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A DETAILED ARGUMENT FOR THE POST-LINIAL THEOREMS[1]

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In 1946 at the Princeton Bicentennial, Tarski proposed problems about fragments of the propositional calculus. In 1948 Post and Linial gave the solutions to these problems. They showed that there exists a partial propositional calculus with an unsolvable decision problem, and that the problems of determining, of an arbitrary propositional calculus, whether or not it is complete and whether or not its axioms are independent are recursively unsolvable.[2] Only an abstract of their work has been published. Davis, in his book,[3] uses the methods of Post and Linial to demonstrate their results. In his brief argument, however, he reaches conclusions which are not immediately obvious.

This paper deals with the first two problems, the decision problem and that of completeness. It uses Davis' construction with a modified axiom set. The axioms were chosen to parallel a possible definition of proof in a semi-Thue system. The definition is not the usual one and will be given later. Perhaps the most crucial points in the paper are the technical definitions of validity. For a proof by mathematical induction to be successful these definitions had to have just the right degree of restrictiveness. The attempt to find such definitions was started by a suggestion of Professor William W. Boone that a validity argument might be fruitful.

Some introductory definitions should be given.

A *partial propositional calculus* is a system having $\sim$, $\supset$, [, and] as primitive symbols along with the propositional variables $p_1, q_1, r_1, p_2, q_2, r_2, p_3, \ldots$. Its well formed formulas are (1) a propositional variable, (2) $[A \supset B]$, where A and B are well formed formulas, and (3) $\sim A$, where A is a well formed formula. (In this paper the abbreviations and grouping conventions of Church[4] will be used). It has a finite set of axioms, all of which are tautologies, and its two rules of inference are modus ponens and substitution.

Since the axioms of a partial propositional calculus are tautologies, and the rules of inference preserve tautologies, it follows that all theorems of a partial propositional calculus are tautologies. A partial propositional calculus P is *complete* if every tautology is a theorem of P. Hence in a complete partial propositional calculus the set of theorems is identical to the set of tautologies. There is a mechanical way of determining whether a

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given well formed formula is a tautology or not. So the decision problem for a complete partial propositional calculus is recursively solvable.

Since all complete partial propositional calculi have the same theorems, it is reasonable to talk about *the complete propositional calculus* and its different formulations. One such formulation is given by the following three axioms.

$$p_1 \supset [q_1 \supset p_1]$$

$$[p_1 \supset [q_1 \supset r_1]] \supset \blacksquare [p_1 \supset q_1] \supset [p_1 \supset r_1]$$

$$[\sim q_1 \supset \sim p_1] \supset [p_1 \supset q_1]$$

A definition of a semi-Thue system can be found in a paper by Boone.[5] Using his notation, a semi-Thue system is specified by a finite alphabet Z, and a finite set of word pairs U.

$$Z\colon\ a_1, a_2, \ldots, a_n$$

$$U\colon\ A_1 \to B_1,\ A_2 \to B_2, \ldots, A_m \to B_m$$

A word is a finite string of symbols of Z, with possible repetitions, which may be empty. Boone defines $C \vdash D$, where C and D are words on Z, as the assertion that there exists a finite sequence of words, $C_1, C_2, \ldots, C_\ell$, such that C_1 is C, C_ℓ is D, and for each pair (C_i, C_{i+1}) C_i is XA_jY and C_{i+1} is XB_jY for some words X and Y and for some j, $1 \leqslant j \leqslant m$.

In this paper a different, but equivalent, concept of $C \vdash D$ is used. $C \vdash D$ if and only if there exists a finite sequence of statements $C_1 \vdash D_1$, $C_2 \vdash D_2, \ldots, C_\ell \vdash D_\ell$ such that C_ℓ is C and D_ℓ is D, and such that each statement $C_i \vdash D_i$ is justified by one of the following rules.

1. C_i is AC_j, D_i is AD_j for some j, $1 \leqslant j < i$, and for some word A.
2. C_i is C_jA, D_i is D_jA for some j, $1 \leqslant j < i$, and for some word A.
3. C_i is D_i.
4. C_i is A_j and D_i is B_j for some j, $1 \leqslant j \leqslant m$.
5. C_i is C_j, D_i is D_k, and D_j is C_k for some j and k, $1 \leqslant j < i$, $1 \leqslant k < i$.

A less explicit, but possibly clearer, summary of these rules follows:

1. If $G \vdash D$, then $AC \vdash AD$.
2. If $C \vdash D$, then $CA \vdash DA$.
3. $C \vdash C$.
4. If $C \to D$, then $C \vdash D$.
5. If $C \vdash E$ and $E \vdash D$, then $C \vdash D$.[6]

It has been shown that there exists a semi-Thue system σ_o such that Z_{σ_o} contains exactly two letters, all the words in the word pairs of U_{σ_o} are non-empty, and σ_o has a recursively unsolvable word problem.[7]

Theorem 1. *There exists a partial propositional calculus with a recursively unsolvable decision problem.*

This theorem is proved by constructing a partial propositional calculus P_σ from a semi-Thue system σ on two letters. P_σ and σ are related by a one-to-one mapping from the non-empty words of σ onto a subset of the well formed formulas of P_σ such that $C \vdash_\sigma D$ if and only if it is a theorem of P_σ that the well formed formula associated with C implies the well formed formula associated with D. If σ is a semi-Thue system on two letters such that U_σ contains no empty words and σ has a recursively unsolvable word problem, then P_σ must have a recursively unsolvable decision problem. The proof consists of constructing P_σ and showing that there is a mapping with the desired properties.

