(n) \to M(n)$. Because of the orientability of $\tilde{M}(n)$, the boundary $\partial U(S)$ is a torus.

Since a compressible, two-sided torus in an irreducible 3-manifold must bound a disk bundle over S^1, the torus $\partial U(S)$ is incompressible in M(n). Together with u, an element of the form vt^k with $v \in K_n$ generates $\pi_1(S)$ in G_n. Hence t^{2k} commutes with u, so that $k = 0$ by Lemma 7. Since $\pi_1(S)$ is a subgroup of K_n, the surface S is not a Klein bottle. Therefore S is a torus and is covered by an incompressible, one-sided torus in $T^2 \times R^1$. This is impossible, since every incompressible torus in $T^2 \times R^1$ is parallel to $T^2 \times \{0\}$.

LEMMA 10. *Let* $\psi: T^2 \to T^2$ *be a homeomorphism such that* $T^2 \times R^1/\psi$ *is homeomorphic to* M(n). *Then* ψ *is isotopic to a conjugate of* $\Phi(n)$.

Proof. Let $f: M(n) \to T^2 \times R^1/\psi$ be a homeomorphism preserving the base points. A presentation for $\pi_1(T^2 \times R^1/\psi)$ is given by

$$G_n' = \langle x', y', t' : [x', y'] = 1,\ t'x't'^{-1} = \psi_*(x'),\ t'y't'^{-1} = \psi_*(y') \rangle .$$

We have the automorphism $f_*: G_n \to G_n'$. By Lemma 5, the subgroup $f_*(K_n)$ of G_n' is generated by $\{x', y'\}$. Suppose $f_*(t) = ut'$, where $u \in f_*(K_n)$. Then for $w \in K_n$ we have the relations

$$f_*\Phi(n)_*(w) = f_*(twt)^{-1} = ut'f_*(w)(ut')^{-1} = \psi_*(f_*(w)).$$

That is, $\psi_* = f_*\Phi(n)_*f_*^{-1}$. Therefore ψ is isotopic to a homeomorphism that is conjugate to $\Phi(n)$.

4. INVOLUTIONS OF M(n)

We are now ready to consider our main result.

THEOREM 3. *For each odd integer* $n \geq 1$ *there exists a* PL *involution on* $M(n)$, *unique up to conjugation.*

The involution of $M(n)$ can be viewed in the following way. Let r_1 and r_2 denote the rotations on T^2 defined by

$$r_1(z_1, z_2) = (-z_1, z_2) \quad \text{and} \quad r_2(z_1, z_2) = (z_1, -z_2).$$

The involutions h_0, h_1, h_2, h_3 on $M(n)$, $T^2 \times R^1/\Phi(n)r_1$, $T^2 \times R^1/\Phi(n)r_2$, $T^2 \times R^1/\Phi(n)r_1 r_2$, respectively, are defined by $g \times 1$, where $g(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$. The involution h_0 is merely the standard involution on $M(n)$. Observe that these four involutions are all conjugates of h_0:

$$h_0 = r_2 h_1 r_2 = r_1 r_2 h_2 r_2 r_1 = r_1 h_3 r_1.$$

The next two lemmas are essential to the proof of Theorem 3.

LEMMA 11. *Let* h *be a* PL *involution on* $M(n)$. *Then there exists a nonseparating, two-sided torus* T *in* $M(n)$ *such that either* $h(T) \cap T = \emptyset$ *or* $h(T) = T$, *and* T *is in general position with respect to* Fix(h).

Proof. The proof follows that of Theorem B in [6] after the following observations are made. To follow the argument in [6], we must verify that a two-sided, nonseparating torus is retained after each construction. This becomes obvious if we make a cut along a simple closed curve in T bounding an innermost disk in h(T). If we are cutting along the boundary of an innermost annulus, it is always possible in M(n) to obtain a nonseparating surface (see Lemma 8). However, upon attaching two annuli together along their boundaries, it is possible in general to obtain either a torus or a Klein bottle carrying a generator of K_n. According to Lemma 9, among such surfaces, M(n) contains only two-sided tori. Therefore the reduction in [6] can be carried through to obtain the desired T.

We recall the following special case of Lemma 6.3 of [5].

LEMMA 12. *Let* $\alpha: T^2 \times [0, 1] \to T^2 \times [0, 1]$ *be an involution. Then there exist an involution* β *of* T^2 *and a product structure of* $T^2 \times [0, 1]$ *such that* $\alpha(x, t) = (\beta(x), t)$ *or* $\alpha(x, t) = (\beta(x), 1 - t)$, *for* $x \in T^2$ *and* $0 \leq t \leq 1$.

