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DECOMPOSABLE ORTHOLOGICS

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1 *Introduction* The inability of Boolean algebras to represent the logic of quantum mechanics, as observed by Birkhoff and von Neuman [1], Mackey [11], and others [6], [12], [13], has created interest in logics which represent the propositions of empirical science affiliated with more than one physical operation.

In [4] and [5] Foulis and Randall introduced the concept of a *manual* of physical operations whose elements are identified with the sets of outcomes for physical experiments. The logic affiliated with such a manual is called an *orthologic*. Most, if not all, of the logics which have been proposed for quantum mechanics by the above authors have been orthologics.

In [2], Dacey introduced the *Dacey sum* of a collection of manuals, which is a manual whose elements are identified with the outcome sets of two stage physical experiments. A manual is said to be *indecomposable* if it cannot be written in a non-trivial fashion as a Dacey sum. In [8] the author defines the *composite product* of a family of orthologics, which corresponds to the logic of the Dacey sum of manuals whence the orthologics came. Here we define an orthologic to be *indecomposable* if it cannot be written in a non-trivial fashion as a composite product of orthologics. Our main theorem states that an orthologic is indecomposable if and only if every member of a wide class of manuals whence it came is indecomposable. Our definition of decomposability of an orthologic appears to be the natural generalization of the concept of irreducibility in lattice theory. In particular, we also show that every indecomposable orthologic is irreducible.

The majority of the results appearing here may be found in the author's dissertation, submitted to the graduate school of the University of Massachusetts in 1972, and written under the direction of Professor D. J. Foulis. The author would like to thank Professor Foulis for his assistance in this work.

2 *Definitions and Motivation* Let $\mathfrak{A}$ denote a non-empty set of non-empty sets. Write $A = \bigcup \mathfrak{A}$. For $x, y \in A$, write $x \perp y$ to mean that $x \neq y$ and there

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exists $E \in \mathfrak{A}$ with $x, y \in E$. Define $D \subseteq A$ to be an *event* for $\mathfrak{A}$ if there exists $E \in \mathfrak{A}$ with $D \subseteq A$. Define $B \subseteq A$ to be an *orthogonal set* if $x \perp y$ holds for all $x, y \in B$ with $x \neq y$. Then $\mathfrak{A}$ is said to be a *manual* if and only if the following conditions are satisfied.

M1 If $E, F \in \mathfrak{A}$ with $E \subseteq F$, then $E = F$.
M2 If $E, F \in \mathfrak{A}$ and if B is an orthogonal set contained in $E \cup F$, then B is an event.

In general we will use script letters to denote manuals and the corresponding capital Roman letters to represent the union of the manual.

Each $x \in A$ is called an *outcome* for the manual. Each $E \in \mathfrak{A}$ is called an *operation* for the manual, and its elements are precisely the outcomes for a physical operation identified with E. If $x, y \in A$ with $x \perp y$, then we say the x is *orthogonal* to y or x *refutes* y. Physically, if $x \perp y$ then x and y are distinct outcomes for the same physical operation. If $x \perp y$ holds for all $x, y \in A$ with $x \neq y$, then by M1 and M2 $\mathfrak{A}$ consists of precisely one operation and A is the classical sample space for an experiment. In this case, $\mathfrak{A}$ is called a κ-*manual*. At the other extreme, if $x \not\perp y$ for any $x, y \in A$, then $\mathfrak{A}$ is called a *scattered* manual.

Write $O(\mathfrak{A})$ for the set of all events for $\mathfrak{A}$. If $B \subseteq A$ write $B^{\perp}$ for $\{x \in A\colon\ x \perp b \text{ for all } b \in B\}$, $B^{\perp\perp}$ for $(B^{\perp})^{\perp}$, etc. If $x \in A$ we will write $x^{\perp}$ for $\{x\}^{\perp}$. Let $\mathcal{L}(\mathfrak{A}) = \{D^{\perp\perp}\colon\ D \in O(\mathfrak{A})\}$. Partially order $\mathcal{L}(\mathfrak{A})$ by containment. If $B, C \in O(\mathfrak{A})$ define $B^{\perp\perp}$ to be *orthogonal* to $C^{\perp\perp}$ in $\mathcal{L}(\mathfrak{A})$, in symbols $B^{\perp\perp} \perp C^{\perp\perp}$, if and only if $B \subseteq C^{\perp}$. Write 0 for $\emptyset$, the empty set, as a member of $\mathcal{L}(\mathfrak{A})$. Write 1 for A as a member of $\mathcal{L}(\mathfrak{A})$. It is straightforward to verify that the system $\langle \mathcal{L}(\mathfrak{A}), \leq, 0, 1, \perp \rangle$ satisfies the following conditions:

