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UNIVERSAL PAIRS OF REGRESSIVE ISOLS

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1 *Introduction* Universal isols were first introduced by E. Ellentuck in [4] to provide a uniform source of counter-examples for proposed arithmetic statements in Λ. Prof. Ellentuck was also the first to prove, in unpublished notes, the existence of regressive universal isols, which provide a source for counter-examples in Λ_R; his proof is essentially a category argument. The present paper generalizes this argument to prove the existence of universal pairs of regressive isols which can serve as a source of counter-examples for proposed properties of Λ_R^2.

For f a recursive combinatorial function, let C_f denote the canonical extension of f to the isols; if f is recursive, then D_f denotes the canonical extension. From [4] we have the following definition: An isol T is *universal* if for each pair of recursive, combinatorial functions f and g,

$$C_f(T) = C_g(T) \rightarrow \{x \mid f(x) \neq g(x)\} \text{ is finite}$$
$$\text{or}$$
$$\text{there exists a number } n \text{ such that } x \geqslant n \rightarrow f(x) = g(x).$$

We are interested here in pairs of regressive isols (S, T) that have the property that if $f(x, y)$ and $g(x, y)$ are any recursive, combinatorial functions of x and y, then the identity $C_f(S, T) = C_g(S, T)$ will imply certain non-trivial similarities between the two functions f and g.

One analogue of the above definition would require a universal pair (S, T) of regressive isols to have the property that for $f(x, y)$ and $g(x, y)$ any recursive, combinatorial functions,

$$C_f(S, T) = C_g(S, T) \rightarrow \{(x, y) \mid f(x, y) \neq g(x, y)\} \text{ is finite.}$$

However, it is not difficult to construct recursive combinatorial functions $\tilde{f}$ and $\tilde{g}$ having the property that for all infinite regressive isols S and T,

$$C_{\tilde{f}}(S, T) = C_{\tilde{g}}(S, T) \text{ and } \{(x, y) \mid \tilde{f}(x, y) \neq \tilde{g}(x, y)\} \text{ is infinite;}$$

even easier functions refute the implication if S or T is taken to be finite. Thus we see that this analogue of the one-dimensional definition is too stringent, and we are led to the following definition: A pair of regressive

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isols (S, T) is *universal* if for each pair of recursive, combinatorial functions $f(x, y)$ and $g(x, y)$,

$$C_f(S, T) = C_g(S, T) \rightarrow \exists \text{ numbers } m \text{ and } n \text{ such that}$$
$$x \geqslant m \text{ and } y \geqslant n \rightarrow f(x, y) = g(x, y).$$

2 *Universal Pairs* We will outline the proof of the existence of universal pairs of regressive isols. Following E. Ellentuck, we let $\mathbf{E}$ denote the non-negative integers and define a function f from a subset of $\mathbf{E}$ into $\mathbf{E}$ to be *initial* if the domain of f, δf, is empty or of the form $\{0, 1, \ldots, k\}$ for some $k \in \mathbf{E}$. A function f: $\mathbf{E} \rightarrow \mathbf{E}$ is e-*initial.* Let

$$\mathbf{X} = \text{the collection of all e-initial functions,}$$
$$\mathbf{G} = \text{the collection of all initial functions,}$$
$$\mathbf{F} = \mathbf{X} \cup \mathbf{G}.$$

For functions f and g in $\mathbf{F}$, $f \leqslant g$ denotes the function g an extension of the function f. For $f \in \mathbf{F}$, let

$$N_f = \{g \in \mathbf{X} \mid f \leqslant g\}$$

and

$$\mathscr{B} = \{(N_f \times N_g) \mid f, g \in \mathbf{G}\}.$$

Then $\mathscr{B}$ serves as a base for a topology on $\mathbf{X}^2$; we let $\mathbf{X}^2$ indicate the space with this topology.

Lemma 1 $\mathbf{X}^2$ *is Category* II.