Let σ be defined by:

Z_σ: $1, b$

U_σ: $G_i \rightarrow \bar{G}_i, \quad i = 1, 2, \ldots, m.$

G_i and $\bar{G}_i$, $i = 1, 2, \ldots, m$, are non-empty words on Z_σ. $G_i \rightarrow \bar{G}_i$ is used, instead of $A_i \rightarrow B_i$, to conform to Davis' notation.

If W is a non-empty word of σ, then define W' to be the well formed formula of the partial propositional calculi, as follows:

$1'$ is $\sim\,\sim [\sim p_2 \supset \sim p_2]$,
b' is $\sim\,\sim\,\sim\,\sim [\sim p_2 \supset \sim p_2]$,
$(V1)'$ is $[V' \;\&\; 1']$, and
$(Vb)'$ is $[V' \;\&\; b']$, where $[A \;\&\; B]$ is an abbreviation for $\sim [A \supset \sim B]$.

For example, $(1b1)'$ is $[[1' \;\&\; b'] \;\&\; 1']$ or $[[\sim\sim[\sim p_2 \supset \sim p_2] \;\&\; \sim\sim\sim\sim[\sim p_2 \supset \sim p_2]] \;\&\; \sim\sim[\sim p_2 \supset \sim p_2]]$.

For any non-empty word W of σ, W' is well defined. W' is also a tautology, since $1'$ and b' are tautologies and the conjunction of tautologies is a tautology.

Now P_σ can be defined by the following set of axioms.

1. $[p_1 \;\&[q_1 \;\&\; r_1]] \supset [[p_1 \;\&\; q_1] \;\& r_1]$
2. $[[p_1 \;\&\; q_1] \;\&\; r_1] \supset [p_1 \;\&\; [q_1 \;\&\; r_1]]$
3. $[p_1 \supset q_1] \supset \blacksquare [r_1 \;\&\; p_1] \supset [r_1 \;\&\; q_1]$
4. $[p_1 \supset q_1] \supset \blacksquare [p_1 \;\& r_1] \supset [q_1 \;\&\; r_1]$
5. $p_1 \supset p_1$
6. $G_i' \supset \bar{G}_i' \quad i = 1, 2, \ldots, m$
7. $[p_1 \supset q_1] \supset \blacksquare [q_1 \supset r_1] \supset [p_1 \supset r_1]$

Notice that axiom 6 is actually m axioms, one for each pair of U_σ.

These axioms seem reasonable in a system which is to be closely related to σ. Axioms 1 and 2 have no counterparts in σ, but this is to be expected, since the letters of a word of σ are not grouped. Axioms 3-7, on the other hand, correspond respectively to rules 1-5 for deriving a statement $C \vdash_\sigma D$. By including these axioms, the rules of σ are built into P_σ.

It will be convenient to have some notation and terminology defined before going on. If X is a well formed formula of P_σ, then X is *regular* if and only if (1) X is $1'$, or X is b', or (2) X is of the form $[X_1 \;\& X_2]$, where X_1 and X_2 are regular well formed formulas. It should be noticed that the only propositional variable occurring in a regular well formed formula is p_2.

If X is regular, then $\langle X \rangle$ is the unique word of σ obtained by the following procedure:

(1) abbreviating X so that it contains only [,] &, $1'$, and b', (2) removing all occurrences of [,], and &, (3) replacing $1'$ by 1, and b' by b. Use induction on the number n of occurrences of $\supset$ in X to show that $<X>$ is unique. If $n = 1$, then, since it is regular, X must be either $\sim\sim[\sim p_2 \supset \sim p_2]$ or $\sim\sim\sim\sim[\sim p_2 \supset \sim p_2]$. That is, X is $1'$ or b'. In either case it is well defined. For the induction step, assume that $<Y>$ is a unique word of σ for all regular well formed formulas Y containing less than n $\supset$'s. Since X is regular, X is $[X_1 \& X_2]$, where X_1 and X_2 are regular. By the induction hypothesis $<X_1>$ and $<X_2>$ are unique words of σ. By an argument originally due to Kleene, the arrangement of [,], $\supset$, and $\sim$'s is unique.[8] Hence there is only one way in which X can be written, as a conjunction. So X_1 and X_2 are well defined, and hence $<X>$ is well defined, also. $<X>$ is $<X_1> <X_2>$.

Two regular well formed formulas, X and Y, of P_σ are *associates* if and only if $<X>$ is $<Y>$.

LEMMA 1. *If X and Y are associates, then* $\vdash_{P_\sigma} X \supset Y$ *and* $\vdash_{P_\sigma} Y \supset X$.

The proof is by strong induction on the number n of occurrences of $1'$ and b' in X.

If $n = 1$, then X is $1'$ or X is b'. Hence $<X>$ is 1 and $<Y>$ is 1, or $<X>$ is b and $<Y>$ is b. In either case, X is Y. $[p_1 \supset p_1]$ is an axiom of P_σ. Hence, by substitution, $\vdash_{P_\sigma} X \supset Y$ and $\vdash_{P_\sigma} Y \supset X$.

For the induction step, call the number of occurrences of $1'$ and b' in X the length of X and let ℓ_X = length of X. Since X and Y are associates $\ell_X = \ell_Y$. Assume that, if W_1 and W_2 are associates such that $\ell_{W_1} < \ell_X$, then $\vdash_{P_\sigma} W_1 \supset W_2$.

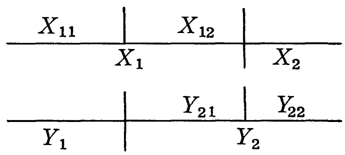
If $\ell_X > 1$, then X is $[X_1 \& X_2]$ and Y is $[Y_1 \& Y_2]$ for some regular formulas X_1, X_2, Y_1, and Y_2 of P_σ. There are two cases to consider, either $\ell_{X_1} = \ell_{Y_1}$ or $\ell_{X_1} \neq \ell_{Y_1}$.