L1 $\langle \mathcal{L}(\mathfrak{A}), \leq \rangle$ is a partially ordered set satisfying $0 \leq x \leq 1$ for all $x \in \mathcal{L}(\mathfrak{A})$.
L2 $\perp$ is a symmetric binary relation on $\mathcal{L}(\mathfrak{A})$ satisfying the property that whenever $x \in \mathcal{L}(\mathfrak{A})$ with $x \perp x$ then $x = 0$.
L3 If $x, y \in \mathcal{L}(\mathfrak{A})$ with $x \perp y$ then the supremum of x and y in $\mathcal{L}(\mathfrak{A})$, $x \vee y$, exists in $\mathcal{L}(\mathfrak{A})$.
L4 If $x, y, z \in \mathcal{L}(\mathfrak{A})$ with $x \perp y$, $y \perp z$, and $x \perp z$, then $x \perp (y \vee z)$.
L5 If $x \in \mathcal{L}(\mathfrak{A})$ then there exists $y \in \mathcal{L}(\mathfrak{A})$ with $x \perp y$ and $x \vee y = 1$.
L6 If $x, y \in \mathcal{L}(\mathfrak{A})$ then $x \leq y$ holds if and only if for all $z \in \mathcal{L}(\mathfrak{A})$ the condition $z \perp y$ implies $z \perp x$.

The system $\langle \mathcal{L}(\mathfrak{A}), \leq, 0, 1, \perp \rangle$, as described above, is called the *logic affiliated with the manual* $\mathfrak{A}$. The elements of $\mathcal{L}(\mathfrak{A})$ are called *propositions*. We say that $B^{\perp\perp} \in \mathcal{L}(\mathfrak{A})$ is *confirmed* by the outcome $x \in A$ exactly when $x \in B^{\perp\perp}$, and *refuted* by the outcome $x \in A$ exactly when $x \in B^{\perp}$.

Any system $\langle L, \leq, 0, 1, \perp \rangle$, where L is any abstract set, satisfying L1-L6 is called an *orthologic*. Hence the logic affiliated with every manual is an orthologic. Conversely, it is shown in [3] that all orthologics arise from manuals in this way. That is, given an orthologic L, there exists a manual $\mathfrak{A}$ (not necessarily unique) such that L is (up to isomorphism) the logic affiliated with $\mathfrak{A}$. Furthermore, if F is any undirected graph with no loops, and we define $\mathfrak{A}$ by $E \in \mathfrak{A}$ if and only if E is the set of vertices of a clique of F, then $\mathfrak{A}$ is a manual. Every finite manual and every finite orthologic arise in this way.

Two manuals $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *isomorphic*, in symbols $\mathfrak{A} \approx \mathfrak{B}$, if and only if there exists a bijection φ: $A \to B$ satisfying $E \in \mathfrak{A}$ if and only if $\varphi(E) \in \mathfrak{B}$. Two orthologics L and L' are said to be isomorphic, in symbols $L \approx L'$, if and only if there exists a bijection φ: $L \to L'$ satisfying the condition that for all $x, y \in L$, $x \perp y$ if and only if $\varphi(x) \perp \varphi(y)$. Clearly isomorphisms preserve all existing structure.

All the above definitions, and further motivation for manuals and orthologics may be found in [4] and [5].

Let $\mathfrak{B}$ be a manual called the *conditioned manual* and let $\left\{\mathfrak{A}_b\text{: } b \in B = \bigcup \mathfrak{B}\right\}$ be a family of manuals called the *conditioning manuals*, indexed by B. Then the *Dacey sum* of the conditioning manuals over the conditioned manual, denoted by $\sum \mathfrak{B} | \mathfrak{A}_b$ is defined to be the manual $\mathfrak{C}$, where $E \in \mathfrak{C}$ if and only if the following conditions hold:

D1 $E \subseteq \bigcup_{b \in B} \{b\} \times A_b$, where $A_b = \bigcup \mathfrak{A}_b$ for each $b \in B$.

D2 The projection on the first factor of E, denoted $\pi_1(E)$, is an element of $\mathfrak{B}$.

D3 For each $b \in \pi_1(E)$, $\pi_2(E) \cap A_b \in \mathfrak{A}_b$, where $\pi_2(E)$ denotes the projection on the second factor of E.

It is straightforward to verify that $\mathfrak{C}$ is indeed a manual,

$$C = \bigcup \mathfrak{C} = \bigcup_{b \in B} \{b\} \times A_b, \text{ and if } (b,x),\ (b_1, x_1) \in C \text{ then } (b,x) \perp (b_1, x_1)$$

if and only if either $b \perp b_1$ in $\mathfrak{B}$, or $b = b_1$ and $x \perp x_1$ in $\mathfrak{A}_b$. Physically, if $E \in \mathfrak{C}$, then E is the following operation: Perform the experiment with outcome set $\pi_1(E)$. If the outcome $b \in \pi_1(E)$ is obtained then perform the experiment with outcome set $\pi_2(E) \cap A_b$. If the outcome x is obtained, then record the outcome (b, x). Note that two outcomes in $\mathfrak{C}$ refute each other if and only if the outcomes of the first stage refute each other, or the outcomes of the first stage agree and the outcomes of the second stage refute each other.