Proof: Let A be any Category I subset of $\mathbf{X}^2$, i.e.,

$$A = \sum_0^\infty A_i$$

where for each i, A_i is nowhere dense in $\mathbf{X}^2$. We wish to show that $A \not\subseteq \mathbf{X}^2$. Let $(N_f \times N_g) \in \mathscr{B}$. Because A_0 is nowhere dense in $\mathbf{X}^2$, there exist functions f_0 and g_0 in $\mathbf{G}$ such that

$$(N_{f_0} \times N_{g_0}) \subset (N_f \times N_g)$$

and

$$(N_{f_0} \times N_{g_0}) \cap A_0 = \emptyset.$$

Continuing, sequences of functions $\{f_i\}$ and $\{g_i\}$ are obtained with

$$f \leqslant f_0 \leqslant f_1 \leqslant \ldots \text{ and } g \leqslant g_0 \leqslant g_1 \leqslant \ldots$$

and

$$(N_{f_i} \times N_{g_i}) \cap \underset{j \leqslant i}{A_j} = \emptyset.$$

Four possibilities arise concerning these functions:

(i) $(\exists k)(f_k = f_{k+1} = \ldots)$ and $(\exists j)(g_j = g_{j+1} = \ldots)$,

(ii) $(\exists k)(f_k = f_{k+1} = \ldots)$ and $\bigcup_0^\infty \delta g_i = \mathbf{E}$,

(iii) $\bigcup_0^\infty \delta f_i = \mathbf{E}$ and $(\exists j)(g_j = g_{j+1} = \ldots)$,

(iv) both $\bigcup_0^\infty \delta f_i = \mathbf{E}$ and $\bigcup_0^\infty \delta g_i = \mathbf{E}$.

For each of these cases we construct a member of $\mathbf{X}^2 - A$.

(i) Let $m = \max(k, j)$. Then $(\mathrm{N}_{f_m} \times \mathrm{N}_{g_m}) \subset \mathbf{X}^2 - A$.
(ii) Let a function $\tilde{g}$ be defined by $\tilde{g} = \lim_{i\to\infty} g_i$. Then $(\mathrm{N}_{f_k} \times \tilde{g}) \subset \mathbf{X}^2 - A$.
(iii) Similar to case (ii).
(iv) Let $\tilde{f} = \lim_{i\to\infty} f_i$, $\tilde{g} = \lim_{i\to\infty} g_i$; then $(\tilde{f}, \tilde{g}) \in \mathbf{X}^2 - A$.

This completes the proof of Lemma 1.

For $f \in \mathbf{F}$, we define a function f^* with $\delta f^* = \delta f$ by

$$f^*(n) = \prod_0^n q_i^{f(i)+1}$$

where q enumerates the primes in increasing order. Let π_f denote the range of f^*. Then for $f \in \mathbf{X}$, π_f is an infinite retraceable set.

Lemma 2 *Let $\{\alpha_i\}$ be an enumeration of all infinite* r.e. *sets. Let*

$$A_i = \{f \mid f \in \mathbf{X} \text{ and } \alpha_i \subset \pi_f\}$$

and

$$\mathbf{W} = \sum_0^\infty A_i = \{f \mid f \in \mathbf{X} \textit{ and } \pi_f \textit{ contains an infinite } \text{r.e. } \textit{subset}\}.$$

Then both $\mathbf{W} \times \mathbf{X}$ *and* $\mathbf{X} \times \mathbf{W}$ *are Category* I *in* $\mathbf{X}^2$.

Proof: $\mathbf{W} \times \mathbf{X} = \left(\sum_0^\infty A_i\right) \times \mathbf{X} = \sum_0^\infty (A_i \times \mathbf{X})$. If we can prove that $A_i \times \mathbf{X}$ is nowhere dense in $\mathbf{X}^2$, then $\mathbf{W} \times \mathbf{X}$ will be Category I. Let $(\mathrm{N}_f \times \mathrm{N}_g) \in \mathscr{B}$. Then $f \in \mathbf{G}$ with $\delta f = \{0, 1, \ldots, k-1\}$, where this is the empty set if $k = 0$, and π_f is a finite set. Let m be a number such that $m \in \alpha_i$ and $m \notin \pi_f$. Define a function $h(x)$ by

$$\begin{aligned} \delta h &= \{0, \ldots, k\}, \\ h(x) &= f(x) \text{ for } 0 \leq x \leq k-1, \\ h(k) &= m. \end{aligned}$$

Then $(\mathrm{N}_h \times \mathrm{N}_g) \subset (\mathrm{N}_f \times \mathrm{N}_g)$ and $(\mathrm{N}_h \times \mathrm{N}_g) \cap (A_i \times \mathbf{X}) = \emptyset$. Hence $A_i \times \mathbf{X}$ is nowhere dense in $\mathbf{X}^2$ and $\mathbf{W} \times \mathbf{X}$ is Category I in $\mathbf{X}^2$. A similar proof holds for $\mathbf{X} \times \mathbf{W}$.