Assume $\ell_{X_1} = \ell_{Y_1}$. Then $\ell_{X_2} = \ell_{Y_2}$. $<X_1>$ must be the first ℓ_{X_1} letters of $<X>$, and $<X_2>$ the last ℓ_{X_2} letters. Similarly for $<Y>$, $<Y_1>$, and $<Y_2>$. Since $<X>$ and $<Y>$ are the same, it follows that X_1 and Y_1 are associates and X_2 and Y_2 are associates. The rest of the proof for this case can be outlined:

$\vdash_{P_\sigma} X_1 \supset Y_1$	by ind. hyp.
$\vdash_{P_\sigma} [X_1 \& X_2] \supset [Y_1 \& X_2]$	by axiom 4
$\vdash_{P_\sigma} X_2 \supset Y_2$	by ind. hyp.
$\vdash_{P_\sigma} [Y_1 \& X_2] \supset [Y_1 \& Y_2]$	by axiom 3
$\vdash_{P_\sigma} [X_1 \& X_2] \supset [Y_1 \& Y_2]$	by axiom 7

i.e. $\vdash_{P_\sigma} X \supset Y$.

By the symmetry of this case we also have $\vdash_{P_\sigma} Y \supset X$. Next, assume $\ell_{X_1} \neq \ell_{Y_1}$, and without loss of generality assume $\ell_{X_1} = \ell_{Y_1} + k$. Let $[X_{11} \& X_{12}]$ be an associate of X_1 such that $\ell_{X_{11}} = \ell_{Y_1}$ and $\ell_{X_{12}} = k$. Let $[Y_{21} \& Y_{22}]$ be an associate of Y_2 such that $\ell_{Y_{21}} = k$ and $\ell_{Y_{22}} = \ell_{X_2}$. Then $\langle X_{11} \rangle$ is $\langle Y_1 \rangle$, $\langle X_{12} \rangle$ is $\langle Y_{21} \rangle$, and $\langle X_2 \rangle$ is $\langle Y_{22} \rangle$. This can be diagramed as follows.



The proof can now be completed.

$\vdash_{P_\sigma} [X_1 \& X_2] \supset [[X_{11} \& X_{12}] \& X_2]$	by previous case
$\vdash_{P_\sigma} [X_{11} \& X_{12}] \supset [Y_1 \& Y_{21}]$	by previous case
$\vdash_{P_\sigma} [[X_{11} \& X_{12}] \& X_2] \supset [[Y_1 \& Y_{21}] \& X_2]$	by axiom 4
$\vdash_{P_\sigma} X_2 \supset Y_{22}$	by ind. hyp.
$\vdash_{P_\sigma} [[Y_1 \& Y_{21}] \& X_2] \supset [[Y_1 \& Y_{21}] \& Y_{22}]$	by axiom 3
$\vdash_{P_\sigma} [[Y_1 \& Y_{21}] \& Y_{22}] \supset [Y_1 \& [Y_{21} \& Y_{22}]]$	by axiom 2
$\vdash_{P_\sigma} [Y_1 \& [Y_{21} \& Y_{22}]] \supset [Y_1 \& Y_2]$	by previous case
$\vdash_{P_\sigma} [X_1 \& X_2] \supset [Y_1 \& Y_2]$	by axiom 7

i.e. $\vdash_{P_\sigma} X \supset Y$.

For the implication in the other direction:

$\vdash_{P_\sigma} [Y_1 \& Y_2] \supset [Y_1 \& [Y_{21} \& Y_{22}]]$	by previous case
$\vdash_{P_\sigma} [Y_{21} \& Y_{22}] \supset [X_{12} \& X_2]$	by previous case
$\vdash_{P_\sigma} [Y_1 \& [Y_{21} \& Y_{22}]] \supset [Y_1 \& [X_{12} \& X_2]]$	by axiom 3
$\vdash_{P_\sigma} Y_1 \supset X_{11}$	by ind. hyp.
$\vdash_{P_\sigma} [Y_1 \& [X_{12} \& X_2]] \supset [X_{11} \& [X_{12} \& X_2]]$	by axiom 4
$\vdash_{P_\sigma} [X_{11} \& [X_{12} \& X_2]] \supset [[X_{11} \& X_{12}] \& X_2]$	by axiom 1
$\vdash_{P_\sigma} [[X_{11} \& X_{12}] \& X_2] \supset [X_1 \& X_2]$	by previous case

$\vdash_{P_\sigma} [Y_1 \& Y_2] \supset [X_1 \& X_2]$ by axiom 7

i.e. $\vdash_{P_\sigma} Y \supset X$.

LEMMA 2. *If* $X \vdash_\sigma W$, *then* $\vdash_{P_\sigma} X' \supset W'$.

The proof is by strong induction on the number of steps in the proof of $X \vdash_\sigma W$. If there is only one step in that proof, then X is W, or $X \rightarrow W$ is a pair of U_σ. In either case, $\vdash_{P_\sigma} X' \supset W'$ by axiom 5 or axiom 6.

For the induction step, assume that $X_1 \vdash_\sigma W_1$, $X_2 \vdash_\sigma W_2, \ldots,$ $X_{n-1} \vdash_\sigma W_{n-1}$, $X \vdash_\sigma W$ is a proof in σ. Then, by the induction hypothesis, $\vdash_{P_\sigma} X_i' \supset W_i'$, $i = 1, 2, \ldots, n-1$. For each rule of the semi-Thue system which might justify $X \vdash_\sigma W$, there is a corresponding axiom in P_σ. Therefore $\vdash_{P_\sigma} X' \supset W'$, by that axiom, lemma 1, and axiom 7. Lemma 1 is needed when axiom 3 or 4 is applied.