A manual $\mathfrak{C}$ is called *decomposable* if it can be written in a non-trivial fashion as a Dacey sum. More precisely, $\mathfrak{C}$ is decomposable if and only if either it is a K-manual and C contains exactly two points, or there exists a subset R of C containing at least two points satisfying the following properties: (1) For every $x \in C/R$ either $R \cup \{x\}$ is an event or $x \not\perp r$ for any $r \in R$. (2) There exists $x \in C/R$ with $x^{\perp} \neq \emptyset$. The subset R is called a *partitive subset*. Note that if $\mathfrak{C}$ is not a K-manual with C containing exactly two points and $\mathfrak{C}$ is decomposable, then $\mathfrak{C} \approx \sum \mathfrak{B} | \mathfrak{A}_b$, where $\mathfrak{B}$ and $\mathfrak{A}_b$ are defined as follows: $B = \{R\} \cup C/R$, θ: $C \to B$ is defined by $\theta(x) = x$ if $x \in C/R$ and $\theta(x) = R$ if $x \in R$, $\mathfrak{B} = \{\theta(E)\text{: } E \in \mathfrak{C}\}$, $\mathfrak{A}_R = \{R\}$, $\mathfrak{A}_b = \{\{b\}\}$ for all $b \in C/R$. In this case, we will call $\sum \mathfrak{B} | \mathfrak{A}_b$ the *decomposition of $\mathfrak{C}$ induced by R.*

Conversely, if $\mathfrak{C} \approx \sum \mathfrak{B} | \mathfrak{A}_b$ and there exists $b \in B$ with A_b containing at least two points, and there exists $b_0 \in B/\{b\}$ with either $b_0^\perp \neq \emptyset$ or with $x \in A_{b_0}$ and $x^\perp \neq \emptyset$, then $\mathfrak{C}$ is decomposable. A manual is called *indecomposable* if it is not decomposable.

We now turn to a piecing together of orthologics which corresponds to the Dacey sum. Let $\mathfrak{B}$ be a manual. For each $b \in B$ let $\langle L_b, \leqslant_b, 0_b, 1_b, \perp_b \rangle$ be an orthologic. The *composite product* over $\mathfrak{B}$ of $\{L_b\colon\ b \in B\}$, denoted $\pi \mathfrak{B} | L_b$, is defined to be the set of all triples of the form $\langle K, T, \{t_y\colon y \in T\}\rangle$ where:

(i) $T \in O(\mathfrak{B})$.
(ii) There exists $Q \in O(\mathfrak{B})$ with $Q \subseteq T^\perp$ and $K = (T \cup Q)^{\perp\perp}$.
(iii) For each $y \in T$, $t_y \in L_y/\{0_y, 1_y\}$.

For notational convenience, if we write that G is an element of the product we will mean that $G = \langle K_G, T_G, \{g_y\colon\ y \in T_G\}\rangle$.

For $G, H \in \pi \mathfrak{B} | L_b$, we define $G \leqslant H$ if and only if the following conditions hold:

(i) $T_H \subseteq T_G \cup K_G{}^\perp$.
(ii) $K_G \subseteq K_H$.
(iii) If $y \in T_G \cap T_H$ then $g_y \geqslant_y h_y$.

We define $G \perp H$ if and only if the following conditions hold:

(i) $K_G \cap T_G{}^\perp \subseteq K_H{}^\perp$.
(ii) $T_G/T_H \subseteq K_H{}^\perp$.
(iii) If $y \in T_G \cap T_H$ then $g_y \perp_y h_y$.

Then $\langle \pi \mathfrak{B} | L_b, \leqslant, 0, 1, \perp \rangle$ is an orthologic, where $0 = \langle \emptyset, \emptyset, \emptyset \rangle$ and $1 = \langle B, \emptyset, \emptyset \rangle$. Furthermore, if $\{\mathfrak{A}_b\colon b \in B\}$ is *any* collection of manuals satisfying $\mathcal{L}(\mathfrak{A}_b) \approx L_b$ for all $b \in B$ then $\pi \mathfrak{B} | L_b \approx \pi \mathfrak{B} | \mathcal{L}(\mathfrak{A}_b) \approx \mathcal{L}\left(\sum \mathfrak{B} | \mathfrak{A}_b\right)$. These proofs may be found in [8], Theorem 5.1.

An orthologic L is said to be *decomposable* if it can be written non-trivially as a composite product. More precisely, L is decomposable if and only if either L is isomorphic to the Boolean algebra 2^2 or there exists a manual $\mathfrak{B}$ and a family of orthologics $\{L_b\colon b \in B\}$ with $L \approx \pi \mathfrak{B} | L_b$ and at least two of the orthologics in $\{\mathcal{L}(\mathfrak{B})\} \cup \{L_b\colon\ b \in B\}$ are not the Boolean algebra 2. Our choice for non-triviality is due to the facts that if $\mathfrak{B}$ is any manual with $L \approx \mathcal{L}(\mathfrak{B})$, then $L \approx \pi \mathfrak{B} | 2$, and if $\mathfrak{B}$ is any scattered manual then $\mathcal{L}(\mathfrak{B}) \approx 2$ and $L \approx \pi \mathfrak{B} | L_b$ where $L_b = L$ for some fixed $b \in B$ and $L_t = 2$ for all $t \in B/\{b\}$.