Proof of Theorem 3. Let h be an arbitrary involution on M(n). We first show that there exists an invariant torus T in M(n) meeting Fix(h). Let T be the torus obtained in Lemma 11. Suppose that $h(T) \cap T = \emptyset$. Let us view M(n) as $T \times R^1/\psi$ (for a suitable ψ), where $T = T \times 0$ and $h(T) = T \times 1/2$. By Lemma 12, we may assume that $h \mid T \times [0, 1/2]$ is given by $h(x, t) = (\beta(x), 1/2 - t)$ and $h \mid T \times [1/2, 1]$ is given by $h(x, t) = (\bar{\beta}(x), 3/2 - t)$, where β and $\bar{\beta}$ are involutions of T. It follows that $(x, 0)$ is identified with $(\bar{\beta}\beta^{-1}(x), 1)$ for each $x \in T^2$, that is, $\psi = \bar{\beta}\beta^{-1}$. The automorphism of $\pi_1(T^2)$ induced by $\bar{\beta}\beta^{-1}$ has period either two or four, which implies that $\pi_1(M(n))$ has a nontrivial center. Because we have observed that the center of $\pi_1(M(n))$ is in fact trivial, the torus T obtained in Lemma 11 must be invariant under h. Moreover, h does not interchange the sides of T. For if it did, there would be another torus T', parallel to and on one side of T, for which $h(T') \cap T' = \emptyset$.

It follows from Theorem 7.12 of [3] that $\mathrm{Fix}(h) \neq \emptyset$. Since the sides of T are not interchanged, we see from Lemma 12 that $\mathrm{Fix}(h)$ meets T.

Next we show that $\mathrm{Fix}(h)$ is one-dimensional. Suppose that $\mathrm{Fix}(h)$ contains a two-dimensional component F. The inclusion-induced homomorphism $\pi_1(F) \to \pi_1(M(n))$ is injective [5]. In view of Lemma 12, $\mathrm{Fix}(h)$ consists of two annuli attached along the boundaries in some fashion, where each boundary component of the annuli is a simple closed curve meeting T. By Lemma 9, the surface F must be either a torus fiber or a separating torus. If F is a fiber, then we can argue as before that $\pi_1(M(n))$ has a nontrivial center. If F is a separating torus, let A be the closure of one of the components of M - F. Then A is a retract of M(n), and $H_1(A; Z)$ is a direct summand of $H_1(M(n)) \cong Z \oplus Z_n$. Since A has a torus boundary, $H_1(A; Z)$ is infinite. Therefore F carries ut in G_n, for some $u \in K_n$. But $C(ut) = 1$ in G_n, which contradicts the fact that $\pi_1(F)$ is a subgroup of G_n. It follows that $\mathrm{Fix}(h)$ has no 2-dimensional components. More precisely, $\mathrm{Fix}(h)$ is a union of disjoint circles (this follows from Lemma 12).

Now consider the invariant torus fiber T meeting $\mathrm{Fix}(h)$ in four points. If we split M(n) apart along T, we obtain $T^2 \times [0, 1]$. This product structure may be chosen so that h induces the involution $g \times 1$ on $T^2 \times [0, 1]$.

Let ψ denote the homeomorphism repairing the cut made along T. Then $M(n) = T^2 \times R^1/\psi$. According to Lemma 10, there exists a homeomorphism $f: T^2 \to T^2$ such that $\Phi(n)$ is isotopic to $f\psi f^{-1}$. The map f is isotopic to a map $\bar{f}$ that is fiber-preserving with respect to $T^2 \to T^2/g$. Therefore $\Phi(n)$ is isotopic to the fiber-preserving map $\bar{f}\psi\bar{f}^{-1}$. Let h' be the involution on $T^2 \times R^1/\bar{f}\psi\bar{f}^{-1}$ defined by $g \times 1$. It follows from Theorem 2 and the Remark that h' is conjugate to one of the involutions h_0, h_1, h_2 or h_3. We complete the proof of Theorem 3 by observing that h' is conjugate to h. Let $f': T^2 \times R^1/\psi \to T^2 \times R^1/\bar{f}\psi\bar{f}^{-1}$ denote the homeomorphism defined by $\bar{f} \times 1$. Since $\bar{f}$ is fiber-preserving, it follows that $h' = f'hf'^{-1}$.

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