Before continuing, another definition should be given. This is the crucial definition in the proof of theorem 1.

If W is a well formed formula of P_σ, then W is valid if and only if W is of the form $W_1 \supset W_2$ and (1) W_1 is regular, W_2 is regular, and $<W_1> \vdash_\sigma <W_2>$, or (2) W_1 is not regular, W_2 is not regular, and if W_1 is valid then W_2 is valid.

LEMMA 3. *If W is a regular well formed formula of P_σ, and if A is a well formed formula of P_σ such that A is not p_2, then the result V of substituting A for p_2 in W is not regular and is not valid.*

The proof is by strong induction on the number of occurrences of $\supset$ in W. If there is only one $\supset$, then either W is $\sim\sim[\sim p_2 \supset \sim p_2]$, or W is $\sim\sim\sim\sim[\sim p_2 \supset \sim p_2]$, since W is regular. Therefore, V is $\sim\sim[\sim A \supset \sim A]$ or V is $\sim\sim\sim\sim[\sim A \supset \sim A]$. A is not p_2. So in either case V is not $1'$ or b', and V is not of the form $\sim[A_1 \supset \sim A_2]$. Therefore V is not regular. Since V is not of the form $V_1 \supset V_2$, V is not valid.

For the induction step assume that W is $[W_1 \& W_2]$ where W_1 and W_2 are regular. Let V_1 and V_2 be the results of substituting A for p_2 in W_1 and W_2, respectively. Then V is $[V_1 \& V_2]$. By the induction hypothesis V_1 and V_2 are not regular. So V is not regular. $[V_1 \& V_2]$ is $\sim[V_1 \supset \sim V_2]$ which is not of the form $U_1 \supset U_2$. So V is not valid.

LEMMA 4. *The results of substituting well formed formulas for the propositional variables in the axioms are valid.* (In particular, axiom 6 is valid without substitution).

Let P, Q, and R be the well formed formulas substituted for p_1, q_1, and r_1, respectively, and consider the axioms individually.

Axiom 1: $[P \& [Q \& R]] \supset [[P \& Q] \& R]$.

If $P \& [Q \& R]$ is regular, then so is $[P \& Q] \& R$, and $<P \& [Q \& R]>$ is $<[P \& Q] \& R>$. So $<P \& [Q \& R]> \vdash_\sigma <[P \& Q] \& R>$. If $P \& [Q \& R]$

is not regular, then neither is $[P \& Q] \& R$. $P \& [Q \& R]$ is $\sim[P \supset \sim[Q \& R]]$, and is not of the form $W_1 \supset W_2$. So $P \& [Q \& R]$ is not valid. In either case $[P \& [Q \& R]] \supset [[P \& Q] \& R]$ is valid.

Axiom 2: $[[P \& Q] \& R] \supset [P \& [Q \& R]]$.

The proof is similar to that of axiom 1.

Axiom 3: $[P \supset Q] \supset \blacksquare [R \& P] \supset [R \& Q]$.

It is necessary to show that if $P \supset Q$ is valid, then $[R \& P] \supset [R \& Q]$ is valid. So assume that $P \supset Q$ is valid. That is, assume (1) P is regular, Q is regular, and $<P> \vdash_\sigma <Q>$, or (2) P is not regular, Q is not regular, and if P is valid then Q is valid.

There are two possibilities. Either $[R \& P]$ is regular, or it is not regular. First assume that $[R \& P]$ is regular. Then R is regular and P is regular. Since P is regular and $P \supset Q$ is valid, Q must be regular and $<P> \vdash_\sigma <Q>$. Therefore $[R \& Q]$ is regular and $<R \& P> \vdash_\sigma <R \& Q>$. So $[R \& P] \supset [R \& Q]$ is valid.

If $[R \& P]$ is not regular, then R is not regular, or R is regular and P is not regular. If R is not regular, then $[R \& Q]$ is not regular. If P is not regular, then, since $P \supset Q$ is valid, Q is not regular, and $[R \& Q]$ is not regular. $[R \& P]$ is $\sim[R \supset \sim P]$, which is not of the form $W_1 \supset W_2$. Hence $[R \& P]$ is not valid. Therefore $[R \& P] \supset [R \& Q]$ is valid.

Axiom 4: $[P \supset Q] \supset \blacksquare [P \& R] \supset [Q \& R]$.

The proof is similar to that of axiom 3.

Axiom 5: $P \supset P$.

If P is regular, then P is regular and $<P> \vdash_\sigma <P>$. If P is not regular, and if P is valid, then P is valid. So, whether or not P is regular, $P \supset P$ is valid.

Axiom 6: $G_i' \supset \bar{G}_i'$.

There are two cases to consider. In the case of no substitution G_i' and $\bar{G}_i'$ are both regular and $<G_i'> \vdash_\sigma <\bar{G}_i'>$. Therefore, axiom 6 is valid.

The other case is that in which there is substitution. By lemma 3, the results of substituting a well formed formula, not p_2, in the regular formulas G_i' and G_i' are not regular. By the same lemma the result of substituting into G_i' is not valid. Therefore, the result of substituting into $G_i' \supset \bar{G}_i'$ is valid.

Axiom 7: $[P \supset Q] \supset \blacksquare [Q \supset R] \supset [P \supset R]$.

It is necessary to show that, if $P \supset Q$ is valid, then $[Q \supset R] \supset [P \supset R]$ is valid. To show this, it must be shown that, if $Q \supset R$ is valid, then so is $P \supset R$. So, assume $P \supset Q$ is valid and $Q \supset R$ is valid, and prove that $P \supset R$ is valid.