3 *Lemmas* Given any orthologic L, there exists a manual $\mathfrak{A}$ with the property that $L \approx \mathcal{L}(\mathfrak{A})$. The fact that this manual is not unique gives rise to difficulties when we wish to describe the decomposability of an orthologic in terms of the decomposability of the manual whence the logic came. We therefore distinguish certain manuals, those which are called *point-determining*, as being, in some sense, canonical. We begin by defining a manual $\mathfrak{A}'$ to be a *point determinant* of a manual $\mathfrak{A}$ if there exists a surjection $\sigma\colon A \to A'$ satisfying the following conditions:

(i) If $a, b \in A$ then $a^{\perp} = b^{\perp}$ if and only if $\sigma(a) = \sigma(b)$.
(ii) $\mathfrak{A}' = \{\sigma(E): E \in \mathfrak{A}\}$.

Lemma 3.1 *Let* $\mathfrak{A}$, $\mathfrak{A}'$ *and* σ *be as above. If* $a, b \in A$, *then* $a \perp b$ *if and only if* $\sigma(a) \perp \sigma(b)$.

Proof: Assume that $a \perp b$. Then there exists $E \in \mathfrak{A}$ with $a, b \in E$. But then $\sigma(E) \in \mathfrak{A}'$, so either $\sigma(a) \perp \sigma(b)$ or $\sigma(a) = \sigma(b)$. If $\sigma(a) = \sigma(b)$ then $b \in a^{\perp} = b^{\perp}$, a contradiction. Conversely, assume $\sigma(a) \perp \sigma(b)$. Then there exists $E \in \mathfrak{A}$ with $\sigma(a), \sigma(b) \in \sigma(E) \in \mathfrak{A}'$. Here there exists $c, d \in E$ with $\sigma(a) = \sigma(c)$ and $\sigma(b) = \sigma(d)$. If $c = d$ then $\sigma(a) = \sigma(b)$, a contradiction. Hence $c \perp d$. Thus $c \in d^{\perp} = b^{\perp}$, so $b \in c^{\perp} = a^{\perp}$, and $a \perp b$.

Note that given any manual $\mathfrak{A}$, if $\mathfrak{A}' = \{\{[x] \mid x \in E\}: E \in \mathfrak{A}\}$, where for each $x \in A$, $[x]$ is the equivalence class containing x under the equivalence relation $x \equiv y$ if and only if $x^{\perp} = y^{\perp}$, and if σ: $A \to A'$ is the natural map, then $\mathfrak{A}'$ is a point determinant of $\mathfrak{A}$. Hence every manual has a point determinant. Furthermore, the point determinant of a manual is unique up to isomorphism. For if $\mathfrak{A}'$ and σ are defined as above, and $\mathfrak{A}''$ is another point determinant for $\mathfrak{A}$, with σ_1: $A \to A''$ the surjection, then φ: $A' \to A''$, defined by $\varphi([x]) = \sigma_1(x)$ is an isomorphism of manuals. We will therefore speak of *the* point determinant of $\mathfrak{A}$, denote it by $\mathfrak{A}'$, and write σ: $A \to A'$ for the surjection.

Lemma 3.2 *Let* $\mathfrak{A}$ *be a manual. Then the following hold:*

(i) $O(\mathfrak{A}') = \{\sigma(B) \mid B \in O(\mathfrak{A})\}$.
(ii) *If* $B \subseteq A$, *then* $\sigma(B^{\perp}) = \sigma(B)^{\perp}$ *and* $\sigma(B^{\perp\perp}) = \sigma(B)^{\perp\perp}$.
(iii) $\mathcal{L}(\mathfrak{A})' = \{\sigma(G): G \in \mathcal{L}(\mathfrak{A})\}$.
(iv) $\mathcal{L}(\mathfrak{A}) \approx \mathcal{L}(\mathfrak{A}')$.