Lemma 3 *Let $h_1(x, y)$ and $h_2(x, y)$ be two recursive combinatorial functions of two variables which are induced by the normal recursive combinatorial operations Φ_1 and Φ_2, respectively. Let $p(x)$ be a one-to-one partial recursive function. Define a set λ*

$$\lambda = \{(x,y) \mid h_1(x,y) \neq h_2(x,y)\}$$

and a set H

$$\mathsf{H} = \mathsf{H}(p, h_1, h_2) = \{(f, g) \in \mathsf{X}^2 \mid \Phi_1(\pi_f, \pi_g) \subset \delta p \wedge p\Phi_1(\pi_f, \pi_g) = \Phi_2(\pi_f, \pi_g)\}.$$

If λ *is totally unbounded, then* H *is nowhere dense in* X^2.

Proof: Let $(\mathsf{N}_f \times \mathsf{N}_g) \in \mathscr{B}$. Then $f, g \in \mathsf{G}$ with $\delta f = \{0, 1, \ldots, n-1\}$ and $\delta g = \{0, 1, \ldots, m-1\}$ (these are empty if $n = 0$ or $m = 0$, respectively). We may assume $(\mathrm{card}\ \delta f, \mathrm{card}\ \delta g) \in \lambda$. If not, since λ is totally unbounded, extensions f' and g' of f and g fulfill this property and $(\mathsf{N}_{f'} \times \mathsf{N}_{g'}) \subset (\mathsf{N}_f \times \mathsf{N}_g)$; the proof could proceed on $(\mathsf{N}_{f'} \times \mathsf{N}_{g'})$. If $(\mathsf{N}_f \times \mathsf{N}_g) \cap \mathsf{H} = \emptyset$, the proof is complete, so assume the existence of $(\tilde{f}, \tilde{g}) \in (\mathsf{N}_f \times \mathsf{N}_g) \cap \mathsf{H}$. $(\tilde{f}, \tilde{g}) \in (\mathsf{N}_f \times \mathsf{N}_g) \rightarrow \pi_f \subset \pi_{\tilde{f}}$ and $\pi_g \subset \pi_{\tilde{g}}$, so that $\Phi_1(\pi_f, \pi_g) \subset \Phi_1(\pi_{\tilde{f}}, \pi_{\tilde{g}})$ and $\Phi_2(\pi_f, \pi_g) \subset \Phi_2(\pi_{\tilde{f}}, \pi_{\tilde{g}})$. $(\tilde{f}, \tilde{g}) \in \mathsf{H} \rightarrow \Phi_1(\pi_{\tilde{f}}, \pi_{\tilde{g}}) \subset \delta p$ and $p\Phi_1(\pi_{\tilde{f}}, \pi_{\tilde{g}}) = \Phi_2(\pi_{\tilde{f}}, \pi_{\tilde{g}})$. Thus $\Phi_1(\pi_f, \pi_g) \subset \delta p$ and $\Phi_2(\pi_f, \pi_g) \subset \rho p$. However, $(\mathrm{card}\ \delta f, \mathrm{card}\ \delta g) = (\mathrm{card}\ \pi_f, \mathrm{card}\ \pi_g) \in \lambda$, so that $h_1(\mathrm{card}\ \pi_f, \mathrm{card}\ \pi_g) \neq h_2(\mathrm{card}\ \pi_f, \mathrm{card}\ \pi_g)$ or, by a property of Φ_1 and Φ_2, $\mathrm{card}\ \Phi_1(\pi_f, \pi_g) \neq \mathrm{card}\ \Phi_2(\pi_f, \pi_g)$. Since p is one-to-one, we cannot have $p\Phi_1(\pi_f, \pi_g) = \Phi_2(\pi_f, \pi_g)$. Two cases may obtain:

(i) $\exists x \in \Phi_1(\pi_f, \pi_g)$ and $\exists y \in \Phi_2(\pi_{\tilde{f}}, \pi_{\tilde{g}}) - \Phi_2(\pi_f, \pi_g)$, $y = p(x)$,
(ii) $\exists x \in \Phi_1(\pi_{\tilde{f}}, \pi_{\tilde{g}}) - \Phi_1(\pi_f, \pi_g)$ and $\exists y \in \Phi_2(\pi_f, \pi_g)$, $y = p(x)$.

In each case we construct a member of $\mathscr{B}$ which is a subset of $(\mathsf{N}_f \times \mathsf{N}_g)$ and whose intersection with H is empty.