First consider the case in which P is regular. Since $P \supset Q$ is valid, it follows that Q is regular and $<P> \vdash_\sigma <Q>$. Since Q is regular and $Q \supset R$

is valid, R must be regular and $<Q> \vdash_\sigma <R>$. Therefore, R is regular and $<P> \vdash_\sigma <R>$. So $P \supset R$ is valid.

Next consider the possibility that P is not regular. In this case, since $P \supset Q$ is valid, Q is not regular. Since $Q \supset R$ is valid, it follows that R is not regular. If P is valid, then Q is valid. If Q is valid, then R is valid. Hence, if P is valid, then R is valid. Therefore, $P \supset R$ is valid.

LEMMA 5. *If W_1 and W_2 are well formed formulas of P_σ such that W_1 is valid and $W_1 \supset W_2$ is valid, then W_2 is valid.*

W_1 is not regular, since, if it were, it would not be of the form $V_1 \supset V_2$, and hence not valid. Since $W_1 \supset W_2$ is valid and W_1 is not regular, it follows that if W_1 is valid then W_2 is valid. By hypothesis, W_1 is valid. Therefore, W_2 is valid.

LEMMA 6. *If X and W are regular and $\vdash_{P_\sigma} X \supset W$, then $<X> \vdash_\sigma <W>$.*

The proof of $\vdash_{P_\sigma} X \supset W$ can be rearranged so that all of the substitutions precede all of the uses of modus ponens. The result of substituting into the result of a substitution can be achieved by a single substitution. Hence the only substitutions necessary are direct substitution into the axioms.[9] By lemma 4, the results of such substitutions are valid. By lemma 5, modus ponens preserves validity. Therefore $X \supset W$ is valid. Since X is regular, it must follow that $<X> \vdash_\sigma <W>$.

LEMMA 7. *$W_1 \vdash_\sigma W_2$ if and only if $\vdash_{P_\sigma} W_1' \supset W_2'$.*

This lemma is a restatement of the combination of lemmas 2 and 6.

Lemma 7 shows that the decision problem for a certain class of well formed formulas of P_σ is equivalent to the word problem for σ. Since there exists a semi-Thue system on two letters for which the word problem is recursively unsolvable, this completes the proof of theorem 1.

THEOREM 2. *The problem of determining of an arbitrary partial propositional calculus whether or not it is complete is recursively unsolvable.*

To prove this theorem let σ_o be a semi-Thue system on two letters such that U_{σ_o} contains no empty words and σ_o has a recursively unsolvable word problem, and let P_{σ_o} be the corresponding partial propositional calculus constructed as in theorem 1. Let W_1 and W_2 be arbitrary regular well formed formulas of P_{σ_o}. Then construct a partial propositional calculus $R(W_1, W_2)$ such that $R(W_1, W_2)$ is complete if and only if $\vdash_{\bar{P}_{\sigma_o}} W_1 \supset W_2$. Actually $R(W_1, W_2)$ represents a class of partial propositional calculi, one for each pair of regular well formed formulas W_1 and W_2, for which the problem of determining, for an arbitrary member of the class, if it is complete is recursively unsolvable.

As in theorem 1, this proof consists of a construction and proofs that the construction satisfies the necessary requirements.

After fixing W_1 and W_2, it is only necessary to give the axioms to define $R(W_1, W_2)$. There are 10 axioms. Axioms 1-7 are the axioms of P_{σ_o}, as defined in the proof of theorem 1. The other axioms are listed below.

8. $[W_1 \supset W_2] \supset_\blacksquare p_1 \supset [q_1 \supset p_1]$

9. $[W_1 \supset W_2] \supset_\blacksquare [p_1 \supset [q_1 \supset r_1]] \supset_\blacksquare [p_1 \supset q_1] \supset [p_1 \supset r_1]$

10. $[W_1 \supset W_2] \supset_\blacksquare [\sim q_1 \supset \sim p_1] \supset [p_1 \supset q_1]$

LEMMA 1. *If* $\vdash_{P_{\sigma_0}} W_1 \supset W_2$, *then* $R(W_1, W_2)$ *is complete.*

Since the axioms of P_{σ_0} are also axioms of $R(W_1, W_2)$, if $\vdash_{P_{\sigma_0}} W_1 \supset W_2$, then $\vdash_{R(W_1, W_2)} W_1 \supset W_2$. Hence, by modus ponens and axioms 8, 9, and 10

$$\vdash_{R(W_1, W_2)} p_1 \supset [q_1 \supset p_1],$$

$$\vdash_{R(W_1, W_2)} [p_1 \supset [q_1 \supset r_1]] \supset_\blacksquare [p_1 \supset q_1] \supset [p_1 \supset r_1],$$

and

$$\vdash_{R(W_1, W_2)} [\sim q_1 \supset \sim p_1] \supset [p_1 \supset q_1].$$

These three theorems are the axioms of the complete propositional calculus. Therefore, all tautologies are theorems of $R(W_1, W_2)$, and $R(W_1, W_2)$ is complete.

To prove the converse of lemma 1 a technical validity concept is used, as in theorem 1. So this definition is given next, and lemmas proved about it.

A well formed formula X of P_{σ_0} is

A-regular if and only if there is a regular well formed formula W such that X is the result of substituting the well formed formula A for p_2 *in W.* This will be symbolized by $X = W^A$.

A well formed formula X of P is A-valid if and only if X is of the form $X_1 \supset X_2$ *and* (1) *there are regular well formed formulas* V_1 *and* V_2 *such that* $X_1 = V_1{}^A$, $X_2 = V_2{}^A$, *and* $\vdash_{P_{\sigma_0}} V_1 \supset V_2$, *or* (2) X_1 *is not A-regular,* X_2 *is not A-regular, and if* X_1 *is A-valid, then* X_2 *is A-valid.*

LEMMA 2. *If* X *is A-regular, then there is one and only one regular well formed formula W such that* $X = W^A$.