Proof: (i) Let $B \in O(\mathfrak{A})$. Then there exists $E \in \mathfrak{A}$ with $B \subseteq E$. Then $\sigma(B) \subseteq \sigma(E) \in \mathfrak{A}'$, so $\sigma(B) \in O(\mathfrak{A}')$. Conversely, let $B' \in O(\mathfrak{A}')$. Then there exists $E \in \mathfrak{A}$ with $B' \subseteq \sigma(E) \in \mathfrak{A}'$. Then $B' = \sigma(\sigma^{-1}(B') \cap E)$, where $\sigma^{-1}(B') \cap E \subseteq E \in \mathfrak{A}$ implies that $\sigma^{-1}(B') \cap E \in O(\mathfrak{A})$.
(ii) It follows immediately from Lemma 3.1 that $\sigma(B^{\perp}) \subseteq \sigma(B)^{\perp}$. So let $z \in \sigma(B)^{\perp}$. Then there exists $r \in A$ with $\sigma(r) = z$. Let $b \in B$ be arbitrary. Then $\sigma(b) \in \sigma(B)$ implies $\sigma(r) \perp \sigma(b)$ so by Lemma 3.1, $r \perp b$. Hence $r \in B^{\perp}$ and $z \in \sigma(B^{\perp})$. Hence $\sigma(B^{\perp}) = \sigma(B)^{\perp}$. Then $\sigma(B^{\perp\perp}) = \sigma(B^{\perp})^{\perp} = \sigma(B)^{\perp\perp}$.
(iii) The proof of (iii) follows from (i) and (ii) and the fact that $G \in \mathfrak{A}$ (respectively $\mathfrak{A}'$) if and only if $G = D^{\perp\perp}$ for some $D \in O(\mathfrak{A})$ (respectively $O(\mathfrak{A}')$).
(iv) Define φ: $\mathcal{L}(\mathfrak{A}) \to \mathcal{L}(\mathfrak{A}')$ by $\varphi(G) = \sigma(G)$ for $G \in \mathcal{L}(\mathfrak{A})$. By (iii), φ is well defined and a surjection. Assume $G, H \in \mathcal{L}(\mathfrak{A})$ with $\sigma(G) = \sigma(H)$. Then by (ii) and Lemma 3.1, $x \in G^{\perp}$ if and only if $\sigma(x) \in \sigma(G)^{\perp} = \sigma(H)^{\perp} = \sigma(H^{\perp})$, if and only if $x \in H^{\perp}$. Hence $G^{\perp} = H^{\perp}$, and $G = G^{\perp\perp} = H^{\perp\perp} = H$. Thus φ is a bijection. The fact that it is an isomorphism follows from (ii) and Lemma 3.1.

If $\mathfrak{A}$ is any manual which is isomorphic to its own point determinant, then $\mathfrak{A}$ is called *point determining*. Clearly $(\mathfrak{A}')' \approx \mathfrak{A}'$, that is, the point determinant of every manual is point determining. Since, as mentioned earlier, every manual has a point determinant, and since $\mathcal{L}(\mathfrak{A}) \approx \mathcal{L}(\mathfrak{A}')$, from the viewpoint of the logics, there is no loss in dealing with point determining manuals.

Note also that a manual $\mathfrak{A}$ is point determining if and only if the identity map σ: $A \to A$ satisfies the conditions of the definition of $\mathfrak{A}$ being a point determinant of $\mathfrak{A}$. Hence it follows immediately that $\mathfrak{A}$ is point determining if and only if whenever $a, b \in A$ and $a^\perp = b^\perp$ then $a = b$.

If $\mathfrak{A}$ is a manual, then $x \in A$ is an *isolated point* if $x^\perp = \emptyset$. Note that if $\mathfrak{A}$ is point determining it can have at most one isolated point.

Lemma 3.3 *Let $\mathfrak{A}$ be a manual. Then either $\mathfrak{A}$ is scattered or there exists a point determining manual $\mathfrak{A}_0$, having no isolated points, with $\mathcal{L}(\mathfrak{A}) \approx \mathcal{L}(\mathfrak{A}_0)$.*

Proof: Assume that $\mathfrak{A}$ is not scattered. By Lemma 3.2 (iv) we may assume that $\mathfrak{A}'$ has isolated points. Since $\mathfrak{A}'$ is point determining, it has exactly one isolated point, call it z. Then $\{z\} \in \mathfrak{A}'$. Let $\mathfrak{A}_0 = \mathfrak{A}'/\{z\}$. Since $\mathfrak{A}$ is not scattered $\mathfrak{A}' \neq \{z\}$, so $\mathfrak{A}_0 \neq \emptyset$. It is straightforward to show that $\mathfrak{A}_0$ is the required manual.

4 Main Theorem *An orthologic L is indecomposable if and only if for all point determining manuals $\mathfrak{A}$ for which $L \approx \mathcal{L}(\mathfrak{A})$, $\mathfrak{A}$ is indecomposable.*

Proof: Assume first that for all point determining manuals $\mathfrak{A}$ for which $L \approx \mathcal{L}(\mathfrak{A})$, $\mathfrak{A}$ is indecomposable. Suppose that L is decomposable. We wish to obtain a contradiction. We may assume that $L \not\approx 2^2$, for if so then $L \approx \mathcal{L}(\mathfrak{A})$ where $\mathfrak{A}$ is a K-manual and A contains exactly two points, so $\mathfrak{A}$ is point determining and decomposable, which is a contradiction. Hence there exists a manual $\mathfrak{B}$, and a collection of orthologics $\{L_b\colon b \in B\}$ such that at least two of the orthologics in $\{\mathcal{L}(\mathfrak{B})\} \cup \{L_b\colon b \in B\}$ are not isomorphic to 2, and $L \approx \pi\mathfrak{B} \mid L_b$.