(i) Define function $\bar{f}$ by

$$\begin{aligned} &\delta\bar{f} = \{0, \ldots, n\}, \\ &\bar{f}(x) = f(x) \text{ for } 0 \leqslant x \leqslant n-1, \\ &\bar{f}(n) \quad \text{is such that } \bar{f}^*(n) > \max\,(\text{1'st components in } \Phi_2^{-1}(y)). \end{aligned}$$

Define function $\bar{g}$ by

$$\begin{aligned} &\delta\bar{g} = \{0, \ldots, m\}, \\ &\bar{g}(x) = g(x) \text{ for } 0 \leqslant x \leqslant m-1, \\ &\bar{g}(m) \quad \text{is such that } \bar{g}^*(m) > \max\,(\text{2'nd components in } \Phi_2^{-1}(y)). \end{aligned}$$

Then $(\mathsf{N}_{\bar{f}} \times \mathsf{N}_{\bar{g}}) \subset (\mathsf{N}_f \times \mathsf{N}_g)$ and $(\mathsf{N}_{\bar{f}} \times \mathsf{N}_{\bar{g}}) \cap \mathsf{H} = \emptyset$.

(ii) Define function $\bar{f}$ by

$$\begin{aligned} &\delta\bar{f} = \{0, \ldots, n\}, \\ &\bar{f}(x) = f(x) \text{ for } 0 \leqslant x \leqslant n-1, \\ &\bar{f}(n) \quad \text{is such that } \bar{f}^*(n) > \max\,(\text{1'st components in } \Phi_1^{-1}(x)). \end{aligned}$$

Define function $\bar{g}$ by

$$\begin{aligned} &\delta\bar{g} = \{0, \ldots, m\}, \\ &\bar{g}(x) = g(x) \text{ for } 0 \leqslant x \leqslant m-1, \\ &\bar{g}(m) \quad \text{is such that } \bar{g}^*(m) > \max\,(\text{2'nd components of } \Phi_1^{-1}(x)). \end{aligned}$$

Then $(\mathsf{N}_{\bar{f}} \times \mathsf{N}_{\bar{g}}) \subset (\mathsf{N}_f \times \mathsf{N}_g)$ and $(\mathsf{N}_{\bar{f}} \times \mathsf{N}_{\bar{g}}) \cap \mathsf{H} = \emptyset$.

This completes the proof of Lemma 3.

Theorem 1 *A universal pair of regressive isols exists.*

Proof: Let (h_{1_k}, h_{2_k}) be an enumeration of all pairs of recursive combinatorial functions of two variables such that for each k, $\lambda_k = \{(x,y) \mid h_{1_k}(x,y) \neq h_{2_k}(x,y)\}$ is a totally unbounded set. For each k and each one-to-one partial recursive function p, we have from Lemma 3 that the set $\mathsf{H}(p, h_{1_k}, h_{2_k})$ is nowhere dense in $\mathbf{X}^2$. Let $\mathbf{W}$ be defined as in Lemma 2. Then using Lemma 2, the set $\mathbf{M}$,

$$\mathbf{M} = \sum_{p,k} \mathsf{H}(p, h_{1_k}, h_{2_k}) \cup (\mathbf{W} \times \mathbf{X}) \cup (\mathbf{X} \times \mathbf{W}),$$

is Category I in $\mathbf{X}^2$. Since $\mathbf{X}^2$ is Category II by Lemma 1, let $(s,t) \in \mathbf{X}^2 - \mathbf{M}$. Then $s, t \in \mathbf{X}$ so π_s and π_t are infinite retraceable sets. Also, $s \notin \mathbf{W}$ so that π_s contains no infinite r.e. subset, i.e., π_s is immune. Similarly π_t is immune and if $\mathsf{S} = \mathrm{Req}\ \pi_s$, $\mathsf{T} = \mathrm{Req}\ \pi_t$, we have $\mathsf{S}, \mathsf{T} \in \Lambda_R - \mathbf{E}$.