The proof is by strong induction on the number n of $\supset$'s which occur in X, but do not appear in an occurrence of A. If there is only one such $\supset$, then W is $1'$ or b'. In either case it is well defined.

If $n > 1$, then W^A is $V_1{}^A$ & $V_2{}^A$ for some V_1 and V_2. By the uniqueness of the principal $\supset$, the & is uniquely determined. By the hypothesis of induction V_1 and V_2 are well defined. Hence W is well defined, and W is $[V_1$ & $V_2]$.

LEMMA 3. *The axioms of* P_{σ_0} *are A-valid under substitution.*

As one would expect from the similarity of the definitions of validity in theorem 1 and A-validity in theorem 2, the proof of this lemma parallels the proof of lemma 4, theorem 1. It procedes by consideration of each axiom. As before, let P, Q, and R be the well formed formulas substituted for p_1, q_1, and r_1.

Axiom 1: $[P \,\&\, [Q \,\&\, R]] \supset [[P \,\&\, Q] \,\&\, R]$.

If $P \,\&\, [Q \,\&\, R] = V_1^A$, then $[P \,\&\, Q] \,\&\, R = V_2^A$, and $<V_1>$ is $<V_2>$. Hence $\vdash_{P_{\sigma_o}} V_1 \supset V_2$, and $V_1^A \supset V_2^A$ is A-valid. If $P \,\&\, [Q \,\&\, R]$ is not A-regular, then $[P \,\&\, Q] \,\&\, R$ is not A-regular. Since $P \,\&\, [Q \,\&\, R]$ is $\sim[P \supset \sim[Q \,\&\, R]]$, which is not A-valid, $[P \,\&\, [Q \,\&\, R]] \supset [[P \,\&\, Q] \,\&\, R]$ is A-valid.

Axiom 2: $[[P \,\&\, Q] \,\&\, R] \supset [P \,\&\, [Q \,\&\, R]]$.

The proof is similar to that of axiom 1.

Axiom 3: $[P \supset Q] \supset \blacksquare [R \,\&\, P] \supset [R \,\&\, Q]$.

Assume that $P \supset Q$ is A-valid, and show that $[R \,\&\, P] \supset [R \,\&\, Q]$ is A-valid. Consider first the case in which $[R \,\&\, P]$ is A-regular. Then $[R \,\&\, P] = Y_1^A$ for some regular well formed formula Y_1, such that Y_1 is $[V_1 \,\&\, V_2]$, $R = V_1^A$, and $P = V_2^A$. Since $P \supset Q$ is A-valid, $Q = V_3^A$ and $[R \,\&\, Q] = [V_1^A \,\&\, V_3^A]$. Hence $[R \,\&\, Q] = Y_2^A$, where Y_2 is $[V_1 \,\&\, V_3]$. Since $P \supset Q$ is A-valid, $\vdash_{P_{\sigma_o}} V_2 \supset V_3$. Therefore, $\vdash_{P_{\sigma_o}} [V_1 \,\&\, V_2] \supset [V_1 \,\&\, V_3]$ by axiom 3. That is, $\vdash_{P_{\sigma_o}} Y_1 \supset Y_2$. Hence $[R \,\&\, P] \supset [R \,\&\, Q]$ is A-valid.

Next consider the case in which $[R \,\&\, P]$ is not A-regular. In this case either R is not A-regular, or P is not A-regular. If R is not A-regular, then $[R \,\&\, Q]$ is not A-regular. If P is not A-regular, then, since $P \supset Q$ is A-valid, Q is not A-regular, and $[R \,\&\, Q]$ is not A-regular. $[R \,\&\, P]$ is $\sim[R \supset \sim P]$. So $[R \,\&\, P]$ is not A-valid. Therefore, $[R \,\&\, P] \supset [R \,\&\, Q]$ is A-valid.

Axiom 4. $[P \supset Q] \supset \blacksquare [P \,\&\, R] \supset [Q \,\&\, R]$.

The proof is similar to that of axiom 3.

Axiom 5. $P \supset P$.

The proof follows immediately from the definitions.

Axiom 6. $G_i' \supset \overline{G}_i'$.

Substitution yields $G_i'^A \supset \overline{G}_i'^A$. Since G_i' and $\overline{G}_i'$ are regular and $\vdash_{P_{\sigma_o}} G_i' \supset \overline{G}_i'$, this is A-valid.

Axiom 7. $[P \supset Q] \supset \blacksquare [Q \supset R] \supset [P \supset R]$.

Assume that $P \supset Q$ and $Q \supset R$ are both A-valid, and show that $P \supset R$ is A-valid. First suppose that P is A-regular and $P = V_1^A$. Then $Q = V_2^A$ and $R = V_3^A$. Since $\vdash_{P_{\sigma_o}} V_1 \supset V_2$ and $\vdash_{P_{\sigma_o}} V_2 \supset V_3$, it follows by axiom 7 that $\vdash_{P_{\sigma_o}} V_1 \supset V_3$, and $P \supset R$ is A-valid.

Next, suppose that P is not A-regular. Then Q is not A-regular, and hence R is not A-regular. If P is A-valid, then Q is A-valid. If Q is A-valid, then R is A-valid. Therefore, if P is A-valid so is R. Hence $P \supset R$ is A-valid.