Define an equivalence relation, $\equiv$, on B as follows: if $b, t \in B$, then $b \equiv t$ if and only if either $b^\perp = t^\perp$ *and* $L_b \approx 2$ *and* $L_t \approx 2$, or $b = t$. For each $b \in B$, write $[b]$ for the equivalence class containing b. Let $\mathfrak{B}_0 = \{\{[b]\colon b \in E\}\colon E \in \mathfrak{B}\}$. Then $B_0 = \bigcup \mathfrak{B}_0$ is the induced partition of B. It is straightforward to verify that $\mathfrak{B}_0$ is a manual. For each $[b] \in B_0$ with $L_b \approx 2$, let $A_{[b]} = \{x_{[b]}\}$, a singleton, and let $\mathfrak{A}_{[b]}$ be the corresponding manual. For each $[b] \in B_0$ with $L_b \not\approx 2$ let $\mathfrak{A}_{[b]}$ be a point determining manual with no isolated points satisfying $L_b \approx \mathcal{L}(\mathfrak{A}_{[b]})$. Note that this is well defined since $L_b \not\approx 2$ implies $[b] = \{b\}$, and $\mathfrak{A}_{[b]}$ exists by Lemma 3.3. Let $\mathfrak{A}_0 = \sum \mathfrak{B}_0 \mid \mathfrak{A}_{[b]}$.

For each $b \in B$ with $L_b \approx 2$ let $A_b = \{x_b\}$, a singleton, and let $\mathfrak{A}_b$ be the corresponding manual. For each $b \in B$ with $L_b \not\approx 2$, let $\mathfrak{A}_b = \mathfrak{A}_{[b]}$. Let $\mathfrak{A} = \sum \mathfrak{B} \mid \mathfrak{A}_b$.

We claim that $\mathfrak{A}_0$ is the point determinant of $\mathfrak{A}$. To see this, define σ: $A \to A_0$ by $\sigma(b, t) = ([b], t)$ if $L_b \not\approx 2$, and $\sigma(b, t) = ([b], x_{[b]})$ if $L_b \approx 2$. Assume first that (b, t), $(a, s) \in A$ with $(b, t)^\perp = (a, s)^\perp$. If $b = a$ then $t^\perp = s^\perp$ in $\mathfrak{A}_b$, which is point determining. In this case, if $L_b \not\approx 2$ then $\sigma(b, t) = ([b], t) = ([a], s) = \sigma(a, s)$, and if $L_b \approx 2$ then $\sigma(b, t) = ([b], x_{[b]}) = ([a], x_{[a]}) = \sigma(a, s)$. Hence we may assume that $b \neq a$ in which case $b^\perp = a^\perp$. If there exists $r \in A_b$ with $r \perp t$ then $(b, r) \in (b, t)^{\perp}$ implies $(b, r) \in (a, s)^{\perp}$, so $b \perp a$, which is a contradiction. Hence t is an isolated point of A_b. Similarly, s is an isolated point of A_a. Now if $L_b \not\approx 2$ then A_b contains no isolated points, a contradiction. Hence $L_b \approx 2$. Similarly, $L_a \approx 2$. Thus $a \equiv b$. Thus $\sigma(b, t) = ([b], x_{[b]}) = ([a], x_{[a]}) = \sigma(a, s)$. Conversely, assume that $\sigma(b, t) = \sigma(a, s)$. Then either $[a] = [b]$ and $L_a \approx L_b \approx 2$, in which case $a^{\perp} = b^{\perp}$ and $(b, t)^{\perp} = (a, s)^{\perp}$, or $[a] = [b]$ and $t = s$ and $L_b \not\approx 2$, $L_a \not\approx 2$ implies $a = b$, in which case $(b, t) = (a, s)$, so $(b, t)^{\perp} = (a, s)^{\perp}$.

We have left to show that $\mathfrak{A}_0 = \{\sigma(E) \mid E \in \mathfrak{A}\}$. Assume first that $E \in \mathfrak{A}$. Then D1 is satisfied for $\sigma(E)$ by the definition of σ and D2 is satisfied by the definition of $\mathfrak{B}_0$. To verify D3, let $[b] \in \pi_1(\sigma(E))$. If $L_b \approx 2$, then $\pi_2(\sigma(E)) \cap A_{[b]} = \{x_{[b]}\} \in \mathfrak{A}_{[b]}$, and if $L_b \not\approx 2$, then $\pi_2(\sigma(E)) \cap A_{[b]} = \pi_2(E) \cap A_b \in \mathfrak{A}_b = \mathfrak{A}_{[b]}$. Conversely, let $F \in \mathfrak{A}_0$. Then by D2, there exists $G \in \mathfrak{B}$ with $\pi_1(F) = \{[b] \colon b \in G\}$. By D3, for each $[b] \in \pi_1(F)$ with $L_b \not\approx 2$, there exists $C_b \in \mathfrak{A}_b = \mathfrak{A}_{[b]}$ with $\pi_2(F) \cap A_{[b]} = C_b$. For each $b \in \pi_1(F)$ with $L_b \approx 2$, let $C_b = \{x_b\} = A_b$. Let $E = \bigcup_{b \in G} \{b\} \times C_b$. It is straightforward to verify that $E \in \mathfrak{A}$ and $\sigma(E) = F$. We have shown that $\mathfrak{A}_0$ is the point determinant of $\mathfrak{A}$.