We will show that (S, T) is a universal pair. Let $h_1(x,y)$ and $h_2(x,y)$ be two recursive combinatorial functions such that $\mathsf{C}_{h_1}(\mathsf{S}, \mathsf{T}) = \mathsf{C}_{h_2}(\mathsf{S}, \mathsf{T})$. Let Φ_1 and Φ_2 be the operations inducing h_1 and h_2, respectively. Then $\mathrm{Req}\ \Phi_1(\pi_s, \pi_t) = \mathrm{Req}\ \Phi_2(\pi_s, \pi_t)$ so that there exists a one-to-one partial recursive function $p(x)$ such that

$$\Phi_1(\pi_s, \pi_t) \subset \delta p \text{ and } p\Phi_1(\pi_s, \pi_t) = \Phi_2(\pi_s, \pi_t).$$

But since $(s,t) \notin \mathsf{H}(p, h_1, h_2)$, the set $\lambda = \{(x,y) \mid h_1(x,y) \neq h_2(x,y)\}$ cannot be totally unbounded. Thus there exist numbers m and n such that $x \geq m$ and $y \geq n$ imply $h_1(x,y) = h_2(x,y)$. This completes the proof.

We summarize some easily shown properties of universal pairs of regressive isols.

Proposition 1 *Let* (S, T) *be a universal pair of regressive isols. Then*

a) *both* S *and* T *are universal,*
b) $\mathsf{S} \neq \mathsf{T}$,
c) (T, S) *is also a universal pair.*

3 *An Application* The $\leq^*$ relation between isols was introduced in [3], where it was shown that there are pairs of regressive isols incomparable relative to $\leq^*$. (This result also appears in [1].) The use of universal pairs of regressive isols to contradict universal properties of $\Lambda_R{}^2$ is illustrated below in a third proof of this result.

First we characterize universal pairs in terms of the canonical extension to $\Lambda_R{}^2$, α_{R^2}, of a recursive relation α in $\mathbf{E}^2$.

Proposition 2 *Let* $\mathsf{S}, \mathsf{T} \in \Lambda_R - \mathbf{E}$. *Then* (S, T) *is a universal pair* $\leftrightarrow (\mathsf{S}, \mathsf{T}) \notin \alpha_{R^2}$ *for all sets* $\alpha \subset \mathbf{E}^2$ *such that* α *is recursive and* $\mathbf{E}^2 - \alpha$ *is totally unbounded.*

We will make use of the following result due to J. Barback:

Lemma (Barback) *Let a recursive set* $\alpha \subset \mathbf{E}^2$ *be defined by*

$$\alpha = \{(x,y) \mid x \leq y\}.$$

Then for $X, Y \in \Lambda_R$, $(X, Y) \in \alpha_{R^2} \leftrightarrow X \leq^* Y$.

Proof: Since the statement

$$x \leq y \leftrightarrow \min(x, y) = x$$

is valid in E, we apply a well-known result of A. Nerode to extend to Λ_R and get

$$(X, Y) \in \alpha_{R^2} \leftrightarrow D_{\min}(X, Y) = X.$$

But in Λ_R, by a result in [2]

$$D_{\min}(X, Y) = \min(X, Y)$$

and from [3], Theorem T4(c)

$$\min\,(X, Y) = X \leftrightarrow X \leq^* Y.$$

Therefore

$$(X, Y) \in \alpha_{R^2} \Leftrightarrow X \leq^* Y.$$

Theorem 2 *There exist regressive isols* S *and* T *that are incomparable relative to* $\leq^*$.

Proof: Let (S, T) be a universal pair of regressive isols. By Proposition 1(c), (T, S) is a universal pair. Again let

$$\alpha = \{(x, y) \,|\, x \leq y\}.$$

Then α is recursive and $\mathsf{E}^2 - \alpha$ is totally unbounded. By Proposition 2, $(S, T) \notin \alpha_{R^2}$ and $(T, S) \notin \alpha_{R^2}$. Now apply the preceding Lemma to get

$$S \not\leq^* T \text{ and } T \not\leq^* S.$$

This completes the proof.

REFERENCES

[1] Barback, J., "Two notes on regressive isols," *Pacific Journal of Mathematics*, vol. 16 (1966), pp. 407-420.

[2] Barback, J., "Double series of isols," *Canadian Journal of Mathematics*, vol. 19 (1967), pp. 1-15.

[3] Dekker, J. C. E., "The minimum of two regressive isols," *Mathematischke Zeitschrift*, vol. 83 (1964), pp. 345-366.

[4] Ellentuck, E., "Universal isols," *Mathematischke Zeitschrift*, vol. 98 (1967), pp. 1-8.

[5] Myhill, J., "Recursive equivalence types and combinatorial functions," *Bulletin of the American Mathematical Society*, vol. 64 (1958), pp. 373-376.

[6] Nerode, A., "Extensions to isols," *Annals of Mathematics*, vol. 73 (1961), pp. 362-403.

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