LEMMA 4. *If V_1 and V_2 are well formed formulas of P_{σ_o} such that V_1 is A-valid and $V_1 \supset V_2$ is A-valid, then V_2 is A-valid.*

V_1 is not A-regular, since, if it were, it would not be of the form $Y_1 \supset Y_2$, and hence would not be A-valid. Since $V_1 \supset V_2$ is A-valid, if V_1 is A-valid then V_2 is A-valid. By hypothesis, V_1 is A-valid. Therefore, V_2 is A-valid.

LEMMA 5. *All theorems of P_{σ_o} are A-valid.*

The proof of a theorem of P_{σ_o} can be rearranged so that the substitutions precede the uses of modus ponens, and so that substitutions are made only into axioms.[10] By lemma 3, the results of such substitutions are A-valid, and by lemma 4, the subsequent results of modus ponens are also A-valid.

LEMMA 6. *If V_1, V_2, and A are regular well formed formulas of P_{σ_o}. then $\vdash_{P_{\sigma_o}} V_1{}^A \supset V_2{}^A$ if and only if $\vdash_{P_{\sigma_o}} V_1 \supset V_2$.*

If $\vdash_{P_{\sigma_o}} V_1 \supset V_2$, then substitution yields $\vdash_{P_{\sigma_o}} V_1{}^A \supset V_2{}^A$. If $\vdash_{P_{\sigma_o}} V_1{}^A \supset V_2{}^A$, then, by lemma 5, $V_1{}^A \supset V_2{}^A$ is A-valid. V_1^A and $V_2{}^A$ are both A-regular. Therefore, by the definition of A-validity, $\vdash_{P_{\sigma_o}} V_1 \supset V_2$.

The remainder of this paper was revised in proof March 7, 1964. In the original version it was claimed that axioms 8, 9, and 10 could not be used as modus ponens antecedents for axioms 1-7 to yield $W_1 \supset W_2$. Wilson E. Singletary pointed out that although the claim was correct it was not sufficient. He produced a substitution instance of axiom 7 which might lead to $W_1 \supset W_2$ by modus ponens. It is $[P \supset Q] \supset_{\blacksquare} [Q \supset R] \supset [P \supset R]$, where P, Q, and R as follows:

P: $W_1^A \supset W_2^A \supset_{\blacksquare} 1' \supset_{\blacksquare} q_2 \supset q_2 \supset 1'$

Q: $1' \supset_{\blacksquare} q_2 \supset q_2 \supset 1'$

R: $W_1 \supset W_2$.

It has not been shown that $P \supset Q$ and $Q \supset R$ are not theorems of P_{σ_o}. P is a substitution instance of axiom 8. This criticism in no way affects the proof of theorem 1 or the first six lemmas of theorem 2.

Singletary proposed systems P'_{σ_o} with seven axioms and $R'(W_1, W_2)$ with ten axioms to demonstrate theorem 2. These systems are defined as follows: Axioms 1', 2', 5', 6', and 7' of P'_{σ_o} are identical to axioms 1, 2, 5, 6, and 7 of P_{σ_o}. The axioms of $R'(W_1, W_2)$ are the axioms of P'_{σ_o} with 8', 9', and 10' added. Axioms 8', 9', and 10' are the same as axioms 8, 9, and 10 of $R(W_1, W_2)$. Axioms 3', 4', and 7' of P'_{σ_o} and $R'(W_1, W_2)$ are as follows:

3'. $[p_1 \& p_3 \supset_{\blacksquare} q_1 \& q_2] \supset_{\blacksquare} [r_1 \& [p_1 \& p_3]] \supset [r_1 \& [q_1 \& q_2]]$

4'. $[p_1 \& p_3 \supset_{\blacksquare} q_1 \& q_2] \supset_{\blacksquare} [[p_1 \& p_3] \& r_1] \supset [[q_1 \& q_2] \& r_1]$

7'. $[p_1 \& p_3 \supset_{\blacksquare} q_1 \& q_2] \supset_{\blacksquare} [q_1 \& q_2 \supset_{\blacksquare} r_1 \& r_2] \supset [p_1 \& p_3 \supset_{\blacksquare} r_1 \& r_2]$

Use of these axioms requires that $<W_1>$ and $<W_2>$ each have at least two letters, and that U_{σ_0} contain no words of less than two letters. Such semi-Thue systems with recursively unsolvable word problems do exist.[11]

It should be noted that the axioms of P'_{σ_0} are substitution instances of the axioms of P_{σ_0}. Hence the theorems of P'_{σ_0} are theorems of P_{σ_0}.

LEMMA 1a. *If* $\vdash_{P'_{\sigma_0}} W_1 \supset W_2$, *then* $R'(W_1, W_2)$ *is complete.*

The proof is exactly the same as for lemma 1, with P'_{σ_0} in place of P_{σ_0} and $R'(W_1, W_2)$ in place of $R(W_1, W_2)$.

LEMMA 7. $<W_1> \vdash_{\sigma_0} <W_2>$ *if and only if* $\vdash_{P'_{\sigma_0}} W_1 \supset W_2$.

If $\vdash_{P'_{\sigma_0}} W_1 \supset W_2$, then $\vdash_{P_{\sigma_0}} W_1 \supset W_2$, and hence $<W_1> \vdash_{\sigma_0} <W_2>$ by lemma 7, theorem 1.

The proof of lemma 2, theorem 1 can be paralleled to show that if $<W_1> \vdash_{\sigma_0} <W_2>$ then $\vdash_{P'_{\sigma_0}} W_1 \supset W_2$, since for each rule of the semi-Thue system using words of two or more letters in its defining relations, there is a corresponding axiom in P'_{σ_0}.

LEMMA 8. *A theorem of* P'_{σ_0} *is a substitution instance of an axiom or has one of the following forms:*

Form I: $[P \,\&\, Q] \supset [R \,\&\, S]$
Form II: $[P_1 \& Q_1] \supset [R_1 \& S_1] \supset \blacksquare [P_2 \& Q_2] \supset [R_2 \& S_2]$.