Since $\mathfrak{A} = \sum \mathfrak{B} \mid \mathfrak{A}_b$, $L \approx \pi \mathfrak{B} \mid L_b$, and $\mathcal{L}(\mathfrak{A}_b) \approx L_b$ for all $b \in B$, it follows that $L \approx \mathcal{L}(\mathfrak{A})$. Then by Lemma 3.2 (iv), $L \approx \mathcal{L}(\mathfrak{A}_0)$. Since L is decomposable, there exists at least one $z \in B$ with $L_z \not\approx 2$. Hence $\mathfrak{A}_z = \mathfrak{A}_{[z]}$ is a point determining manual with no isolated points, so $A_{[z]}$ contains at least two orthologonal points. Hence $R = \{[z]\} \times A_{[z]}$ is a subset of A_0 containing at least two points. Now for each $([b], t) \in A_0/R$, if $([b], t) \notin R^{\perp}$ then $[b] \neq [z]$ implies $([b], t) \not\perp r$ for any $r \in R$. But since $\mathfrak{A}_0$ is point determining, it is indecomposable by hypothesis. Hence R is not a partitive subset of $\mathfrak{A}_0$. Thus $t^{\perp} = \emptyset$ for all $t \in A_0/R$.

Now again since L is decomposable, either there exists $w \in B/\{z\}$ with $L_w \not\approx 2$ or $\mathcal{L}(\mathfrak{B}) \not\approx 2$. Suppose first that there exists $w \in B/\{z\}$ with $L_w \not\approx 2$. Then $A_{[w]}$ contains at least two orthogonal points, so there exists $([w], s) \in A_{[w]}$ with $([w], s)^{\perp} \neq \emptyset$. But $L_w \not\approx 2$ and $w \neq z$ implies $[w] \neq [z]$, so $([w], s) \in A_0/R$, with $([w], s)^{\perp} \neq \emptyset$, which is a contradiction. Hence we may suppose that $L_w \approx 2$ for all $w \in B/\{z\}$ and $\mathcal{L}(\mathfrak{B}) \not\approx 2$. Hence there exists, b_1, $b_2 \in B$ with $b_1 \perp b_2$. Choose and fix $x_1 \in A_{b_1}$ and $x_2 \in A_{b_2}$. Then $(b_1, x_1) \perp (b_2, x_2)$, so by Lemma 3.1, $\sigma(b_1, x_1) \perp \sigma(b_2, x_2)$. Since not both b_1 and b_2 can be equivalent to (and hence equal to) z, there exists $i = 1$ or 2 with $\sigma(b_i, x_i) \in A_0/R$ and $\sigma(b_i, x_i)^{\perp} \neq \emptyset$, which is again a contradiction. Hence L is indecomposable.

Conversely, assume that L is indecomposable. Suppose there exists a point determining manual $\mathfrak{A}$ for which $L \approx \mathcal{L}(\mathfrak{A})$ and $\mathfrak{A}$ is decomposable. We wish to obtain a contradiction. Since $L \not\approx 2^2$, $\mathfrak{A}$ is not a K-manual with A containing exactly two points. So there exists a partitive subset R of A. Let $\sum \mathfrak{B} \mid \mathfrak{A}_b$ be the decomposition of $\mathfrak{A}$ induced by R. Then $\mathfrak{A} \approx \sum \mathfrak{B} \mid \mathfrak{A}_b$, so $L \approx \pi \mathfrak{B} \mid \mathcal{L}(\mathfrak{A}_b)$.

Suppose that R is scattered. Let $x, y \in R$. Then $x^{\perp} = \{z \in A/R \mid z \in R^{\perp}\} = y^{\perp}$. But $\mathfrak{A}$ is point determining, so $x = y$. But this is a contradiction since R contains at least two points. Hence R is not scattered and $\mathcal{L}(\mathfrak{A}_R) \not\approx 2$.

Now there exists $b \in A/R$ with $b^{\perp} \neq \emptyset$. Say $e \in b^{\perp}$. Then there exists $E \in \mathfrak{A}$ with $b, e \in E$. But $\theta(E) \in \mathfrak{B}$ by definition of the decomposition induced by R, so either $\theta(e) \perp \theta(b)$ or $\theta(e) = \theta(b)$. If $\theta(e) = \theta(b)$, then $b \in R$, which is a contradiction. Hence $\theta(e) \perp \theta(b)$. But $\theta(e)$, $\theta(b) \in B$, so $\mathcal{L}(\mathfrak{B}) \not\approx 2$. But $\mathcal{L}(\mathfrak{A}_R) \not\approx 2$ and $L \approx \pi\mathfrak{B} | \mathcal{L}(\mathfrak{A}_b)$, so L is decomposable, which is the required contradiction.

5 *Concluding Remarks* Our concept of decomposibility in an orthologic is consistent with the notion of reducibility in lattice theory. Firstly, as the familiar cartesian product of a family of more than one orthomodular poset is reducible [7], the cartesian product of a family of more than one orthologic is decomposable. (Incidentally, every orthomodular poset is an orthologic, and numerous examples may be found in [9] to show that the converse fails.)

Lemma **5.1** *Let B be a set containing at least two elements, and let $\{L_b: b \in B\}$ be a family of orthologics indexed by B. Then the cartesian product of $\{L_b: b \in B\}$, denoted $\bigtimes_{b \in B} L_b$, is a decomposable orthologic.*

Proof: Let $\mathfrak{B} = \{B\}$, a K-manual. Define $\varphi: \bigtimes_{b \in B} L_b \to \pi\,\mathfrak{B} | L_b$ by $\varphi(\{x_b\}) = \langle K, T, \{t_b: b \in T\}\rangle$ where $T = \{b \in B \,|\, x_b \notin \{0, 1\} \subseteq L_b\}$, $K = \{b \in B \,|\, x_b \neq 0\}$, $t_b = x_b$ for all $b \in T$. Then φ is an isomorphism of orthologics [9], p. 89. Now $\mathcal{L}(\mathfrak{B}) \not\approx 2$, so if there exists at least one $b \in B$ with $L_b \not\approx 2$ then we are done. Thus we are entitled to assume that $L_b \approx 2$ for all $b \in B$. If B contains exactly two elements, then $\bigtimes_{b \in B} L_b \approx 2^2$, which is decomposable by definition. Otherwise, fix $x, y \in B$ with $x \neq y$. Let $L_a = \bigtimes_{\substack{b \in B \\ b \neq x, y}} L_b$, $L_d = \bigtimes_{b \in \{x, y\}} L_b$, and $\mathfrak{C} = \{\{a, d\}\}$, a K-manual. Then by the same argument as above, $\bigtimes_{b \in B} L_b \approx \bigtimes_{c \in C} L_c \approx \pi\,\mathfrak{C} | L_c$. But then $\mathcal{L}(\mathfrak{C}) \not\approx 2$ and $L_d \not\approx 2$ so $\bigtimes_{b \in B} L_b$ is decomposable.

As a corollary to the above lemma, we conclude that every reducible orthologic is decomposable. We say that two elements x and y of an orthologic L commute, in symbols $x \mathrm{C} y$, if they are compatible in the sense of Mackey [11], that is, there exists $a, b, c \in L$, which are pairwise orthogonal, satisfying $x = a \vee b$ and $y = c \vee b$. Then the *center* of L, denoted $\mathrm{C}(L)$, is $\{x \in L | x \mathrm{C} y \text{ for all } y \in L\}$. An orthologic L is said to be *reducible* if $\mathrm{C}(L) \not\approx 2$. In [10] we show that $\mathrm{C}(L)$ is always a Boolean algebra.

Corollary **5.2** *Every reducible orthologic is decomposable.*

Proof: Choose and fix $e \in \mathrm{C}(L)$ with $e \notin \{0, 1\}$. We claim first that $[0, e] = \{x \in L \,|\, 0 \leq x \leq e\}$, with operations restricted from L, is an orthologic. It is enough to verify L5 and the sufficiency of the condition in L6 for $x \leq y$. L5 follows immediately from the fact that if $a \in [0, e]$ then $e \in \mathrm{C}(L)$ implies that $e \mathrm{C} a$. So suppose $a, b \in [0, e]$ and that for all $f \in [0, e]$, $f \perp b$ implies $f \perp a$. We must show that $a \leq b$. Assume that $t \in L$ with $t \perp b$. It is enough to show that $t \perp a$. By [10], Lemma 4, $t = (e \wedge t) \vee (e' \wedge t)$, where e' is the orthocomplement of e in the Boolean algebra $\mathbf{C}(L)$. Since $e \wedge t \leq e$ and $(e \wedge t) \perp b$, by hypothesis we have that $(e \wedge t) \perp a$. Since $e' \wedge t \leq e'$ and $a \perp e'$, it follows that $a \perp (e' \wedge t)$. Hence $t \perp a$.

Since $e' \in \mathbf{C}(L)$ enjoys the same properties as e, $[0, e]$ and $[0, e']$ are orthologics. By the lemma, it is sufficient to show that $L \approx [0, e] \times [0, e']$. The isomorphism is precisely the one used in the orthomodular case, and the argument is similar. The lemmas required to justify some of the properties of orthomodular posets for orthologics used in the proof may be found in [10].

We remark that the converse of Corollary **5.2** fails, as expected, with this weaker-than-orthomodular structure. Indeed, the horizontal sum [7] of a family of non-trivial orthologics is always decomposable, while it certainly is not reducible.

Lemma **5.3** *Let B be a set containing at least two elements, and let $\{L_b : b \in B\}$ be a family of orthologics indexed by B, at least two of which are not isomorphic to* 2. *Then the horizontal sum, denoted $\underset{b \in B}{+} L_b$, is decomposable.*

Proof: Let $\mathfrak{B} = \{\{b\} : b \in B\}$, a scattered sample space. Define $\varphi : \pi \mathfrak{B} | L_b \to \underset{b \in B}{+} L_b$ by $\varphi\langle B, \emptyset, \emptyset\rangle = 1$, $\varphi\langle\emptyset, \emptyset, \emptyset\rangle = 0$ and $\varphi\langle B, \{b\}, \{t_b\}\rangle = t_b$ for each $b \in B$. Then φ is an isomorphism of orthologics [9], p. 95, and hence $\underset{b \in B}{+} L_b$ is decomposable.

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