The proof is by induction on the number of steps in a proof in P'_{σ_0}. If there is only one step, then the theorem must be an axiom.

For the induction hypothesis, assume that $\vdash_{P'_{\sigma_0}} A_1$, $\vdash_{P'_{\sigma_0}} A_2, \ldots,$ $\vdash_{P'_{\sigma_0}} A_k$, $\vdash_{P'_{\sigma_0}} A_{k+1}$ is a proof, and that the lemma holds for A_i, $1 \leq i \leq k$. If A_{k+1} is the result of substitution, or if it is an axiom, it has the desired form. Modus ponens cannot be used with substitition instances of axioms 1', 2', 6' or Form I, since their antecedents are conjunctions and A_i, $1 \leq i \leq k$, are all implications. Modus ponens on a substitution instance of axiom 5' yields nothing new. Modus ponens on substitution instances of axioms 3', 4', or Form II give results of Form I. Modus ponens on a substitution instance of axiom 7' yields a formula of Form II. Hence A_{k+1} has the desired form.

LEMMA 9. *If* $R'(W_1, W_2)$ *is complete, then* $<W_1> \vdash_{P'_{\sigma_0}} <W_2>$.

W_1 and W_2 are regular words, and therefore tautologies. Hence $W_1 \supset W_2$ is a tautology. If $R'(W_1, W_2)$ is complete, then $\vdash_{R'(W_1, W_2)} W_1 \supset W_2$.

Examination of axioms 8', 9', and 10' shows that substitution into them will not yield a regular word implying a regular word. Neither are substitution instances of axioms 8', 9', and 10' of value as modus ponens antecedents. They obviously cannot be used with each other, and since they are not conjunctions they cannot be antecedents for substitution instances of axioms 1', 2', 6' or Form I. Axiom 5' would yield nothing new. Since $W_1 \supset W_2$ is not a conjunction, they cannot be used as antecedents for substitution instances of axioms 3', 4', 7' or Form II.

Axioms 8', 9', and 10' can be used to obtain theorems shorter than themselves only if $W_1 \supset W_2$, or a substitution instance of it, is already available as a theorem. Therefore, by using only axioms 1' - 7' it must be possible to prove $\vdash_{P'_{\sigma_o}} W_1 \supset W_2$, or $\vdash_{P'_{\sigma_o}} W_1^A \supset W_2^A$ for some well formed formula A. Hence $\vdash_{P_{\sigma_o}} W_1 \supset W_2$ or $\vdash_{P_{\sigma_o}} W_1^A \supset W_2^A$. By lemma 6, the latter case also yields $\vdash_{P_{\sigma_o}} W_1 \supset W_2$. By lemma 7, theorem 1, this gives the desired result, $<W_1> \vdash_{P_{\sigma_o}} <W_2>$.

LEMMA 10. *$R'(W_1, W_2)$ is complete if and only if $<W_1> \vdash_{\sigma_o} <W_2>$.*

The implication in one direction is lemma 9. For the other implication assume $<W_1> \vdash_{\sigma_o} <W_2>$. Then by lemma 7, $\vdash_{P'_{\sigma_o}} W_1 \supset W_2$. Hence by lemma 1a, $R'(W_1, W_2)$ is complete.

By lemma 10, the problem of determining, of an arbitrary partial propositional calculus of the class represented by $R'(W_1 W_2)$, whether or not it is complete is equivalent to the word problem of σ_o. Since σ_o has a recursively unsolvable word problem, the problem of determining, of an arbitrary partial propositional calculus, whether or not it is complete is recursively unsolvable. This completes the proof of theorem 2.

The system P'_{σ_o} and the kind of analysis made of that system are closely related to the question of specifying a partial propositional calculus whose decision problem is of an arbitrarily assigned recursively enumerable degree of unsolvability. M. D. Gladstone and Ann H. Ihrig had independently of each other specified such constructions prior to Singletary's suggestion that P'_{σ_o} be used in proving theorem 2.

NOTES

1. Presented in the University of Illinois Logic Seminar in October, 1962.
2. Linial, Samuel, and Post, Emil L., Recursive Unsolvability of the Deducibility, Tarski's Completeness and Independence of Axioms Problems of the Propositional Calculus (Abstract), *Bulletin of the American Mathematical Society*, vol. 55, p. 50, 1949.
3. Davis, Martin, *Computability and Unsolvability*, McGraw-Hill Book Co., pp. 137-142, 1958.
4. Church, Alonzo, *Introduction to Mathematical Logic*, vol. 1, Princeton University Press, pp. 74-75, 1956.
5. Boone, William W., The Word Problem, *Annals of Mathematics*, vol. 70, no. 2, p. 207, 1959. A semi-Thue system is defined on page 213.
6. The addition of the rule, if $C \vdash D$ then $D \vdash C$, would make the system a Thue system.

7. Davis, Martin, *op. cit.*, p. 93.

   Hermes, H., *Aufzählbarkeit, Entscheidbarkeit, Berechenbarkeit*, Springer-Verlag, p. 147-156, 1961.

   Kleene, S. C., *Introduction to Metamathematics*, Van Nostrand Co., pp. 382-386, 1952.

   Post, Emil L., Recursive Unsolvability of a Problem of Thue, *The Journal of Symbolic Logic*, vol. 12, pp. 1-11, 1947.

8. Church, Alonzo, *op. cit.*, pp. 122-123.

9. Church, Alonzo, *op. cit.*, pp. 149-150.

10. See note 9.

11. Boone, William W., *op. cit.*, p. 210, footnote 7.

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